Strongly κ^+-cc forcing

Mirna Džamonja, Tutorial 3, including joint work with J. Cummings and I. Neeman

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κ^+-cc and $(< \kappa)$-closure

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Remark (2) uses the fact that for any function $f : \mathrm{Ord} \to \mathrm{Ord}$ in the extension by a κ^+-cc forcing, there is a function F in the ground model such that $f(\alpha) \in F(\alpha)$ and $|F(\alpha)| \leq \kappa$.

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Definition

$\mathbb{P}$ is κ^+-stationary-cc if for every $\langle p_i : i < \kappa^+\rangle$ in $\mathbb{P}$ there is a club C in κ^+ and a regressive function f, such that

$$i, j \in S^{\kappa^+}_\kappa \cap C \text{ and } f(i) = f(j) \implies p_i, p_j \text{ countable } .$$

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Theorem (Shelah 1978)

An iteration with $(<\kappa)$-supports of κ^+-stationary-cc $(<\kappa)$-closed forcing in which every two conditions have lub, is κ^+-cc.

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Let κ satisfy $\kappa = \kappa^{<\kappa}$, say $\kappa \geq \aleph_1$. Consider elementary submodels $M \prec \mathcal{H}(\chi) = \langle H(\chi), \in, <^* \rangle$ with $|M| = \kappa$, ${}^{<\kappa}M \subseteq M$, $\mathbb{P}$=the forcing in question $\in M$.

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Note If $r|M$ is a residue for r and $r \geq t$, then $r|M$ is also a residue for t. Hence, to prove that a forcing is strongly κ^+-cc, it suffices to show that there is a dense set of conditions which are strongly $(M, \mathbb{P})$-generic, for relevant Ms.

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Proof Let $M \prec H(\chi)$, $|M| = \kappa$, ${}^{<\kappa}M \subseteq M$ and $\kappa, \lambda \in M$. Then for any condition r in $\mathrm{Add}(\kappa, \lambda)$, it suffices to let $r|M = r \restriction (\mathrm{dom}(r) \cap M)$.

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Classical methods show that this property is preserved by iteration with $(< \kappa)$-supports.

Main result

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Note For a filter *G* to be *M*-generic, it suffices that it intersects all open dense sets in *M*.

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and then letting $q = \mathrm{lub}_{i<\omega} p_i$. Let $H = \bigcup_{i<\omega} H_i$.

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5. for all i, $H_i \cup \{p_{i+1}, H_i\} \cup \mathrm{supt}(p_{i+1}) \subseteq H_{i+1}$,
6. $\langle p_i : i < \omega \rangle$ admits a least upper bound,

and then letting $q = \mathrm{lub}_{i<\omega} p_i$. Let $H = \bigcup_{i<\omega} H_i$.

Note In our context, every p allows a canonical extension.

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Given $p \in \mathbb{P}$, let $q \geq p$ be a canonical extension of p, we shall prove that q has a residue over M.

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For each $i < \omega$, let $\mathcal{D}_i$ collect all $u \in \mathbb{P}$ such that there is $\underset{\sim}{r}^* \in M \cap \underset{\sim}{Q}_\alpha$ with

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