Games orbits play
&
obstructions to Borel reducibility

Aristotelis Panagiotopoulos
joint with M.Lupini

California Institute of Technology

2019 Panhellenic Logic Symposium

Classification problems

Definition

A **classification problem** is a pair (X, E), where X is a Polish space and E is an analytic equivalence relation on X.

Classification problems

Definition

A **classification problem** is a pair (X, E), where X is a Polish space and E is an analytic equivalence relation on X.

Recall. A **Polish space** is a separable and completely metrizable topological space. An equivalence relation E is **analytic** if it is the continuous image of some Polish space under a continuous map.

Classification problems

Definition

A **classification problem** is a pair (X, E), where X is a Polish space and E is an analytic equivalence relation on X.

Recall. A **Polish space** is a separable and completely metrizable topological space. An equivalence relation E is **analytic** if it is the continuous image of some Polish space under a continuous map.

Examples. This formal setup encompasses many natural problems:

- $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ the isomorphism problem between countable graphs.
- $(\mathcal{U}(\mathcal{H}), \simeq_U)$ the problems of classifying unitary operators of a separable Hilbert space $\mathcal{H}$ up to unitary equivalence.

Classification problems

Definition

A **classification problem** is a pair (X, E), where X is a Polish space and E is an analytic equivalence relation on X.

Recall. A **Polish space** is a separable and completely metrizable topological space. An equivalence relation E is **analytic** if it is the continuous image of some Polish space under a continuous map.

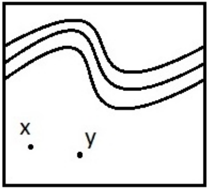
Examples. This formal setup encompasses many natural problems:

- $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ the isomorphism problem between countable graphs.
- $(\mathcal{U}(\mathcal{H}), \simeq_U)$ the problems of classifying unitary operators of a separable Hilbert space $\mathcal{H}$ up to unitary equivalence.

More generally. (X, E_X^G), where G is a Polish group acting continuously on a Polish space X and E_X^G is the associated **orbit equivalence relation**:

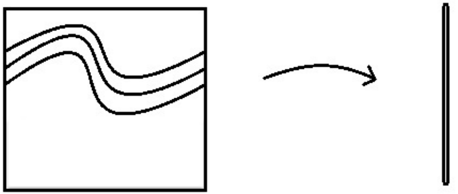
$$xE_X^G x' \iff [x]_G = [x']_G \iff \exists g \in G \;\; gx = x'.$$

Classification problems

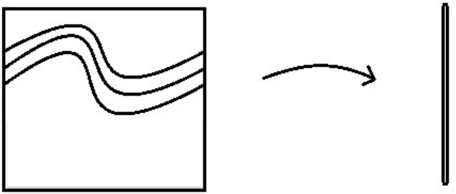


Are x and y equivalent?

Assigning invariants



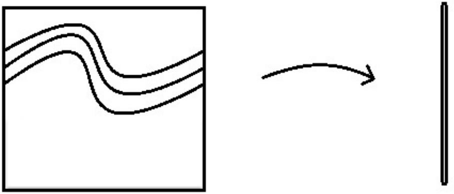
Assigning invariants



Invariants for graph isomorphism $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$:

- $G \mapsto \mathrm{maxdeg}(G)$, mapping G to its max degree;
- $G \mapsto \mathrm{conn}(G)$, mapping G to the #(connected components);

Assigning invariants

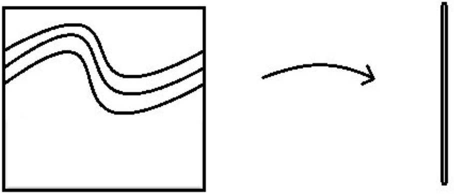


Invariants for graph isomorphism $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$:

- $G \mapsto \mathrm{maxdeg}(G)$, mapping G to its max degree;
- $G \mapsto \mathrm{conn}(G)$, mapping G to the $\#$(connected components);

Notice that $\{\mathrm{maxdeg}(\cdot), \mathrm{conn}(\cdot)\}$ is **not** a complete set of invariants.

