Variations on Δ^1_1 Determinacy and $\aleph_{\omega_1}$

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History

Theorem (Friedman, 1971)

In order to prove the determinacy of Borel games one needs $\aleph_1$ iterations of the Power Set operation.

Harvey M. Friedman, *Higher set theory and mathematical practice*, Ann. Math. Logic (1971).

Friedman produces models of Z (Zermelo's set theory), where infinite Borel games have no winning strategy (relative consistency).

He also considers fragments of ZF.

It was observed that the existence of trantitive models of ZF^-, with $\aleph_\xi$'s can be obtained from (somewhat) similar arguments.

Theorem (Martin, 1975. ZF, using $\aleph_\xi$, $\xi < \aleph_1$)

Infinite Borel games are determined.

Borel determinacy, Annals of Mathematics (1975).

AIMS

1. A new (simple) proof of Friedman's result,
2. ... from a "weaker" hypothesis,
3. ... but, omitting the level-by-level analysis.
4. Get weak determinacy properties for Σ^1_1 sets.

Basics and Notation

$\mathcal{N} = \omega^\omega =$ "the reals". $\mathcal{D} =$ the set of Turing degrees.

Theories

ZF	= Zermelo-Fraenkel Set Theory
ZF^-	= ZF without the Power Set Axiom
KP_∞	= Kripke-Platek Set Theory + Axiom of infinity.

We always work in ZF^-: $\mathcal{P}(\omega)$ may be a proper class, still we can talk about Σ^1_1, Π^1_1, and Δ^1_1 sets of reals through their integer codes.

Infinite games

For $A \subseteq \mathcal{N}$, $G_\omega(A)$ is the infinite game of perfect information, with 2 players:

Player $\mathbf{I}$	n_0		n_2		$\cdots$	$n_i \in \omega$
Player $\mathbf{II}$		n_1		n_3		

Player $\mathbf{I}$ wins if $(n_i)_{i<\omega} \in A$, else $\mathbf{II}$ wins.

$G_\omega(A)$ is **determined** if one the players has a winning strategy.

Determinacy and **Turing-Determinacy**, for Γ a class of sets:

Det(Γ) : All $G_\omega(A)$, for $A \in \Gamma$, are determined.
Turing-Det(Γ) : All $G_\omega(A)$, for **Turing-closed** $A \in \Gamma$, are deternined.

Turing cone: $\text{Cone}(c) = \{x \in \mathcal{N} \mid c \leqslant_T x\}$, where $c \in \mathcal{N}$.

Martin's Lemma. For $A \subseteq \mathcal{N}$, Turing-closed:
$G_\omega(A)$ is determined **iff** A or $\sim A$ contains a Turing cone.

Turing-Det(Γ) equivalents: [Γ^{T} = the Turing-closed sets of Γ]
TD$_1$: For all $A \in \Gamma^{\mathsf{T}}$, $G_\omega(A)$ is determined.
TD$_2$: For all $A \in \Gamma^{\mathsf{T}}$, A or $\sim A$ contains a Turing cone.
TD$_3$: For all $A \in \Gamma^{\mathsf{T}}$, A is cofinal in the degrees $\Rightarrow A \supseteq$ a Turing cone

Theorem (ZF$^-$. Friedman, unpublished.)

Assume Turing-Det(Δ^1_1).
For all $\nu < \omega_1^{\mathrm{CK}}$, *there is a transitive model* $M \models$ ZF$^-$+ "$\aleph_\nu$ exists".

Hence: The $\aleph_\xi$, $\xi < \omega_1^{\mathrm{CK}}$, are needed for a proof of Det(Δ^1_1).

Weak Turing Determinacy

For $1 \leqslant \rho < \omega_1^{\mathrm{CK}}$, $x \equiv_\rho y$ denotes: $x \in \Delta^0_\rho(y)$ & $y \in \Delta^0_\rho(x)$.
$\equiv_1$ is just Turing equivalence.

Definition

For a class Γ, and $2 \leqslant \rho < \omega_1^{\mathrm{CK}}$, define Weak-Turing-Det$_\rho(\Gamma)$ as:
For every set of reals $A \in \Gamma$ ***cofinal in the degrees****,*
there are two Turing distinct reals $x, y \in A$ such that $x \equiv_\rho y$.

