Sharply 2-transitive Groups of Finite Morley Rank

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12th Panhellenic Logic Symposium

27 June 2018

(joint work with T. Altınel and A. Berkman)

The classification of simple groups

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Finite	Finite Morley rank
Gorenstein programme	Borovik programme
1965–1983 (2004)	1977–
10.000 pages	556 pages for the even type (Altınel, Borovik, Cherlin)
Undergoing the second revision	Still open
Analysis of the centralisers of involutions	Analysis of the centralisers of involutions
Based on the Feit-Thompson (odd order) theorem	No Feit-Thompson available (degenerate case possible)
Heavy use of character theory	No character theory

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Thompson (1960): The Frobenius kernel of a finite Frobenius group is nilpotent.

Terence Tao:

It seems to me that the above four theorems (Frobenius, Suzuki, Feit-Thompson, and CFSG) provide a ladder of sorts (with exponentially increasing complexity at each step) to the full classification, and that any new approach to the classification might first begin by revisiting the earlier theorems on this ladder and finding new proofs of these results first (in particular, if one had a "robust" proof of Suzuki's theorem that also gave non-trivial control on "almost CA-groups" — whatever that means — then this might lead to a new route to classifying the finite simple groups of Lie type and bounded rank). But even for the simplest two results on this ladder — Frobenius and Suzuki — it seems remarkably difficult to find any proof that is not essentially the character-based proof.

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In a later blog entry Tao gives a proof of Frobenius’ Theorem using commutative representation theory rather than non-commutative one. But it is is heavily based on averages.

Permutation groups

If G is a Frobenius group with (definable) Frobenius complement H, the left action on the coset space G/H yields a (definable) transitive permutation group such that the stabiliser of any two points is trivial. Conversely, for a permutation group G acting transitively on X such that the stabiliser of any two distinct points is trivial, the centraliser of any one point is malnormal, and G is a Frobenius group.

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Borvik and Nesin have conjectured that Frobenius' and Thompson's Theorems hold when replacing *finite* by *finite Morley rank*:

Conjecture 1. A Frobenius group G of finite Morley rank with Frobenius complement H splits as $G = N \rtimes H$ for nilpotent $N = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$.

Sharp 2-transitivity

A permutation group is *sharply* 2*-transitive* if for any two pairs of distinct points there is a unique permutation exchanging the pairs. The *standard* example is the group of affine transformations of some field K, i.e. the group $K^+ \rtimes K^\times$.

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Now any permutation $g \in G$ exchanging two points x and y must have order 2; if g' is an involution exchanging x' and y', then $g' = g^h$ for the unique $h \in G$ with $h(x') = x$ and $h(y') = y$. Thus all involutions are conjugate.

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Conjecture 2. An infinite sharply 2-transitive permutation group of finite Morley rank is standard for some algebraically closed field. More precisely:

(i) A sharply 2-transitive permutation group of finite Morley rank splits.

(ii) A sharply 2-transitive split permutation group of finite Morley rank is standard.

Permutation characteristic

If the Frobenius complement does not contain an involution, the *permutation characteristic* of G is 2. Otherwise, all products of two distinct involutions are conjugate, and the *permutation characteristic* of G is the order of ij, for any two distinct involutions i and j (or 0, if the order is infinite).

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They might, however, contribute to the study of the *odd type* case of the Algebraicity Conjecture.

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Proof.

Let G be the group, H its Frobenius complement, and $N \trianglelefteq G$ the definable normal subgroup given by the Theorem. If $i \in N$ is an involution, then $RM(N) \geq RM(i^H) = RM(H)$. Thus

$$2RM(H) \leq RM(HN) \leq RM(G) = 2RM(H).$$

It follows that $G = N \rtimes H$ splits.

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Deloro and Wiscons have recently obtained this Theorem as a corollary of a more general result on the 2-structure of a connected group of finite Morley rank.

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The *kernel* $\ker(K)$ of a near-field K is the set of elements with respect to which multiplication is left distributive:

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It need not be in the centre of K, however, since conjugation is not an automorphism of K and need not stabilize the kernel.

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In characteristic $p > 0$, note first that K is additively connected, as for any additive proper subgroup H of finite index the intersection $\bigcap_{x \in K^\times} xH$ is trivial, but equals a finite subintersection, and hence is of finite index, a contradiction. There is thus a unique type of maximal Morley rank, so $K^\times$ is multiplicatively connected as well.

Let A be a definable connected infinite abelian multiplicative subgroup, and M_0 an A-minimal additive subgroup. Then M_0 is additively isomorphic to the additive group of an algebraically closed field K_0, and A embeds multiplicatively into $K_0^\times$.

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In fact, for any $e_0 \in M_0 \setminus \{0\}$ and $a_1, \ldots, a_n \in A$ such that $a_1 e_0 + \cdots + a_n e_0 = 0$, we have by right distributivity that $a_1^{e_0} + \cdots + a_n^{e_0} = 0$, so the addition induced on A by K_0 is the one inherited from K-addition on A^{e_0}.

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So we might replace A by A^{e_0}, M_0 by $e_0^{-1} M_0$ and e_0 by 1. Then $A \subseteq M_0 = K_0^+$, and field multiplication on K_0 is induced from K on $A \times K_0$, but does not necessarily agree with multiplication from K if the left factor is in $K_0 \setminus A$.

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In particular, A is a decent torus. Hence the centre $Z(K^\times)$ contains the Sylow 2-subgroup, which is infinite in permutation characteristic different from 2.

We finish by the first paragraph. □

Corollary

A sharply 2-transitive group of finite Morley rank and permutation characteristic 3 is the group of affine transformations of an algebraically closed field of characteristic 3.

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Proof.

A sharply 2-transitive permutation group of permutation characteristic 3 splits (Kerby, Wefelscheid). Now use the Fact and Theorem above. □

Merci !

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