A Theory of Truth with a Determinacy Operator

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12th Panhellenic Logic Symposium,
Anogeia, Crete, June 2019

(L) (L) is not true.

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(T) **S** is true iff *p*.

(L) (L) is not true.

(T) **S** is true iff p.

(M) (M) is true.

(C) If (C) is true, then $\bot$.

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(C) is true iff (if (C) is true, then $\bot$).

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There are two kinds of models, valuations$_*$ and valuations. A valuation$_*$ is an assignment, to each formula, of one of the three values. The connectives are governed by the following tables:

A	$\neg$**A**
1	0
1/2	1/2
0	1

A **B**	**A** $\vee$ **B**
at least one has 1	1
both have 1/2, or the one has 1/2 and the other 0	1/2
both have 0	0

A B	A ∧ B	
both have 1	1	
both have 1/2, or the one has 1/2 and the other 1	1/2 or 0	(i)
at least one has 0	0	

A	B	A → B	
1	1	1	
1	1/2	1/2	
1	0	0	
1/2	1	1	
1/2	1/2	1 or 1/2	(ii)
1/2	0	1/2	
0	1	1	
0	1/2	1	
0	0	1	

There are additional rules for cases (i)–(ii). As regards conjunction, their effect is that, in every valuation$_*$ and for all $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the conjunction $\mathbf{A} \wedge \mathbf{A}$, as well as $[\mathbf{A} \wedge \mathbf{A}] \wedge \mathbf{A}$, has the same value as $\mathbf{A}$, while $\mathbf{A} \wedge \mathbf{B}$ has the same value as $\mathbf{B} \wedge \mathbf{A}$, and $[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}$ has the same value as $\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]$. As regards implication, their effect is that, in every valuation$_*$ and for all $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, if $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{C}$ get 1, then so does $\mathbf{A} \rightarrow \mathbf{C}$, and if $\mathbf{A} \rightarrow \mathbf{B}$ gets 1, then so does $\mathbf{B}' \rightarrow \mathbf{A}'$, where $\mathbf{B}' \rightarrow \mathbf{A}'$ is a *contrapositive* of $\mathbf{A} \rightarrow \mathbf{B}$, that is, one formula in the pair $\{\mathbf{A}, \mathbf{A}'\}$ is the negation of the other and, also, one formula in the pair $\{\mathbf{B}, \mathbf{B}'\}$ is the negation of the other.

Valuations are the valuations$_*$ that conform with two more rules, one for $\wedge$ and one for $\rightarrow$.

D is a *deep conjunct* of **C** iff **D** is not a conjunction and there is an occurrence $\mathcal{O}$ of **D** in **C** such that every symbol, in **C** but outside $\mathcal{O}$, in whose scope $\mathcal{O}$ lies is a $\wedge$.

The rules are the following: (α) If **A** $\wedge$ **B** falls under case (i), and there is no valuation$_*$ in which all its deep conjuncts have 1, and there is also no valuation$_*$ in which they all have 0, then **A** $\wedge$ **B** must get 0. And (β) if **A** $\rightarrow$ **B** falls under (ii), and in each valuation$_*$ the value of **A** is smaller than, or equal to, the value of **B**, then **A** $\rightarrow$ **B** must get 1.

An inference from premisses $\mathbf{A}_1, \ldots, \mathbf{A}_j$ to conclusion $\mathbf{B}$ is *valid* ($\mathbf{A}_1, \ldots, \mathbf{A}_j \vDash \mathbf{B}$) iff $\mathbf{B}$ has 1 in every valuation in which $\mathbf{A}_1, \ldots, \mathbf{A}_j$ have 1. A formula $\mathbf{A}$ is *valid* ($\vDash \mathbf{A}$) iff it has 1 in every valuation.

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We want to be able to assign 0 to a contradiction $\mathbf{A} \wedge \neg\mathbf{A}$ even if $\mathbf{A}$ and $\neg\mathbf{A}$ have $1/2$. On the other hand, if $\mathbf{A}$ has $1/2$, we should give the same value to the conjunction $\mathbf{A} \wedge \mathbf{A}$. Thus value-functionality is eschewed for $\wedge$.

