Dependence and the method of definition by recursion

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Montague's theorem - I

Fact (Montague (1955))

Let R be a well-founded relation on a set A and let $G : A \times {}^{[A]}B \to B$ be any function.
There exists one and only one function $f : A \to B$ such that, for every $x \in A$,

(Rec) $$f(x) = G(x, f \restriction x^R),$$

- ${}^{[A]}B$ denotes the set of all partial functions from A to B.
- x^R denotes the set of all R-predecessors of x.
- $f \restriction x^R$ denotes the restriction of f to x^R.

Montague's theorem - II

Definition

Let $R \subseteq A \times A$ and $G : A \times {}^{[A]}B \to B$. Let $p \in {}^{[A]}B$ and $x \in \mathrm{dom}(p)$. We say that p is (G, R)*-recursive at* x iff

1. $x^R \subseteq \mathrm{dom}(p)$, and
2. $p(x) = G(x, p \restriction x^R)$.

p is (G, R)*-recursive* iff p is (G, R)-recursive at x for every $x \in \mathrm{dom}(p)$.

Fact

Let R be any binary relation on A and let $G : A \times {}^{[A]}B \to B$ be any function.
There exists one and only one (G, R)-recursive function defined on the well-founded part of R.

Dependence and recursion

Two parameters in Montague's theorem:

- The set x^R represents the set of objects on which the value of x *depends on*.
- The function G formalises a way of *evaluating* x given the course of values of f at the objects in x^R.

In the logico-philosophical literature we find:

- Dependence represented by a *monotone operator* Δ.
- Functionality represented by an *evaluation system* Γ.

Question: How can we generalise Montague's theorem to allow us to define a unique "(Γ, Δ)-recursive" function, given Γ and Δ?

Valuation systems - I

Definition

A *valuation system* is a triple $(\mathcal{F}, \mathcal{F}', \Gamma)$, where both $\mathcal{F}$ and $\mathcal{F}'$ are non-empty sets of functions and Γ is an operator $\Gamma : \mathcal{F} \to \mathcal{F}'$.

- $q \in \mathrm{Q}_\Gamma(x, p)$ iff $q \in \mathcal{F} \wedge p \subseteq q \wedge x \in \mathrm{dom}(\Gamma(q))$.
- x is Γ-determined by p iff $\exists q \in \mathrm{Q}_\Gamma(x, p)$ and $\forall q, q' \in \mathrm{Q}_\Gamma(x, p)\, (\Gamma(q)(x) = \Gamma(q')(x))$.
- $(x, p) \in \mathrm{dom}(\mathrm{G}_\Gamma)$ iff $p \in \mathrm{Q}_\Gamma(x, p)$ or x is Γ-determined by p.

$$\mathrm{G}_\Gamma(x, p) = \begin{cases} \Gamma(p)(x) & \text{if } p \in \mathrm{Q}_\Gamma(x, p) \\ \Gamma(q)(x), \text{ where } q \in \mathrm{Q}_\Gamma(x, p) & \text{otherwise.} \end{cases}$$

Valuation systems - II

Definition

We say that a valuation system $(\mathcal{F}, \mathcal{F}', \Gamma)$ is *complete* whenever

1. $\mathcal{F} = {}^{[A]}B$.
2. $\mathcal{F}' = {}^{A}B$.

Fact

The map $\Gamma \mapsto G_\Gamma$ witnesses a one to one correspondence between complete valuation systems on A and total functions from $A \times {}^{[A]}B$ to B.

Dependence operators - I

Definition

A *monotone operator* on A is a function $\Delta : \mathcal{P}(A) \to \mathcal{P}(A)$ such that, for every $X, Y \subseteq A$,

$$X \subseteq Y \Rightarrow \Delta(X) \subseteq \Delta(Y).$$

Given a binary relation $R \subseteq A \times A$, we can define a monotone operator Δ_R on A by putting

$$\Delta_R(X) = \{x \in A \mid x^R \subseteq X\},$$

for every $X \subseteq A$.

Dependence operators - II

- $\mathrm{S}_\Delta(x) = \{X \subseteq A \mid x \in \Delta(X)\}$.
-
$$\mathrm{E}_\Delta(x) = \begin{cases} \bigcap \mathrm{S}_\Delta(x) & \text{if } \mathrm{S}_\Delta(x) \neq \emptyset \\ A & \text{otherwise.} \end{cases}$$

Definition

A monotone operator Δ is an *essential-dependence operator* iff $\mathrm{E}_\Delta(x) \in \mathrm{S}_\Delta(x)$ holds for every $x \in A$.

