Diophantine Approximation and Recursion Theory

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Abstract

Parallels between the recursion theoretic study of randomness and Diophantine approximation.

- Avoiding definable null sets / Normality
- Incompressibility / Irrationality exponent

Randomness

formulated by measure

Definition

A real number ξ is *Martin-Löf random* iff it does not belong to any effectively-null G_δ set. Precisely, if $(O_k : k \in \mathbb{N})$ is a uniformly computably enumerable sequence of open sets such that for all k, O_k has measure less than $1/2^k$, then $\xi \notin \cap_{k\in\mathbb{N}} O_k$.

This is not mysterious: Identify a family of sets of measure 0, and say that ξ is random if it does not belong to any set in the family.

Randomness

formulated by compressibility

Definition

A real number ξ is *algorithmically incompressible* iff there is a C such that for all ℓ, $K(\xi \upharpoonright \ell) > \ell - C$, where K denotes prefix-free Kolmogorov complexity and $\xi \upharpoonright \ell$ denotes the first ℓ bits in the base 2 representation of ξ.

This is also not mysterious: Say that ξ is incompressible when for all ℓ, it takes ℓ plus a constant number of bits of information to describe $\xi \upharpoonright \ell$. One can interpret *description* in a variety of ways and obtain reasonable characteristics of ξ.

Schnorr's Theorem

Theorem (Schnorr 1973)

ξ is Martin-Löf random iff it is algorithmically incompressible.

Émile Borel (1909): Normal Numbers

A Diophantine characteristic of randomness

Definition

Let ξ be a real number.

- ξ is *simply normal to base b* iff in its base-b expansion, $(\xi)_b$, each digit appears with limiting frequency equal to $1/b$.
- ξ is *normal to base b* iff in $(\xi)_b$ every finite block pattern of digits occurs with limiting frequency equal to the expected value $1/b^\ell$, where ℓ is the block length.
- ξ is *absolutely normal* iff it is normal to every base b.

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Exercise

ξ is normal to base b iff the sequence $(b^k\xi : k \in \mathbb{N})$ is uniformly distributed in the unit interval mod 1.

Normality

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Disanalogous: Unlike in recursion theory, the integer bases provide a one-parameter family of randomness criteria.

Examples

First constructions of absolutely normal numbers by Lebesgue and Sierpiński, independently, 1917.

Theorem (Champernowne 1933)

$0.123456789101112131415161718192021222324\ldots$ *is normal to base ten.*

An elementary but intricate counting argument shows that Champernowne's number is normal to base 10, but it is not known whether it is absolutely normal.

A Computable Example

Theorem (Turing c. 1938 (see Becher, Figueira and Picchi 2007))

There is a computable absolutely normal number.

Other computable instances Schmidt 1961/1962, Becher and Figueira 2002.

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Theorem (Becher, Heiber and Slaman; Lutz and Mayerdomo; Figueira and Nies 2013)

There is an absolutely normal number ξ such that for any base b, the sequence of digits in $(\xi)_b$ is computable in polynomial time.

- Lutz and Mayerdomo gave the fastest algorithm, which runs in nearly linear time.

Normality to Different Bases

Question (Steinhaus c. 1950)

Does normality in one base imply normality in others?

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There is one readily identified connection between normality to one base and normality to another.

Definition

For natural numbers b_1 and b_2 greater than 0, we say that b_1 and b_2 are *multiplicatively dependent* if they have a common power.

Theorem (Maxfield 1953)

If b_1 and b_2 are multiplicatively dependent bases, then, for any real ξ, ξ is normal to base b_1 iff it is normal to base b_2.

Normality and Multiplicative Independence

Theorem (Schmidt 1961/62)

Let R be a subset of the natural numbers greater than or equal to 2 that is closed under multiplicative dependence. There is a real ξ such that ξ is normal to every base in R and not normal to any base in the complement of R.