Modus ponens is valid. (For any $\mathbf{A}$ and $\mathbf{B}$) $\mathbf{A} \rightarrow \mathbf{B}, \mathbf{A} \vDash \mathbf{B}$. Thus, if the conditional $\mathbf{A} \rightarrow \mathbf{B}$ is valid, so is the inference from $\mathbf{A}$ to $\mathbf{B}$. On the other hand, it may be that the inference is valid, but the conditional is not. Generally, if $\vDash \mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i \rightarrow \mathbf{B}$, then $\mathbf{A}_1, \ldots, \mathbf{A}_i \vDash \mathbf{B}$. But it may be that $\mathbf{A}_1, \ldots, \mathbf{A}_i \vDash \mathbf{B}$, yet $\nvDash \mathbf{A}_1 \wedge \cdots \wedge \mathbf{A}_i \rightarrow \mathbf{B}$.

(β) If $\mathbf{A} \to \mathbf{B}$ falls under (ii), and in each valuation$_*$ the value of $\mathbf{A}$ is smaller than, or equal to, the value of $\mathbf{B}$, then $\mathbf{A} \to \mathbf{B}$ must get 1.

Rule (β) validates various conditionals in which it is valid to infer the consequent from the antecedent. For example, it ensures that $\vDash \mathbf{A} \to \mathbf{A}$, $\vDash \mathbf{A} \to \neg\neg\mathbf{A}$ and $\vDash \neg\neg\mathbf{A} \to \mathbf{A}$.

(α) If $\mathbf{A} \wedge \mathbf{B}$ falls under case (i), and there is no valuation$_*$ in which all its deep conjuncts have 1, and there is also no valuation$_*$ in which they all have 0, then $\mathbf{A} \wedge \mathbf{B}$ must get 0.

Rule (α) ensures that $\vDash \neg[\mathbf{A} \wedge \neg\mathbf{A}]$. Generally, the effect of (α) is to make various conjunctions whose conjuncts clearly seem incompatible take on the value 0 in all valuations. In this way, it validates the negations of those conjunctions.

Since we validate $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ and do not permit a conjunction $\mathbf{A} \wedge \mathbf{A}$ to have 0 in a valuation where $\mathbf{A}$ has $1/2$, we abandon substitution of equivalents. $\ldots\mathbf{B}\ldots, \mathbf{A} \leftrightarrow \mathbf{B} \not\models \ldots\mathbf{A}\ldots$.

Since we validate $\neg[\mathbf{A} \wedge \neg\mathbf{A}]$ and do not permit a conjunction $\mathbf{A} \wedge \mathbf{A}$ to have 0 in a valuation where $\mathbf{A}$ has $1/2$, we abandon substitution of equivalents. $\ldots\mathbf{B}\ldots, \mathbf{A} \leftrightarrow \mathbf{B} \nvDash \ldots\mathbf{A}\ldots$.

The main difference between Field's logic and the system presented here is that his logic validates the substitution of equivalents but does not include the law of non-contradiction, while the current system does the opposite.

The logic that is being sketched out validates many classical principles, but not all of them. Some noteworthy deviations from classical logic are the following:

$\nvDash \mathbf{A} \vee \neg \mathbf{A}$.

$\neg[\mathbf{A} \wedge \mathbf{B}] \nvDash \neg \mathbf{A} \vee \neg \mathbf{B}$.

Although $\neg \mathbf{A} \vee \mathbf{B} \vDash [\mathbf{A} \rightarrow \mathbf{B}]$, the converse does not hold: $\mathbf{A} \rightarrow \mathbf{B} \nvDash \neg \mathbf{A} \vee \mathbf{B}$.

And there are many cases where a classical inference is validated, but the corresponding conditional is not.

We can consider inferences of the form '$\Gamma_1, \ldots, \Gamma_k, \mathbf{A}_1, \ldots, \mathbf{A}_n$; hence, $\mathbf{B}$' ($k \geq 1$, $n \geq 0$). Here, $\mathbf{A}_1, \ldots, \mathbf{A}_n$ and $\mathbf{B}$ are formulae, but each one of $\Gamma_1, \ldots, \Gamma_k$ is an inference in which the premisses and conclusion are formulae.