Assigning invariants



Invariants for graph isomorphism $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$:

- $G \mapsto \mathrm{maxdeg}(G)$, mapping G to its max degree;
- $G \mapsto \mathrm{conn}(G)$, mapping G to the $\#$(connected components);

Notice that $\{\mathrm{maxdeg}(\cdot), \mathrm{conn}(\cdot)\}$ is **not** a complete set of invariants.

Definition

A classification problem (X, E) is concretely classifiable if there is a Borel map f from X to some Polish space Y so that $xEx' \iff f(x) = f(x')$.

Concrete Classification

Definition

A classification problem (X, E) is concretely classifiable if there is a Borel map f from X to some Polish space Y so that $xEx' \iff f(x) = f(x')$.

Friedman and Stanley: $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ is not concretely classifiable.

Concrete Classification

Definition

A classification problem (X, E) is concretely classifiable if there is a Borel map f from X to some Polish space Y so that $xEx' \iff f(x) = f(x')$.

Friedman and Stanley: $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ is not concretely classifiable.

What about $(\mathcal{U}(\mathcal{H}), \simeq_U)$?

Concrete Classification

Definition

A classification problem (X, E) is concretely classifiable if there is a Borel map f from X to some Polish space Y so that $xEx' \iff f(x) = f(x')$.

Friedman and Stanley: $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ is not concretely classifiable.

What about $(\mathcal{U}(\mathcal{H}), \simeq_U)$?

- When $\mathcal{H}$ is of **finite** dimension n, then assignment $\mathcal{U}(\mathcal{H}) \mapsto \mathbb{T}^n$ which maps each element of $\mathcal{U}(\mathcal{H})$ to its eigenvalues $(\lambda_1, \ldots, \lambda_n)$ in increasing order provides a concrete classification.

Concrete Classification

Definition

A classification problem (X, E) is concretely classifiable if there is a Borel map f from X to some Polish space Y so that $xEx' \iff f(x) = f(x')$.

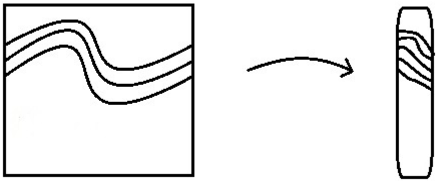
Friedman and Stanley: $(\mathrm{Graphs}(\mathbb{N}), \simeq_{\mathrm{iso}})$ is not concretely classifiable.

What about $(\mathcal{U}(\mathcal{H}), \simeq_U)$?

- When $\mathcal{H}$ is of **finite** dimension n, then assignment $\mathcal{U}(\mathcal{H}) \mapsto \mathbb{T}^n$ which maps each element of $\mathcal{U}(\mathcal{H})$ to its eigenvalues $(\lambda_1, \ldots, \lambda_n)$ in increasing order provides a concrete classification.

- **Choksi, Nadkarni**: when $\mathcal{H}$ is the **infinite** dimensional separable Hilbert space then the problem $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not concretely classifiable.

Relative complexity of classification problems

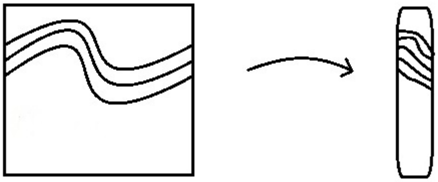


Let (X, E) and (Y, F) be two classification problems.
A **Borel reduction** from E to F is a Borel map $f \colon X \to Y$ with

$$xEx' \iff f(x)Ff(x').$$

We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

Relative complexity of classification problems



Let (X, E) and (Y, F) be two classification problems.
A **Borel reduction** from E to F is a Borel map $f\colon X \to Y$ with

$$xEx' \iff f(x)Ff(x').$$

We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

Notice that (X, E) is concretely classifiable iff $(X, E) \leq_B (Y, =)$, for some Polish space Y.

Relative complexity of classification problems

Definition

Let (X, E) and (Y, F) be two classification problems. A **Borel reduction** from E to F is a Borel map $f\colon X \to Y$ with $xEx' \iff f(x)Ff(x')$. We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

Relative complexity of classification problems

Definition

Let (X,E) and (Y,F) be two classification problems. A **Borel reduction** from E to F is a Borel map $f\colon X \to Y$ with $xEx' \iff f(x)Ff(x')$. We write $(X,E) \leq_B (Y,F)$ when such a Borel reduction exists.