An earlier result due to Cassels (1959) is that almost every element of the Cantor middle-third set is normal to every base that is multiplicatively independent of 3.

Simple Normality: extending the Schmidt-Cassels theorem

Theorem (Becher, Bugeaud and Slaman 2013)

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integer m, if $b^m \in M$ then then $b \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Irrationality Exponents

Next, we discuss the Diophantine version of algorithmic compressibility.

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Definition

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

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is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- When z is large, instances of $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$ are instances of algorithmic compression.
- Every irrational number has irrationality exponent greater than or equal to 2. Irrationality exponent is equal to 2 on a set of full measure.

Normality and Irrationality Exponents

Disanalogous

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Theorem (Becher and Slaman, extending results of Amou and Bugeaud)

Suppose $a \in [2, \infty]$ and M is a subset of the integers greater than or equal to 2 which satisfies the conditions for a set of bases of simple normality. Then there is a real number ξ such that ξ is simply normal to exactly the bases in M and ξ has exponent of irrationality a.

Irrationality Exponents Relative to Independent Bases

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Definition (following Amou and Bugeaud 2010)

For a real number ξ, the *base-b irrationality exponent of ξ* is the least upper bound of the set of real numbers z such that

$$0 < \left| \xi - \frac{p}{b^k} \right| < \frac{1}{(b^k)^z}$$

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is satisfied by an infinite number of integer pairs (p, k) with $k > 0$.

- If ξ is simply normal in base b, then the base-b irrationality exponent of ξ is equal to 1.

Irrationality Exponent: Depends on Base

Theorem (Amou and Bugeaud 2010)

Suppose that b_1 and b_2 are multiplicative independent bases, and suppose that a_2 and a_3 are greater than $1 + \frac{1+\sqrt{5}}{2}$. There is a real number whose base-b_1 and base-b_2 exponents of irrationality are a_2 and a_3, respectively.

- The proof relies on the theory of continued fractions. A measure theoretic approach would be welcome.

Random or Compressible: Depends on Base

Theorem

There is a real number ξ which is normal to base 2 and whose base 10 exponent of irrationality is equal to ∞.

Ingredients for the proof

Based on Stoneham numbers (1964)

The example number has the form

$$\xi = \sum_{i=1}^{\infty} \frac{1}{5^{n_i}} \frac{1}{2^{m_i}},$$

where $(n_i : i \in \mathbb{N})$ grows monotonically and $(m_i : i \in \mathbb{N})$ grows quickly. In particular, $m_i > 5^{n_i}$.

Ingredients for the proof

Infinite base 10 irrationality exponent

$$\xi = \sum_{i=1}^{\infty} \frac{1}{5^{n_i}} \frac{1}{2^{m_i}}$$

Provided that $(m_i : i \in \mathbb{N})$ grows quickly enough, the decimal fractions

$$\sum_{i=1}^{\ell} \frac{5^{m_i - n_i}}{10^{m_i}}$$

give arbitrarily accurate approximations to ξ. Hence, the base 10 irrationality exponent of ξ is infinity.

Ingredients for the proof

Normality for base 2

$$\xi = \sum_{i=1}^{\infty} \frac{1}{5^{n_i}} \frac{1}{2^{m_i}}$$

- Stoneham studied the recurring parts in the base 2 expansions of $\frac{1}{p^n}$, when 2 is a primitive root of p^2, as is the case for $p = 5$.
- Stoneham showed that the recurring part, as a finite sequence, is nearly normal with respect to the occurrences of its block subsequences. He extended the result to similar rational numbers, such as the partial sums of the above
- Finally, the structure of ξ ensures that the near normality of the base 2 expansions of the partial sums implies normality of the base 2 expansion of ξ.

A Challenge

Question

What patterns of normality and infinite exponent of irrationality are possible across multiplicatively independent bases?

For example, is there a real number which is normal in base 2 and has infinite base 3 exponent of irrationality?

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