Extending our concept of validity to such inferences, our logic does not validate conditional proof: $(\mathbf{A}; \mathbf{B}) \not\models \mathbf{A} \rightarrow \mathbf{B}$. *Reductio* is not validated either: $(\mathbf{A}; \neg\mathbf{A}) \not\models \neg\mathbf{A}$. We can restore validity to conditional proof and *reductio* if we add a relevant instance of excluded middle as an additional premiss:
$(\mathbf{A}; \mathbf{B}), \mathbf{A} \vee \neg\mathbf{A} \models \mathbf{A} \rightarrow \mathbf{B}$, and $(\mathbf{A}; \neg\mathbf{A}), \mathbf{A} \vee \neg\mathbf{A} \models \neg\mathbf{A}$.

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I will select a set $\mathcal{S}$ of valuations and define the theory as $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ is the class of the formulae that have the designated value, 1, in all valuations in $\mathcal{S}$, and $\mathcal{C}_2$ is the class of the simple inferences where the conclusion has 1 in every valuation belonging to $\mathcal{S}$ in which the premisses have 1.

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$\mathcal{S}$ should contain only valuations that give 1 to all biconditionals $T\langle \mathbf{A} \rangle \leftrightarrow \mathbf{A}$. This is our basic stipulation about $\mathcal{S}$.

There are other statements and inferences about truth which we would like to include in our theory because they seem obvious. Some are automatically included once we make the basic stipulation about $\mathcal{S}$. One example is conditionals of the form $T\langle\mathbf{A}\rangle \to \neg T\langle\neg\mathbf{A}\rangle$. Other statements about truth may not be automatically included. One example is $\neg[T\langle\mathbf{A}\rangle \wedge T\langle\neg\mathbf{A}\rangle]$.

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In order to expand the theory, I make an additional stipulation about the set $\mathcal{S}$ of valuations. Its effect is to include in the theory various negations of conjunctions about truth. One such negation is $\neg[T\langle \mathbf{A} \rangle \wedge T\langle \neg\mathbf{A} \rangle]$. Another is $\neg[T\langle T\langle \mathbf{A} \rangle\rangle \wedge T\langle T\langle \neg\mathbf{A} \rangle\rangle]$. A third negation is $\neg[T\langle \mathbf{A} \rangle \wedge \neg T\langle \neg\neg\mathbf{A} \rangle]$.

Theorem 1

For each assignment K of values to one or more sentential letters, there is a valuation that belongs to $\mathcal{S}$ and incorporates K.

If $\mathbf{A}_1, \ldots, \mathbf{A}_n \models \mathbf{B}$, then the inference from $T\langle \mathbf{A}_1 \rangle, \ldots, T\langle \mathbf{A}_n \rangle$ to $T\langle \mathbf{B} \rangle$ belongs to $\mathcal{C}_2$. If $\models \mathbf{A} \rightarrow \mathbf{B}$, then the conditional $T\langle \mathbf{A} \rangle \rightarrow T\langle \mathbf{B} \rangle$ belongs to $\mathcal{C}_1$.

The inference from $T\langle \mathbf{A} \vee \mathbf{B} \rangle$ to $T\langle \mathbf{A} \rangle \vee T\langle \mathbf{B} \rangle$ belongs to $\mathcal{C}_2$, as does the converse inference, from $T\langle \mathbf{A} \rangle \vee T\langle \mathbf{B} \rangle$ to $T\langle \mathbf{A} \vee \mathbf{B} \rangle$. Our theory also contains the inference from $T\langle \mathbf{A} \rangle \wedge T\langle \mathbf{B} \rangle$ to $T\langle \mathbf{A} \wedge \mathbf{B} \rangle$ and the converse inference, as well as the inference from $T\langle \mathbf{A} \rangle \rightarrow T\langle \mathbf{B} \rangle$ to $T\langle \mathbf{A} \rightarrow \mathbf{B} \rangle$ and the converse.

