Formulas-as-exponential-polynomials correspondence

Applications to normal forms and logical equivalence

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Plan of talk:

1. Equality of exp. polynomials and equivalence of formulas
2. Exp. polynomial normal form and intuitionistic "prenex" form

Formulas as exponential polynomials

A superficial correspondence?

The language of formulas

$$\mathcal{E} \ni f, g ::= f \vee g \mid f \wedge g \mid f \Rightarrow g \mid x_i$$

resembles the language of exponential multivariate polynomials

$$\mathcal{E} \ni f, g ::= f + g \mid fg \mid g^f \mid x_i$$

Formulas as exponential polynomials

Kinds of equivalence we are interested in

$\mathbb{N}^+ \models f = g$ f and g compute to the same positive integer, for every instantiation of the variables by positive integers

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\begin{aligned}
f &= f & f1 &= f \\
f + g &= g + f & f^1 &= f \\
(f + g) + h &= f + (g + h) & 1^f &= 1 \\
fg &= gf & f^{g+h} &= f^g f^h \\
(fg)h &= f(gh) & (fg)^h &= f^h g^h \\
f(g + h) &= fg + fh & (f^g)^h &= f^{gh}
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$f \cong g$ — there are proofs M of $f \Rightarrow g$ and N of $f \Leftarrow g$ such that $\lambda x.M(Nx) =_{\beta\eta} \lambda x.x$ and $\lambda y.N(My) =_{\beta\eta} \lambda y.y$

Formulas as exponential polynomials

$$f \cong g \qquad\qquad \mathbb{N}^+ \models f = g$$

$$f \Leftrightarrow g$$

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Formulas as exponential polynomials

$$f \cong g \longrightarrow \mathbb{N}^+ \models f = g$$

$$\swarrow$$

$$f \Leftrightarrow g$$

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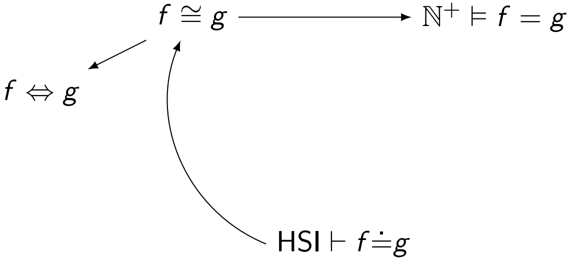
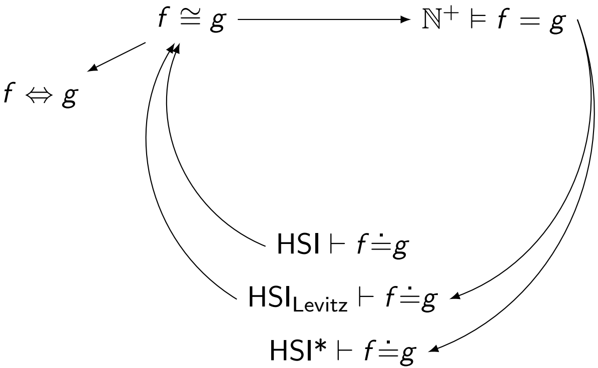
Formulas as exponential polynomials



Formulas as exponential polynomials



Formulas as exponential polynomials

Extension for the quantifiers

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\begin{aligned}
\forall\bar{x}(f \wedge g) &\cong \forall\bar{x}f \wedge \forall\bar{x}g & (fg)^{\bar{x}} &= f^{\bar{x}}g^{x} & & \\
\exists\bar{x}(f \vee g) &\cong \exists\bar{x}f \vee \exists\bar{x}g & \bar{x}(f+g) &= \bar{x}f + \bar{x}g & & \\
\exists\bar{x}f \Rightarrow g &\cong \forall\bar{x}(f \Rightarrow g) & g^{\bar{x}f} &= (g^{f})^{\bar{x}} & & (\text{where } \bar{x} \notin \mathrm{FV}(g)) \\
g \Rightarrow \forall\bar{x}f &\cong \forall\bar{x}(g \Rightarrow f) & (f^{\bar{x}})^{g} &= (f^{g})^{\bar{x}} & & (\text{where } \bar{x} \notin \mathrm{FV}(g))
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Formulas as exponential polynomials

Extension for the quantifiers

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\exists\bar{x}(f \vee g) \cong \exists\bar{x}f \vee \exists\bar{x}g & & \bar{x}(f+g) = \bar{x}f + \bar{x}g & \\
\exists\bar{x}f \Rightarrow g \cong \forall\bar{x}(f \Rightarrow g) & & g^{\bar{x}f} = (g^{f})^{\bar{x}} & (\text{where } \bar{x} \notin \mathrm{FV}(g)) \\
g \Rightarrow \forall\bar{x}f \cong \forall\bar{x}(g \Rightarrow f) & & (f^{\bar{x}})^{g} = (f^{g})^{\bar{x}} & (\text{where } \bar{x} \notin \mathrm{FV}(g))
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$$

We again get that

$$\mathrm{HSI}^{\forall\exists} \vdash f \doteq g \quad \rightarrow \quad f \cong g$$

Intuitionistic "prenex" form

- The prenex normal form can express any formula – classically
  - One gets the ubiquous arithmetical hierarchy!

Intuitionistic "prenex" form

- The prenex normal form can express any formula – classically
  - One gets the ubiquous arithmetical hierarchy!
- In intuitionistic first-order logic, it does not work
  - For instance, the fragment of prenex formulas is decidable

Intuitionistic “prenex” form

How can exponential polynomials help?