Main Theorem (ZF$^-$)

Let $2 \leqslant \rho < \omega_1^{\mathrm{CK}}$, and assume Weak-Turing-Det$_\rho(\Delta^1_1)$*.*
For all $\nu < \omega_1^{\mathrm{CK}}$, there is a transitive model $M \models$ ZF$^-$+ "$\aleph_\nu$ exists"*.*

Corollary

If every cofinal Δ^1_1 set of Turing degrees contains both a degree and its jump, then:
For all $\nu < \omega_1^{\mathrm{CK}}$, there is a transitive model $M \models$ ZF$^-$+ "$\aleph_\nu$ exists"*.*

Weak Turing Determinacy

The "**Weak**" property lifts from Δ^1_1 to Σ^1_1.

Theorem

For $2 \leqslant \rho < \omega_1^{\mathrm{CK}}$, $\text{Weak-Turing-Det}_\rho(\Delta^1_1) \Rightarrow \text{Weak-Turing-Det}_\rho(\Sigma^1_1)$

Proof.

Assume $\text{Weak-Turing-Det}_\rho(\Delta^1_1)$.
Let $S \in \Sigma^1_1$, and suppose there are no Turing distinct $x, y \in S$ such that $x \equiv_\rho y$, that is:

$$\forall x, y (x, y \in S \;\; \& \;\; x \equiv_\rho y \;\Rightarrow\; x \equiv_T y).$$

This is a statement $\Phi(S)$, where $\Phi(X)$ is **a Π^1_1 on Σ^1_1 property**.
[**i.e.** for a Σ^1_1 relation, $R \subseteq \mathcal{N} \times \mathcal{N}$, $\{x \mid \Phi(R_x)\}$ is Π^1_1.]

Reflection yields a Δ^1_1 set $D \supseteq S$ such that $\Phi(D)$.

By $\text{Weak-Turing-Det}_\rho(\Delta^1_1)$, D is not cofinal in the degrees.

A fortiori, S is not cofinal. □

Proof structure

- **The theories T_ν**

Definition

For $\nu < \omega_1^{\mathrm{CK}}$, T_ν is the theory

$\mathsf{KP}_\infty + (\mathbb{V} = \mathbb{L})$ + "For all limit λ, $\aleph_{\nu+1}$ does not exist in $\mathbb{L}_\lambda$".

The definition is somewhat ambiguous ...

But, we are interested only in ω-models of T_ν.

Lemma (Rigidity property)

Let $\nu < \omega_1^{\mathrm{CK}}$, and $\mathcal{M}_1, \mathcal{M}_2$ be ω-models of T_ν.
Let $u \in \mathbb{O}\mathrm{n}^{\mathcal{M}_1}$, and $w, w^* \in \mathbb{O}\mathrm{n}^{\mathcal{M}_2}$, for any two isomorphisms

$$f : \mathbb{L}_u^{\mathcal{M}_1} \xrightarrow{\sim} \mathbb{L}_w^{\mathcal{M}_2} \quad \text{and} \quad f^* : \mathbb{L}_u^{\mathcal{M}_1} \xrightarrow{\sim} \mathbb{L}_{w^*}^{\mathcal{M}_2},$$

$f = f^*$. □

The proof's engine: For u, a limit $\mathcal{M}_1$-ordinal, the cardinals of $\mathbb{L}_u^{\mathcal{M}_1}$ are (truly) wellordered, allowing transfinite induction.

Proof structure

- **Models of** $\mathsf{KP}_\infty + (\mathbb{V} = \mathbb{L})$**.**

For a **complete** extension $U \supseteq \mathsf{KP}_\infty + (\mathbb{V} = \mathbb{L})$, one constructs its **term model**: $\mathcal{M}_U = (\omega, \in^{\mathcal{M}_U})$. $\mathcal{M}_U \leqslant_T U$.

Fact: For cofinally many countable admissibles α, $\mathcal{M}_{\mathsf{Th}(\mathbb{L}_\alpha)} \cong \mathbb{L}_\alpha$.

- **Pseudo-wellfounded models.**

A relation $\lhd \subseteq \omega \times \omega$ is **pseudo-wellfounded** if every nonempty $\Delta^1_1(\lhd)$ subset of ω has a $\lhd$-minimal element. This is a Σ^1_1 property.

Definition

For $\nu < \omega_1^{\mathrm{CK}}$, $\mathcal{S}_\nu$ is the set of theories:
$\mathcal{S}_\nu = \{\, U \mid U \supseteq \mathsf{T}_\nu$ is complete, and $\mathcal{M}_U$ is pseudo-wellfounded$\}$.

Check that $\mathcal{S}_\nu$ is Σ^1_1.

Proposition (The $\mathcal{S}_\nu$'s are **sparse**)

No two distinct members of $\mathcal{S}_\nu$ have the same hyperdegree. □

The proof uses the **Rigidity property** of models of T_ν, stated above.