The theory leaves no room for sentences that are neither true nor false. We can equate the falsity of a sentence with the truth of its negation. Let **A** be any formula. $\mathcal{C}_1$ contains the conditional $\neg T\langle\mathbf{A}\rangle \to T\langle\neg\mathbf{A}\rangle$. The theory also contains $\neg T\langle\neg\mathbf{A}\rangle \to T\langle\mathbf{A}\rangle$. And, just as $\neg[T\langle\mathbf{A}\rangle \wedge T\langle\neg\mathbf{A}\rangle]$ belongs to the theory, so does $\neg[\neg T\langle\mathbf{A}\rangle \wedge \neg T\langle\neg\mathbf{A}\rangle]$: it is not the case that **A** is not true and not false.

On the other hand, our theory does not characterize every sentence as being either true or false. It may not contain the disjunction $T\langle\mathbf{A}\rangle \vee T\langle\neg\mathbf{A}\rangle$ (**A** is true or false).

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I take it that, in some cases, such as vagueness, it is reasonable to claim $\neg\Delta\mathbf{S}$ for some sentence **S**. It has not been noticed that such claims clash with classical logic.

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Thus, using the rule of conditional proof, we may proceed to the conditionals $\mathbf{S} \rightarrow \Delta\mathbf{S}$ and $\neg\mathbf{S} \rightarrow \Delta\mathbf{S}$. If, now, we accept $\neg\Delta\mathbf{S}$, then *modus tollens* leads to the contradictory conclusions $\neg\mathbf{S}$ and $\neg\neg\mathbf{S}$.

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If we grant the need or usefulness of claims of the form $\neg\Delta\mathbf{S}$ and do not question the inference from $\mathbf{S}$ to $\Delta\mathbf{S}$ and from $\neg\mathbf{S}$ to $\Delta\mathbf{S}$, then we have a reason for deviating from classical logic.

We already know that abandoning conditional proof affords a treatment of Curry's paradox. That is how we treated it. For if we take a Curry sentence $T\langle\mathbf{A}\rangle \to \mathbf{B} \wedge \neg\mathbf{B}$, where $\mathbf{A}$ is that very sentence, then the theory of truth that we have seen contains the biconditional $T\langle\mathbf{A}\rangle \leftrightarrow (T\langle\mathbf{A}\rangle \to \mathbf{B} \wedge \neg\mathbf{B})$, and the underlying propositional logic validates some rules which, together with conditional proof, lead from that biconditional to the contradiction $\mathbf{B} \wedge \neg\mathbf{B}$.

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If we combine the possibility of treating Curry's paradox with the need to abandon classical logic in order to be able to make claims of the form $\neg \Delta \mathbf{S}$ without inconsistency, we get a sufficient reason for deviating from conditional proof.

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The definition of *valuation*$_*$ is the same as in our basic propositional language except for the addition of a table and three accompanying rules for the new operator. The additional table is this:

A	**ΔA**
1 or 0	1
$1/2$	$1/2$ or 0

And the three rules are the following: if **A** has $1/2$, then $\Delta\neg\mathbf{A}$ is given the same value as $\Delta\mathbf{A}$; if **A** and **B** get $1/2$, and they have the same deep conjuncts or the same deep disjuncts or are conditionals contrapositive of each other, then $\Delta\mathbf{A}$ is given the same value as $\Delta\mathbf{B}$; and if $\Delta[\mathbf{A} \vee \mathbf{B}]$, $\Delta[\mathbf{A} \wedge \mathbf{B}]$ or $\Delta[\mathbf{A} \rightarrow \mathbf{B}]$ gets 0, then at least one of $\Delta\mathbf{A}$ and $\Delta\mathbf{B}$ has 0.

A is a *deep disjunct* of **B** iff **A** is not a disjunction and there is an occurrence $\mathcal{O}$ of **A** in **B** such that every symbol, in **B** but outside $\mathcal{O}$, in whose scope $\mathcal{O}$ lies is a $\vee$.

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The definition of *valuation* is also the same as in our basic language except for the addition of the table and rules that characterize Δ in valuations$_*$.

The fact that if $\Delta\mathbf{A}$ gets 1 in a valuation, $\mathbf{A}$ must have an integral value there leads to validating the inference from $\Delta\mathbf{B}_1, \ldots, \Delta\mathbf{B}_i$ ($i \geq 1$) to any formula that results from substituting $\mathbf{B}_1, \ldots, \mathbf{B}_i$ for $p_1, \ldots, p_i$ respectively in a classical tautology in which the only sentential letters are $p_1, \ldots, p_i$.