Example. While $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not concretely classifiable, by the spectral theorem we can Borel reduce $(\mathcal{U}(\mathcal{H}), \simeq_U)$ to the problem

$$(\mathrm{Prob}(\mathbb{T}), \simeq_{null})$$

of **measure equivalence** between Borel probability measures on $\mathbb{T}$.

Relative complexity of classification problems

Definition

Let (X, E) and (Y, F) be two classification problems. A **Borel reduction** from E to F is a Borel map $f \colon X \to Y$ with $xEx' \iff f(x)Ff(x')$. We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

Example. While $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not concretely classifiable, by the spectral theorem we can Borel reduce $(\mathcal{U}(\mathcal{H}), \simeq_U)$ to the problem

$$(\mathrm{Prob}(\mathbb{T}), \simeq_{null})$$

of **measure equivalence** between Borel probability measures on $\mathbb{T}$.

Can we find "simpler" invariants than this?

Relative complexity of classification problems

Definition

Let (X, E) and (Y, F) be two classification problems. A **Borel reduction** from E to F is a Borel map $f\colon X \to Y$ with $xEx' \iff f(x)Ff(x')$. We write $(X, E) \leq_B (Y, F)$ when such a Borel reduction exists.

Example. While $(\mathcal{U}(\mathcal{H}), \simeq_U)$ is not concretely classifiable, by the spectral theorem we can Borel reduce $(\mathcal{U}(\mathcal{H}), \simeq_U)$ to the problem

$$(\mathrm{Prob}(\mathbb{T}), \simeq_{null})$$

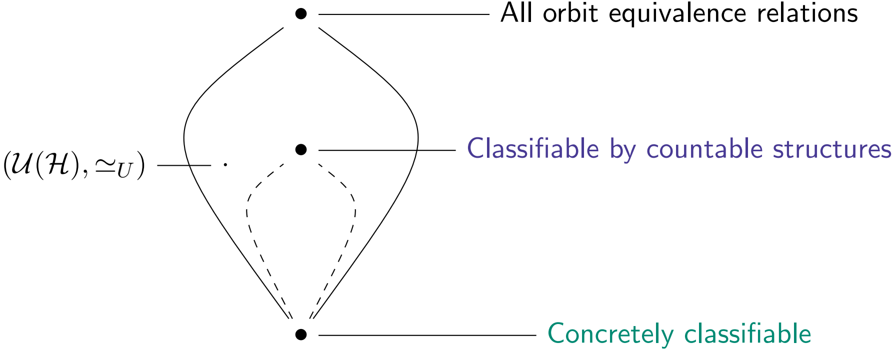
of **measure equivalence** between Borel probability measures on $\mathbb{T}$.

Can we find "simpler" invariants than this?

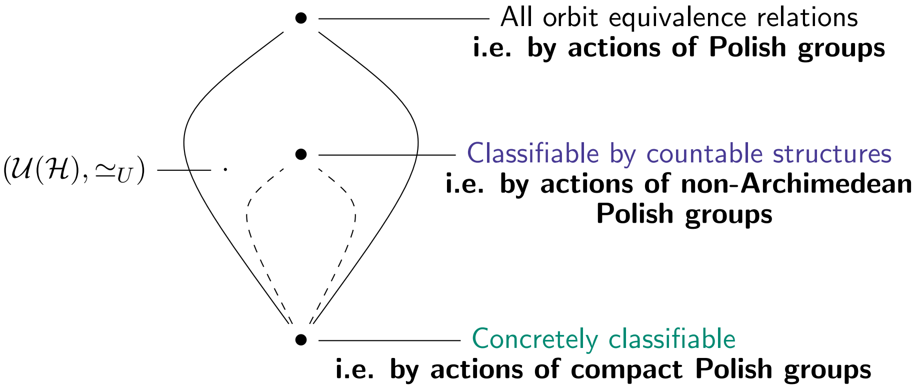
Kechris, Sofronidis: $(\mathcal{U}(\mathcal{H}), \simeq_U)$ does not Borel reduce to any "isomorphism problem between countable structures," e.g.