Fact

The map $R \mapsto \Delta_R$ witnesses a one to one correspondence between binary relations on A and essential-dependence operators on A. Moreover, whenever $\Delta = \Delta_R$,

$$\mathrm{W}(A, R) = \mathrm{lfp}(\Delta).$$

Generalised Montague's theorem - I

Given a monotone operator Δ on A, put

$$\mathsf{C}_\Delta(x) = \{X \in \mathsf{S}_\Delta(x) \mid X \subseteq \mathsf{E}_\Delta(x)\}.$$

Definition

Let $(\mathcal{F}, \mathcal{F}', \Gamma)$ be a valuation system on A and let Δ be a monotone operator on A. Let $p \in {}^{[A]}B$ and $x \in \mathrm{dom}(p)$. We say that p is (Γ, Δ)-*recursive at* x iff

$$\exists X \in \mathsf{C}_\Delta(x)\,(X \subseteq \mathrm{dom}(p) \ \wedge\ (x, p \restriction X) \in \mathrm{dom}(\mathsf{G}_\Gamma) \wedge \\ p(x) = \mathsf{G}_\Gamma(x, p \restriction X)).$$

p is (Γ, Δ)-*recursive* iff p is (Γ, Δ)-recursive at x for every $x \in \mathrm{dom}(p)$.

Generalised Montague's theorem - II

Thm (Generalised Montagues' theorem)

Suppose (Γ, Δ) *admits recursion.*
There exists one and only one (Γ, Δ)*-recursive function* p *defined on the least fixed point of* Δ*.*
Moreover, for every $x \in \mathrm{dom}(p)$*,*

$$p(x) = \mathrm{G}_\Gamma(x, p \restriction X), \tag{E1}$$

where X *is any set in* $\mathrm{C}_\Delta(x)$ *such that* $X \subseteq \Delta(X)$*,* $X \subseteq \mathrm{dom}(p)$ *and* $(x, p \restriction X) \in \mathrm{dom}(\mathrm{G}_\Gamma)$*.*

Recursion on a binary relation

Definition

Let $(\mathcal{F}, \mathcal{F}', \Gamma)$ be any valuation system on A and let $R \subseteq A \times A$. Let $\mathrm{J}_{\Gamma,R} : {}^{[A]}B \to {}^{[A]}B$ denote the operator defined as follows. For every $p \in {}^{[A]}B$,

- $\mathrm{dom}(\mathrm{J}_{\Gamma,R}(p)) = \{x \in A \mid x^R \subseteq \mathrm{dom}(p) \wedge (x, p \restriction x^R) \in \mathrm{dom}(\mathrm{G}_\Gamma)\}$.
- $\mathrm{J}_{\Gamma,R}(p)(x) = \mathrm{G}_\Gamma(x, p \restriction x^R)$, for every $x \in \mathrm{dom}(\mathrm{J}_{\Gamma,R}(p))$.

Thm

Suppose (Γ, R) admits recursion. Then, the unique (Γ, R)-recursive function defined on the well-founded part of R coincides with the least fixed point of the monotone operator $\mathrm{J}_{\Gamma,R}$.

Recursion on the dependence operator - I

Definition

Let $(\mathcal{F}, \mathcal{F}', \Gamma)$ be any valuation system on A. Define

$$\Delta^*_\Gamma(X) = \{x \in A \mid \forall q, q' \in \mathcal{F}\,(q \equiv_X q' \wedge x \in \mathrm{dom}(\Gamma(q)) \cap \mathrm{dom}(\Gamma(q')) \Rightarrow \Gamma(q)(x) = \Gamma(q')(x))\},$$

where $q \equiv_X q'$ is short for
$X \subseteq \mathrm{dom}(q) \cap \mathrm{dom}(q') \wedge q \restriction X = q' \restriction X$.

Definition

Let $(\mathcal{F}, \mathcal{F}', \Gamma)$ be any valuation system on A. We say that

- Γ is *full* iff ${}^A B \subseteq \mathcal{F}$.
- Γ is *regular* iff for every $h \in {}^A B \cap \mathcal{F}$, $\Gamma(h) \in {}^A B$.