It may be asked why we do not give 0 to $\Delta\mathbf{A}$ whenever $\mathbf{A}$ has $1/2$. If we did, we would validate all formulae that begin with two or more Δ's. By validating $\Delta\Delta p_1$, we would leave no room for indeterminacy in questions of the form 'Is it determinate whether . . . ?'. As I said earlier, one may have the intuition that it is indeterminate whether the truth-teller sentence, or the liar sentence, is true. If we follow the intuition and accept that it is indeterminate, we commit ourselves to admitting that it is determinate whether (it is determinate whether the sentence is true). We should not make that commitment. For it may not be a determinate matter whether (it is determinate whether the sentence is true). It is better to say that it may be indeterminate whether the sentence is true, or it may be indeterminate whether (it is determinate whether the sentence is true), or indeterminacy may lurk deeper.

The first of the rules that accompany the table for Δ helps validate the conditionals $\Delta\mathbf{A} \rightarrow \Delta\neg\mathbf{A}$ and $\Delta\neg\mathbf{A} \rightarrow \Delta\mathbf{A}$.

The second rule helps validate the conditionals $\Delta[\mathbf{A} \wedge \mathbf{B}] \rightarrow \Delta[\mathbf{B} \wedge \mathbf{A}]$, $\Delta[[\mathbf{A} \wedge \mathbf{B}] \wedge \mathbf{C}] \rightarrow \Delta[\mathbf{A} \wedge [\mathbf{B} \wedge \mathbf{C}]]$, $\Delta[\mathbf{A} \vee \mathbf{B}] \rightarrow \Delta[\mathbf{B} \vee \mathbf{A}]$ and $\Delta[[\mathbf{A} \vee \mathbf{B}] \vee \mathbf{C}] \rightarrow \Delta[\mathbf{A} \vee [\mathbf{B} \vee \mathbf{C}]]$. Moreover, it contributes to validating $\Delta\mathbf{A} \rightarrow \Delta[\mathbf{A} \wedge \mathbf{A}]$, $\Delta\mathbf{A} \rightarrow \Delta[\mathbf{A} \vee \mathbf{A}]$ and $\Delta[\mathbf{A} \rightarrow \mathbf{B}] \rightarrow \Delta[\mathbf{B}' \rightarrow \mathbf{A}']$ where $\mathbf{B}' \rightarrow \mathbf{A}'$ is a contrapositive of $\mathbf{A} \rightarrow \mathbf{B}$.

Thanks to the third rule accompanying the table for Δ, $\neg\Delta[\mathbf{A} \vee \mathbf{B}] \models \neg\Delta\mathbf{A} \vee \neg\Delta\mathbf{B}$, $\neg\Delta[\mathbf{A} \wedge \mathbf{B}] \models \neg\Delta\mathbf{A} \vee \neg\Delta\mathbf{B}$, and $\neg\Delta[\mathbf{A} \rightarrow \mathbf{B}] \models \neg\Delta\mathbf{A} \vee \neg\Delta\mathbf{B}$.

On the other hand, the tables for Δ and the other connectives suffice, without need for the rules accompanying the tables, to ensure that $\Delta\mathbf{A} \wedge \Delta\mathbf{B} \models \Delta[\mathbf{A} \vee \mathbf{B}]$, $\Delta\mathbf{A} \wedge \Delta\mathbf{B} \models \Delta[\mathbf{A} \wedge \mathbf{B}]$ and $\Delta\mathbf{A} \wedge \Delta\mathbf{B} \models \Delta[\mathbf{A} \rightarrow \mathbf{B}]$.

We should not seek to validate all inferences of the form '$\mathbf{A} \leftrightarrow \neg\mathbf{A}$; hence $\neg\Delta\mathbf{A}$'. Indeed, $\mathbf{A} \leftrightarrow \neg\mathbf{A} \not\models \neg\Delta\mathbf{A}$. On the other hand, $\Delta\Delta\mathbf{A}, \mathbf{A} \leftrightarrow \neg\mathbf{A} \models \neg\Delta\mathbf{A}$.