Proof of Main Theorem.

We may assume $\mathbb{V} = \mathbb{L}$. Fix $\nu < \omega_1^{\mathrm{CK}}$.

Claim. There is a limit λ, such that $\aleph_{\nu+1}$ exists in $\mathbb{L}_\lambda$.

Suppose no such λ exists.

Hence for all admissible $\alpha > \omega$, $\mathbb{L}_\alpha \models \mathsf{T}_\nu$.

Consequence: the Σ^1_1 set $\mathcal{S}_\nu$ is cofinal in the degrees.

Indeed. Given $x \subseteq \omega$,

There is $\alpha > \omega$ admissible, such that $x \in \mathbb{L}_\alpha$ and $\mathcal{M}_{\mathsf{Th}(\mathbb{L}_\alpha)} \cong \mathbb{L}_\alpha$.

Thus $x \leqslant_T \mathsf{Th}(\mathbb{L}_\alpha)$. $\mathcal{M}_{\mathsf{Th}(\mathbb{L}_\alpha)}$ being wellfounded, $\mathsf{Th}(\mathbb{L}_\alpha) \in \mathcal{S}_\nu$.

Now: Weak-Turing-Det$_\rho(\Delta^1_1) \Rightarrow$ Weak-Turing-Det$_\rho(\Sigma^1_1)$.

Hence, there are two distinct $U_1, U_2 \in \mathcal{S}_\nu$ such that $U_1 \equiv_\rho U_2$, contradicting the previous proposition. $\square_{\text{Claim}}$

Let λ be as claimed.

Set $\mu = \aleph_{\nu+1}^{\mathbb{L}_\lambda}$. In $\mathbb{L}_\lambda$, μ is a successor cardinal.

Hence $\mathbb{L}_\mu \models \mathsf{ZF}^-$ [needs an argument].

For all $\xi \leqslant \nu$, $\aleph_\xi^{\mathbb{L}_\lambda}$ is an $\mathbb{L}_\mu$-cardinal: $\mathbb{L}_\mu \models \mathsf{ZF}^- +$ "$\aleph_\nu$ exists". $\square$

After Martin's proof of Borel determinacy, It was observed that Turing-Det(Δ^1_1) $\Rightarrow$ Det(Δ^1_1).

By a similar argument:

Theorem

For $2 \leqslant \rho < \omega_1^{\mathrm{CK}}$, Weak-Turing-Det$_\rho$($\Delta^1_1$) $\Rightarrow$ Det(Δ^1_1).

Proof sketch:

Given a Δ^1_1 game $G_\omega(A)$, say $A \in \Sigma^0_\nu$, ν recursive.

Use Weak-Turing-Det$_\rho$(Δ^1_1), get a transitive $M \models$ ZF$^-$ $+$ "$\aleph_\nu$ exists"

Inside M, $G_\omega(A)$ is determined (Martin's proof of Borel determinacy).

This is a Σ^1_2 fact, true in M.

By Mostowski's absoluteness theorem, it is true in $\mathbb{V}$. □

Effect of $\mathrm{Det}(\Delta^1_1)$ on Σ^1_1 sets

The **hyp-Turing cone** with vertex $d \in \mathcal{D}$ is the set of degrees:

$\mathrm{Cone}_h(d) = \{x \in \mathcal{D} \mid d \leqslant_T x \ \& \ x \leqslant_h d\} = \mathrm{Cone}(d) \cap \Delta^1_1(d)$.

Theorem

Assume Turing-$\mathrm{Det}(\Delta^1_1)$*:*

*If $(S_k)_{k<\omega}$ is a Σ^1_1 sequence of **cofinal** sets of degrees, then:*

$\bigcap_k S_k \neq \varnothing$ *— and, indeed, $\bigcap_k S_k$ contains a hyp-Turing cone.*

Corollary

Assume Turing-$\mathrm{Det}(\boldsymbol{\Delta}^1_1)$*:*

*If $(A_k)_{k<\omega}$ is a sequence of **cofinal** analytic sets of degrees, then:*

$\bigcap_k A_k$ *is cofinal in $\mathcal{D}$.* □

Theorem

Assume $\mathrm{Det}(\Delta^1_1)$*:*

For all Σ^1_1 sets S, one of the following holds for the game $G_\omega(S)$.

(1) *Player* $\mathbf{I}$ *has a strategy σ such that:* $\forall \tau \leqslant_h \sigma (\sigma * \tau \in S)$

(2) *Player* $\mathbf{II}$ *has a winning strategy.* □