Let us use the transformation:

$$g^f = \mathrm{e}^{f \times \log g} \quad \text{also written} \quad g^f = \exp(f \times \log g).$$

Intuitionistic “prenex” form

How can exponential polynomials help?

Let us use the transformation:

$$g^f = \mathrm{e}^{f \times \log g} \quad \text{also written} \quad g^f = \exp(f \times \log g).$$

Algebraically, exp and log are simply defined:

$$\exp(f_1 + f_2) = \exp f_1 \times \exp f_2 \qquad \exp(\log f) = f$$

$$\log(f_1 \times f_2) = \log f_1 + \log f_2 \qquad \log(\exp f) = f.$$

Intuitionistic "prenex" form

The exp-log transformation can be used to obtain the following normal form theorem:

Theorem

Every first-order formula f can be normalized to a formula $\|f\|$, such that $f \cong \|f\|$ and $\|f\| \in \mathbf{\Pi} \cup \mathbf{\Sigma}$, where the classes $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are defined inductively and mutually as follows:

$$\mathbf{\Pi} \ni c ::= \forall \bar{x}_1 (c_1 \Rightarrow b_1) \wedge \cdots \wedge \forall \bar{x}_n (c_n \Rightarrow b_n) \qquad (n \geq 0)$$

$$\mathbf{\Sigma} \ni b ::= p_i \mid c_1 \vee \cdots \vee c_n \mid \exists \bar{x} c \qquad (n \geq 2),$$

where p_i are prime formulas.

Intuitionistic "prenex" form

Assigning levels to the classes $\mathbf{\Sigma}$ and $\mathbf{\Pi}$

Definition

The *intuitionistic arithmetical hierarchy* is defined by assigning levels, $\mathbf{\Sigma}_{\boldsymbol{n}}, \mathbf{\Pi}_{\boldsymbol{n}}$, for $n \in \mathbb{N}$, to the formula classes $\mathbf{\Sigma}$ and $\mathbf{\Pi}$, in the following way:

$$\mathbf{\Pi_0} \ni c ::= \top \Rightarrow p \qquad p \text{ is a prime formula}$$

$$\mathbf{\Sigma_0} \ni b ::= p \qquad p \text{ is a prime formula}$$

$$\mathbf{\Pi}_{\boldsymbol{n+1}} \ni c ::= \forall \bar{x_1}(c_1 \Rightarrow b_1) \wedge \cdots \wedge \forall \bar{x_m}(c_m \Rightarrow b_m) \qquad n = \max_{i=1}^{m}\{k \mid b_i \in \mathbf{\Sigma}_{\boldsymbol{k}}\}$$

$$\mathbf{\Sigma}_{\boldsymbol{n+1}} \ni b ::= p_i \mid c_1 \vee \cdots \vee c_m \mid \exists \bar{x} c \qquad n = \max_{i=1}^{m}\{k \mid c_i \in \mathbf{\Pi}_{\boldsymbol{k}}\} \text{ or } c \in \mathbf{\Pi}_{\boldsymbol{n}}.$$

We also extend the relation "$\in$" from formulas satisfying the inductive definition to all formulas, in the following way: $f \in \mathbf{\Pi}_{\boldsymbol{n}}$ iff $\|f\| \in \mathbf{\Pi}_{\boldsymbol{n}}$; $f \in \mathbf{\Sigma}_{\boldsymbol{n+1}}$ iff $\|f\| \in \mathbf{\Sigma}_{\boldsymbol{n+1}}$.

Intuitionistic "prenex" form

Relation to the classical hierarchy and properness of the intuitionistic one

Say that a formula ϕ is *classically represented in* $\boldsymbol{\Sigma_n}$ *(or* $\boldsymbol{\Pi_n}$*)* when there is a formula $\phi' \in \boldsymbol{\Sigma_n}$ (or $\boldsymbol{\Pi_n}$) such that ϕ and ϕ' are classically equivalent.

Theorem

If $\phi \in \Sigma^0_n$*, then* ϕ *is classically represented in* $\boldsymbol{\Sigma_n}$*. If* $\phi \in \Pi^0_n$*, then* ϕ *is classically represented in* $\boldsymbol{\Pi_n}$*.*
Suppose that ψ *is in prenex normal form and with alternating quantifiers. Then:*
$\psi \in \boldsymbol{\Sigma_n}$ *implies* $\psi \in \Sigma^0_n$*;* $\psi \in \boldsymbol{\Pi_n}$ *implies* $\psi \in \Pi^0_n$*.*

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Theorem

If $\phi \in \Sigma^0_n$, then ϕ is classically represented in $\mathbf{\Sigma_n}$. If $\phi \in \Pi^0_n$, then ϕ is classically represented in $\mathbf{\Pi_n}$.
Suppose that ψ is in prenex normal form and with alternating quantifiers. Then:
$\psi \in \mathbf{\Sigma_n}$ implies $\psi \in \Sigma^0_n$; $\psi \in \mathbf{\Pi_n}$ implies $\psi \in \Pi^0_n$.

Corollary

For $n \geq 0$, $\mathbf{\Sigma_n} \subsetneq \mathbf{\Sigma_{n+1}}$, $\mathbf{\Sigma_n} \subsetneq \mathbf{\Pi_{n+1}}$, $\mathbf{\Pi_n} \subsetneq \mathbf{\Sigma_{n+1}}$, and $\mathbf{\Pi_n} \subsetneq \mathbf{\Pi_{n+1}}$.