$$(\mathcal{U}(\mathcal{H}), \simeq_U) \quad \not\leq_B \quad (\mathrm{Graphs}(\mathbb{N}) \simeq_{\mathrm{iso}})$$

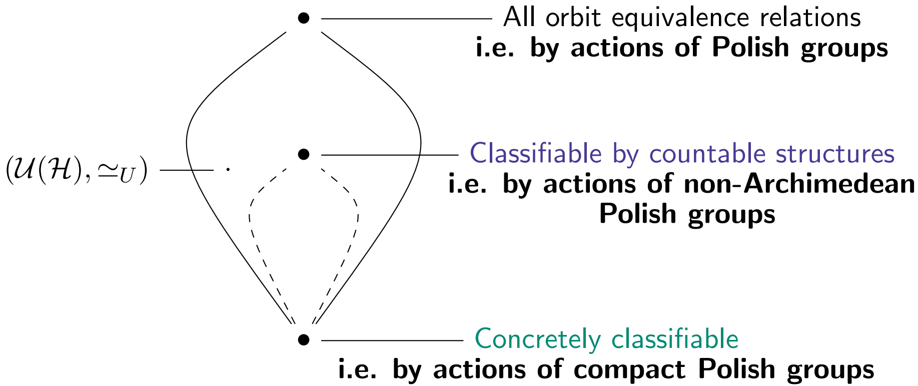
The universe of classification problems (X, E)



The universe with respect to dynamics



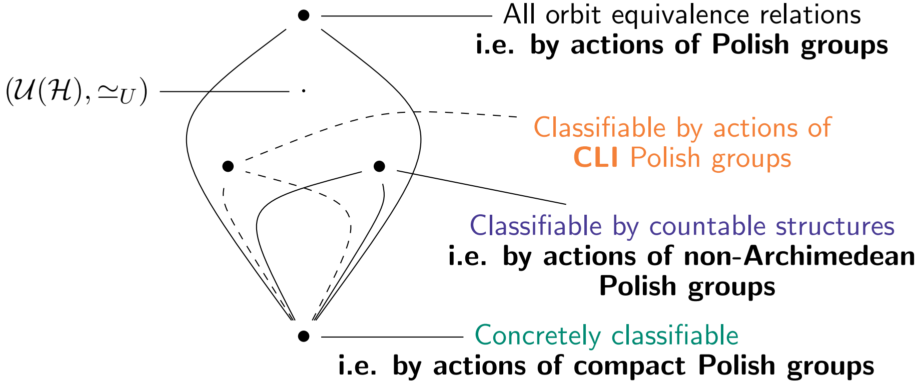
The universe with respect to dynamics



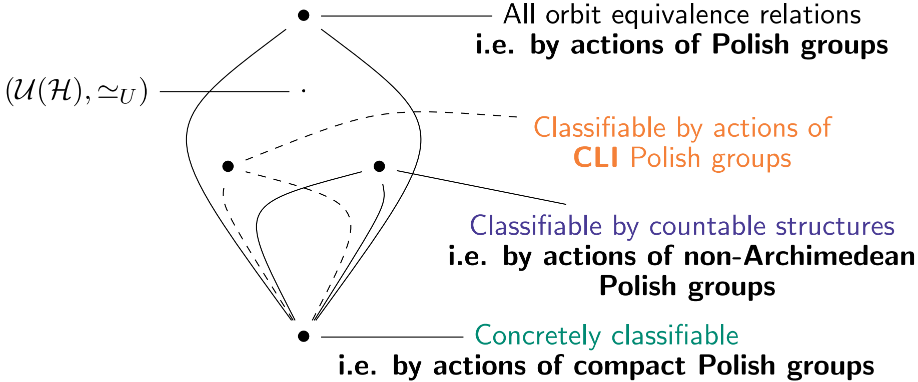
Question.

Can we classify $(\mathcal{U}(\mathcal{H}), \simeq_U)$ using invariants which come from the action of some *"algebraically tame"* Polish group, e.g., Abelian, solvable, etc.?

Yet another complexity class



Yet another complexity class



Theorem (Lupini, P.)