Recursion on the dependence operator - II

Thm

Let Γ be monotone, full and regular. Then, $(\Gamma, \Delta_\Gamma^*)$ admits recursion.
Hence, there exists a unique (Γ, Δ)-recursive function p defined on the least fixed point of Δ_Γ^*.
Moreover, for every $x \in \mathrm{dom}(p)$,

$$p(x) = \mathrm{G}_\Gamma(x, p \restriction X), \tag{E1}$$

where X is any set in $\mathrm{S}_\Delta(x)$ such that $X \subseteq \mathrm{dom}(p)$.

Summary

- There exists unique a (G, R)-recursive function defined on the well-founded part of R.
- G can be identified with a *complete* Γ_G; R can be identified with an *essential-dependence* Δ_R.
- If $\Gamma = \Gamma_G$ and $\Delta = \Delta_R$, then a function is (Γ, Δ)-recursive iff is (G, R)-recursive.
- Generalised Montague's theorem: If (Γ, Δ) admits recursion, then there exists unique a (Γ, Δ)-recursive function defined on the least fixed point of Δ.
- If (Γ, R) admits recursion, then there exists a monotone operator $\mathrm{J}_{\Gamma,R}$ such that $\mathrm{F}_*(\Gamma, R) = \mathrm{lfp}(\mathrm{J}_{\Gamma,R})$ (Example: Kripke's theory of grounded truth).
- If Γ is full and regular and $\Delta = \Delta^*_\Gamma$, then (Γ, Δ) admits recursion and $\mathrm{F}_*(\Gamma, \Delta)$ can be defined by transfinite recursion on the Δ-rank (Example: Leitgeb's theory of grounded truth).

Thank you.

Appendix I

Definition

Let $(\mathcal{F}, \mathcal{F}', \Gamma)$ be a valuation system on A and let Δ be a monotone operator on A. We say that the pair (Γ, Δ) *admits recursion* iff for all $X, Y \subseteq A$,

1. $X \in \mathrm{Dom}(\mathrm{Int}(\mathrm{Rec}(\Gamma, \Delta)))$ implies $\Delta(X) \in \mathrm{Dom}(\mathrm{Int}(\mathrm{Rec}(\Gamma, \Delta)))$ and $X \subseteq \Delta(X)$.
2. $X \subseteq \Delta(X)$, $Y \in \mathrm{Dom}(\mathrm{Int}(\mathrm{Rec}(\Gamma, \Delta)))$, and $X \subseteq Y$ implies $X \in \mathrm{Dom}(\mathrm{Int}(\mathrm{Rec}(\Gamma, \Delta)))$.

Appendix II

Let $\mathcal{L}_{Tr}$ denote the set of all sentences of the first-order language of arithmetic augmented with a unary predicate Tr.

For sentences ϕ, ψ, ξ, formula θ with x the only free variable, and closed term t define:

$$\phi \, \mathrm{R} \, \psi \leftrightarrow \quad \begin{array}{l} \psi = \neg\phi, \text{ or} \\ \psi = \text{“}\phi \wedge \xi\text{” or } \psi = \text{“}\xi \wedge \phi\text{” for some } \xi, \text{ or} \\ \psi = \forall x\, \theta(x) \text{ and } \phi = \theta(t) \text{ for some } \theta, t, \text{ or} \\ \psi = Tr(\ulcorner \phi \urcorner). \end{array}$$

Fact

Let R^* denote the transitive closure of R, and let Γ be the Kripkean monotone operator using the Strong-Kleene evaluation scheme. Then, the pair (Γ, R^*) admits recursion.

Appendix III

- Hypotheses: characteristic functions of subsets of $\mathcal{L}_{Tr}$.
- $\Gamma(h)(\phi)$: the truth value of ϕ interpreting *Tr* by the extension corresponding to its characteristic function h.
- $\Delta(X) = \{\phi \mid \forall h_1, h_2\, (h_1 \equiv_X h_2 \Rightarrow \Gamma(h_1)(\phi) = \Gamma(h_2)(\phi))\}$.
- $\Phi_{lf} = \text{lfp}(\Delta)$: the set of all *grounded* sentences.
- Γ_{lf}: the truth-set for Φ_{lf} defined by transfinite recursion on the Δ-rank of the elements of Φ_{lf}.

Fact

Leitgeb's valuation system is monotone, full and regular, and $\Delta = \Delta^*_\Gamma$. Hence, there exists a unique (Γ, Δ)-recursive function p defined on the least fixed point of Δ. Moreover,

$$\Gamma_{lf} = \{\phi \mid p(\phi) = \mathbf{t}\}.$$