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$\Delta[\mathbf{A} \vee \mathbf{B}] \models \Delta\mathbf{A} \vee \Delta\mathbf{B}$. On the other hand, $\Delta[\mathbf{A} \wedge \mathbf{B}] \not\models \Delta\mathbf{A} \vee \Delta\mathbf{B}$, and rightly so. We should assent to 'The liar sentence (L) is not both true and untrue', and thus to 'It is determinate whether (L) is both true and untrue', but not to 'Either it is determinate whether (L) is true or it is determinate whether (L) is not true'.

Also, $\Delta[\mathbf{A} \rightarrow \mathbf{B}] \not\models \Delta\mathbf{A} \vee \Delta\mathbf{B}$. We should assent to 'It is determinate whether ((L) is true if it is true)', but not to 'Either it is determinate whether (L) is true or it is determinate whether (L) is true'.

Once we have Δ, we can define an operator, $\boxdot$, meaning 'it is determinately the case that ...'. We might define $\boxdot\mathbf{A}$ as $\Delta\mathbf{A} \wedge \mathbf{A}$, but I prefer the definition

$$\boxdot\mathbf{A} \quad =_{\mathrm{df}} \quad \neg[\neg\Delta\mathbf{A} \vee \neg\mathbf{A}].$$

As a result of the definition, $\boxdot$ is governed by the following table in every valuation$_*$:

$\mathbf{A}$	$\Delta\mathbf{A}$	$\boxdot\mathbf{A}$
1	1	1
0	1	0
1/2	1/2	1/2
1/2	0	0

We can also define

$$\Diamond\mathbf{A} \quad =_{\mathrm{df}} \quad \neg\boxdot\neg\mathbf{A}.$$

Our current language possesses paradoxical sentences, involving Δ, that could not be formed in the simple propositional language. They are paradoxical in that we can derive a contradiction about them using only classical logic, the T-schema and possibly the rule, (R), that allows us to infer from $\mathbf{A}$ to $\Delta\mathbf{A}$.

For example, let a_1 name the sentence $\neg Ta_1 \vee \neg\Delta Ta_1$. Assume $\neg Ta_1$; then, $\neg Ta_1 \vee \neg\Delta Ta_1$; hence, by the T-schema, Ta_1. So, by a version of *reductio ad absurdum*, Ta_1. Thus, using the T-schema again, $\neg Ta_1 \vee \neg\Delta Ta_1$. Therefore, by a version of disjunctive syllogism, $\neg\Delta Ta_1$. But also, by (R), ΔTa_1—contradiction. The sentence $\neg Ta_2 \vee \neg\Delta\Delta Ta_2$ is also paradoxical if it is named by a_2, as is $\neg Ta_3 \vee \neg\Delta Ta_3 \vee \neg\Delta\Delta Ta_3$ if it is named by a_3.

As before, I will select a set $\mathcal{S}$ of valuations, and the theory of truth will be $\mathcal{C}_1 \cup \mathcal{C}_2$, where $\mathcal{C}_1$ is the class of the formulae that have value 1 in all valuations in $\mathcal{S}$, and $\mathcal{C}_2$ is the class of the simple inferences whose conclusion has 1 in every valuation belonging to $\mathcal{S}$ in which the premisses have 1.

Our basic stipulation about $\mathcal{S}$ is again that it should contain only valuations that give 1 to all biconditionals $T\langle \mathbf{A}\rangle \leftrightarrow \mathbf{A}$, for any formula $\mathbf{A}$ of the current language. Every valuation in $\mathcal{S}$ will also accord with the additional stipulation that I alluded to when I talked about the theory in the simple propositional language. Further, each valuation in $\mathcal{S}$ should, for every $\mathbf{A}$, give the same value to $\Delta T\langle \mathbf{A}\rangle$ as it gives to $\Delta\mathbf{A}$.

$\mathcal{C}_2$ contains the inference from $T\langle\mathbf{A}\rangle$ to $T\langle\Delta\mathbf{A}\rangle$. Likewise, it contains the inference from $T\langle\neg\mathbf{A}\rangle$ to $T\langle\Delta\mathbf{A}\rangle$. Also, the class $\mathcal{C}_2$ contains the inference from $\Delta T\langle\mathbf{A}\rangle$ to $T\langle\Delta\mathbf{A}\rangle$, as well as the inference from $T\langle\Delta\mathbf{A}\rangle$ to $\Delta T\langle\mathbf{A}\rangle$.