The classification problem $(\mathcal{U}(\mathcal{H}), \simeq_U)$ does not reduce to an orbit equivalence relation induced by an action of a **CLI** *group.*

Note. By a theorem of Solecki solvable Polish groups are CLI.

Dynamical obstructions to classification

Let $G \curvearrowright X$ be a continuous Polish group action and let E_X^G be the associated orbit equivalence relation.

[Folklore] If $G \curvearrowright X$ is **generically ergodic**, i.e., if it has dense and meager orbits, then (X, E_X^G) is **not** concretely classifiable.

Dynamical obstructions to classification

Let $G \curvearrowright X$ be a continuous Polish group action and let E_X^G be the associated orbit equivalence relation.

[Folklore] If $G \curvearrowright X$ is **generically ergodic**, i.e., if it has dense and meager orbits, then (X, E_X^G) is **not** concretely classifiable.
[Choksi, Nadkarni] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is generically ergodic.

Dynamical obstructions to classification

Let $G \curvearrowright X$ be a continuous Polish group action and let E_X^G be the associated orbit equivalence relation.

[Folklore] If $G \curvearrowright X$ is **generically ergodic**, i.e., if it has dense and meager orbits, then (X, E_X^G) is **not** concretely classifiable.
[Choksi, Nadkarni] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is generically ergodic.

[Hjorth] If $G \curvearrowright X$ is **turbulent**, then (X, E_X^G) is **not** classifiable by countable structures.

Dynamical obstructions to classification

Let $G \curvearrowright X$ be a continuous Polish group action and let E^G_X be the associated orbit equivalence relation.

[Folklore] If $G \curvearrowright X$ is **generically ergodic**, i.e., if it has dense and meager orbits, then (X, E^G_X) is **not** concretely classifiable.
[Choksi, Nadkarni] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is generically ergodic.

[Hjorth] If $G \curvearrowright X$ is **turbulent**, then (X, E^G_X) is **not** classifiable by countable structures.
[Kechris, Sofronidis] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is turbulent.

Dynamical obstructions to classification

Let $G \curvearrowright X$ be a continuous Polish group action and let E_X^G be the associated orbit equivalence relation.

[Folklore] If $G \curvearrowright X$ is **generically ergodic**, i.e., if it has dense and meager orbits, then (X, E_X^G) is **not** concretely classifiable.
[Choksi, Nadkarni] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is generically ergodic.

[Hjorth] If $G \curvearrowright X$ is **turbulent**, then (X, E_X^G) is **not** classifiable by countable structures.
[Kechris, Sofronidis] The action $\mathcal{U}(\mathcal{H}) \curvearrowright \mathcal{U}(\mathcal{H})$ by conjugation is turbulent.

We develop dynamical obstructions to classification by **CLI** group actions.

Left completions and CLI groups

Left completions and CLI groups

Let G be a Polish group. A sequence (g_n) in G is **left-Cauchy** if

(g_n) is Cauchy with respect to *some* left-invariant metric on G

$$\iff$$

(g_n) is Cauchy with respect to *any* left-invariant metric on G

Left completions and CLI groups

Let G be a Polish group. A sequence (g_n) in G is **left-Cauchy** if

$$(g_n) \text{ is Cauchy with respect to } \textit{some} \text{ left-invariant metric on } G$$
$$\Longleftrightarrow$$
$$(g_n) \text{ is Cauchy with respect to } \textit{any} \text{ left-invariant metric on } G$$

The **left-completion** $\widehat{G}$ of G is the completion of G with respect to some left-invariant metric. $\widehat{G}$ is always a monoid.

Left completions and CLI groups

Let G be a Polish group. A sequence (g_n) in G is **left-Cauchy** if

$$(g_n) \text{ is Cauchy with respect to } \textit{some} \text{ left-invariant metric on } G$$
$$\Longleftrightarrow$$
$$(g_n) \text{ is Cauchy with respect to } \textit{any} \text{ left-invariant metric on } G$$

The **left-completion** $\widehat{G}$ of G is the completion of G with respect to some left-invariant metric. $\widehat{G}$ is always a monoid.

- If G is CLI then $\widehat{G} = G$;
- $\widehat{S_\infty}$ is the monoid of all injections $\gamma : \mathbb{N} \to \mathbb{N}$;
- $\widehat{\mathcal{U}(\mathcal{H})}$ is the monoid of all linear isometric embeddings $T : \mathcal{H} \to \mathcal{H}$.

Left completions and CLI groups

Let G be a Polish group. A sequence (g_n) in G is **left-Cauchy** if

$$(g_n) \text{ is Cauchy with respect to } \textit{some} \text{ left-invariant metric on } G$$
$$\iff$$
$$(g_n) \text{ is Cauchy with respect to } \textit{any} \text{ left-invariant metric on } G$$

The **left-completion** $\widehat{G}$ of G is the completion of G with respect to some left-invariant metric. $\widehat{G}$ is always a monoid.

- If G is CLI then $\widehat{G} = G$;
- $\widehat{S_\infty}$ is the monoid of all injections $\gamma : \mathbb{N} \to \mathbb{N}$;
- $\widehat{\mathcal{U}(\mathcal{H})}$ is the monoid of all linear isometric embeddings $T : \mathcal{H} \to \mathcal{H}$.

Definition (Becker)

Let X be a Polish G-space. We say that x **left-embeds** in y if for any left-invariant metric d on G there exists a d-Cauchy sequence (g_n) so that $g_n x \to y$.

An obstruction to classification by CLI groups.

Theorem (Lupini, P.)

Let X be a Polish G-space. Assume that for any comeager subset C of X there exist $x, y \in C$ so that:

1. $[x] \neq [y]$;
2. x *left-embeds in* y.

Then E_G^X is not classifiable by CLI group actions.

An obstruction to classification by CLI groups.

Theorem (Lupini, P.)

Let X be a Polish G-space. Assume that for any comeager subset C of X there exist $x, y \in C$ so that:

1. $[x] \neq [y]$;
2. x *left-embeds in* y.

Then E^X_G is not classifiable by CLI group actions.

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \ldots) \; E_{\mathrm{ctbl}} \; (\lambda'_1, \lambda'_2, \ldots)$$

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \ldots)\ E_{\mathrm{ctbl}}\ (\lambda'_1, \lambda'_2, \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} = \{\lambda'_1, \lambda'_2, \ldots\}$$

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \ldots)\ E_{\mathrm{ctbl}}\ (\lambda'_1, \lambda'_2, \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} = \{\lambda'_1, \lambda'_2, \ldots\}$$

$$(\lambda_1, \lambda_2, \ldots) \hookrightarrow_{\mathrm{left}} (\lambda'_1, \lambda'_2, \ldots) \iff$$

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \dots)\; E_{\mathrm{ctbl}}\; (\lambda'_1, \lambda'_2, \dots) \iff \{\lambda_1, \lambda_2, \dots\} = \{\lambda'_1, \lambda'_2, \dots\}$$

$$(\lambda_1, \lambda_2, \dots) \hookrightarrow_{\mathrm{left}} (\lambda'_1, \lambda'_2, \dots) \iff \{\lambda_1, \lambda_2, \dots\} \subseteq \{\lambda'_1, \lambda'_2, \dots\}$$

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \ldots) \; E_{\mathrm{ctbl}} \; (\lambda_1', \lambda_2', \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} = \{\lambda_1', \lambda_2', \ldots\}$$

$$(\lambda_1, \lambda_2, \ldots) \hookrightarrow_{\mathrm{left}} (\lambda_1', \lambda_2', \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} \subseteq \{\lambda_1', \lambda_2', \ldots\}$$

Using the dynamical criterion for non-classifiability by CLI group actions one easily shows that the equivalence relation E_{ctbl} is not classifiable by CLI group actions.