The stipulations we made about the valuations in $\mathcal{S}$ ensure that, for every $\mathbf{A}$, $\boxdot T\langle\mathbf{A}\rangle$ gets the same value as $\boxdot\mathbf{A}$. Thus the theory sanctions the inference from $\boxdot\mathbf{A}$ to $\boxdot T\langle\mathbf{A}\rangle$ and conversely, as well as the inference from $\neg\boxdot\mathbf{A}$ to $\neg\boxdot T\langle\mathbf{A}\rangle$ and conversely.

The stipulations we made about the valuations in $\mathcal{S}$ ensure that, for every **A**, $\boxdot T\langle \mathbf{A} \rangle$ gets the same value as $\boxdot \mathbf{A}$. Thus the theory sanctions the inference from $\boxdot \mathbf{A}$ to $\boxdot T\langle \mathbf{A} \rangle$ and conversely, as well as the inference from $\neg \boxdot \mathbf{A}$ to $\neg \boxdot T\langle \mathbf{A} \rangle$ and conversely.

The class $\mathcal{C}_2$ contains, for any **A**, the inference from $T\langle \mathbf{A} \rangle$ to $T\langle \boxdot \mathbf{A} \rangle$, as well as the converse inference. $\mathcal{C}_2$ also contains the inference from $\neg T\langle \mathbf{A} \rangle$ to $\neg T\langle \boxdot \mathbf{A} \rangle$. It is not, however, the case that, for all **A**, $\mathcal{C}_2$ contains the inference from $\neg T\langle \boxdot \mathbf{A} \rangle$ to $\neg T\langle \mathbf{A} \rangle$.

The stipulations we made about the valuations in $\mathcal{S}$ ensure that, for every $\mathbf{A}$, $\boxdot T\langle\mathbf{A}\rangle$ gets the same value as $\boxdot\mathbf{A}$. Thus the theory sanctions the inference from $\boxdot\mathbf{A}$ to $\boxdot T\langle\mathbf{A}\rangle$ and conversely, as well as the inference from $\neg\boxdot\mathbf{A}$ to $\neg\boxdot T\langle\mathbf{A}\rangle$ and conversely.

The class $\mathcal{C}_2$ contains, for any $\mathbf{A}$, the inference from $T\langle\mathbf{A}\rangle$ to $T\langle\boxdot\mathbf{A}\rangle$, as well as the converse inference. $\mathcal{C}_2$ also contains the inference from $\neg T\langle\mathbf{A}\rangle$ to $\neg T\langle\boxdot\mathbf{A}\rangle$. It is not, however, the case that, for all $\mathbf{A}$, $\mathcal{C}_2$ contains the inference from $\neg T\langle\boxdot\mathbf{A}\rangle$ to $\neg T\langle\mathbf{A}\rangle$.

For each $\mathbf{A}$, both the inference from $\boxdot T\langle\mathbf{A}\rangle$ to $T\langle\boxdot\mathbf{A}\rangle$ and the inference from $T\langle\boxdot\mathbf{A}\rangle$ to $\boxdot T\langle\mathbf{A}\rangle$ belong to $\mathcal{C}_2$ and thus make it to our theory. But also both the inference from $\neg\boxdot T\langle\mathbf{A}\rangle$ to $\neg T\langle\boxdot\mathbf{A}\rangle$ and that from $\neg T\langle\boxdot\mathbf{A}\rangle$ to $\neg\boxdot T\langle\mathbf{A}\rangle$ belong to $\mathcal{C}_2$.

Theorem 2

For each assignment K of values to one or more sentential letters, there is a valuation that belongs to $\mathcal{S}$ and incorporates K.

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Theorem 3

For each assignment K of values to one or more sentential letters, there is a valuation that belongs to $\mathcal{S}$, incorporates K and assigns 0 to $\Delta T\langle \mathbf{A} \rangle$ for some $\mathbf{A}$, including any $\mathbf{A}$ that is a truth-teller sentence or a liar sentence.