The application

Let S be a Polish space and let $\mathrm{Inj}(\mathbb{N}, S)$ be the subspace of $S^{\mathbb{N}}$ consisting of the injective sequences from $\mathbb{N}$ to S. Consider the action of S_∞ on $\mathrm{Inj}(\mathbb{N}, S)$ by permuting coordinates and denote by E_{ctbl} the associated equivalence relation.

$$(\lambda_1, \lambda_2, \ldots)\ E_{\mathrm{ctbl}}\ (\lambda_1', \lambda_2', \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} = \{\lambda_1', \lambda_2', \ldots\}$$

$$(\lambda_1, \lambda_2, \ldots) \hookrightarrow_{\mathrm{left}} (\lambda_1', \lambda_2', \ldots) \iff \{\lambda_1, \lambda_2, \ldots\} \subseteq \{\lambda_1', \lambda_2', \ldots\}$$

Using the dynamical criterion for non-classifiability by CLI group actions one easily shows that the equivalence relation E_{ctbl} is not classifiable by CLI group actions.

Theorem (Lupini, P.)

Let $\mathcal{H}$ be the separable infinite dimensional Hilbert space. Then $\simeq_U$ on $\mathcal{U}(\mathcal{H})$ is **not** *classifiable by CLI group actions.*

Higher dimensional obstructions

Let X be a Polish G-space and let $x, y \in X$.

Definition

The **Becker graph** $\mathcal{B}(X/G)$ associated to $G \curvearrowright X$ is the directed graph:

- $\{[x] \colon x \in X\}$ is the collection of all vertexes of $\mathcal{B}(X/G)$;
- we add an arrow $[x] \to [y]$ whenever x left-embeds into y.

Higher dimensional obstructions

Let X be a Polish G-space and let $x, y \in X$.

Definition

The **Becker graph** $\mathcal{B}(X/G)$ associated to $G \curvearrowright X$ is the directed graph:

- $\{[x] \colon x \in X\}$ is the collection of all vertexes of $\mathcal{B}(X/G)$;
- we add an arrow $[x] \to [y]$ whenever x left-embeds into y.

Theorem (Lupini, P.)

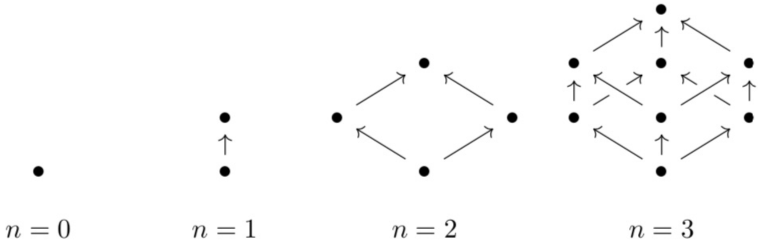
*If the Polish G-space X is generically 1-**dimensional**, i.e., for any comeager subset C of X there exist $x, y \in C$ so that:*

1. $[x] \neq [y]$;
2. x *left-embeds in* y;

*then E_G^X is **not** Borel reducible to an orbit equivalence relation E_H^Y induced by an action of a **CLI** group H.*

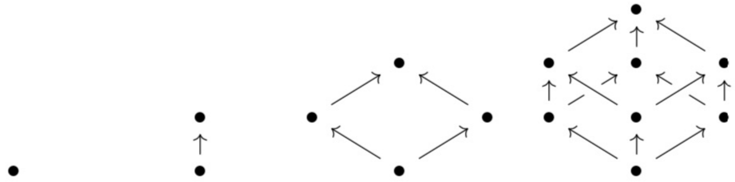
Higher dimensional obstructions

In a recent joint work with A.Kruckman we study higher dimensional obstructions to classification. The notion of dimension we define is based on whether $\mathcal{B}(X/G)$ contains n-cubes as subgraphs.
The n-**cube** is the diagraph $\Delta^n = \big(\mathcal{P}(\{0,\ldots,n-1\}), \subseteq\big)$



Higher dimensional obstructions

In a recent joint work with A.Kruckman we study higher dimensional obstructions to classification. The notion of dimension we define is based on whether $\mathcal{B}(X/G)$ contains n-cubes as subgraphs.
The n-**cube** is the diagraph $\Delta^n = \big(\mathcal{P}(\{0,\ldots,n-1\}), \subseteq\big)$



$n=0$ $n=1$ $n=2$ $n=3$

We use this to obtain anti-classification for isomoprhism relations between certain countable structures which have appeared in the work of Shelah and Baldwin, Koerwien, Laskowski

Thαnk you!