On Zilber's resricted Trichotomy Conjecture

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A theorem of Baldwin and Lachlan

Theorem (Baldwin-Lachlan)

Let T be a countable uncountably categorical theory. Then T has a prime model, $\mathcal{M}_0$. A strongly minimal set S is definable in $\mathcal{M}_0$ and any two models $\mathcal{M}_1, \mathcal{M}_2 \models T$ are isomorphic if and only if $\dim_{\mathcal{M}_1}(S) = \dim_{\mathcal{M}_2}(S)$.

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We remind that:

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A set S definable in an $\aleph_0$-saturated structure $\mathcal{M}$ is strongly minimal if every definable subset of S (not S^n!) is either finite or co-finite.

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A structure $\mathcal{M}$ is strongly minimal if $x = x$ defines a strongly minimal set.

The classical examples

The classical examples of strongly minimal sets are:

Example

1. An infinite set with no structure.
2. A vector space V over a field K in the language $\langle V; 0, +, \lambda\cdot\rangle_{\lambda\in K}$.
3. An algebraically closed field in the language of rings.

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There are many other examples:

1. An infinite regular binary tree.
2. A projective or affine space over a field K.
3. An algebraic curve over an algebraically closed field K.

Pre-geometries

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2. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ for all A.
3. $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ for all $A \subseteq B$.
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5. Exchange: if $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$ then $b \in \mathrm{cl}(Aa)$.

Exchange allows us to show – as in linear algebra – that any two maximal independent sets in a pre-geometry have the same cardinality.

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Zilber's Trichotomy conjecture

If $\mathcal{M}$ is strongly minimal and not locally modular then $\mathcal{M}$ interprets an algebraically closed field.

- The geometry of a pure set or of a binary tree is trivial.
- The geoemtry of a linear space is locally modular.
- The geometry of an algebraic curve over and algerbaically closed field is rich.

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Zilber's Conjecture is true for Zariski Geometries.

Zariski Geometries are a first order topological framework, essentially, axiomatizing the Zariski topology on (Cartesian powers of) regular algebraic curves over algebraically closed fields.

Many far reaching generalisations

- If D is a strongly minimal definable in DCF_0 then D satisfies Zilber's Trichotomy.
- If p is a *thin* type is a separably closed field then p satisfies an appropriate version of Zilber's Trichotomy.
- If p is a minimal type in *ACFA* then p satisfies an appropriate version of Zilber's trichotomy.
- If $\mathcal{M}$ is o-minimal then all 1-types over $\mathcal{M}$ satisfy an appropriate version of Zilber's Trichotomy.

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- If $\mathcal{M}$ is o-minimal then all 1-types over $\mathcal{M}$ satisfy an appropriate version of Zilber's Trichotomy.

Note that the three last examples are not strongly minimal, and the last two are not even stable.

An example of a different flavour

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In all previous examples the topology was available to help produce the field. In Rabinovich's result the topology only exists in the background – restricting the behaviour of definable sets.

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Zilber's Principle

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Ko.-Ran. A strongly minimal set interpretable in ACVF satisfies Zilber's Trichotomy.

A few words on Peterzil's conjecture

A far reaching conjecture:

Peterzil's conjecture covers structures interpretable for example in:

- $\mathbb{R}_{an,exp} := \langle \mathbb{R}; +, \times, \leq, e^x, f : f \text{ analytic on } [0,1] \rangle$.
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An even wider conjecture of Peterzil's:

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An even wider conjecture of Peterzil's:

A geometric structure interpretable in a distal theory satisfies Zilber's Trichotomy.

Example

Every o-minimal theory is distal, and every expansion of an o-minmial theory by externally definable sets is distal. Also, $\mathbb{Q}_p$ is distal.

The restricted Trichotomy in ACF

Theorem (H.-Sustretov)

Let M be a curve over an algebraically closed field K, $\mathcal{M}$ the structrue with universe M and some of the K-induced structure. Then $\mathcal{M}$ satisfies Zilber's Trichotomy.

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- If the conjecture is true then the higher dimensional case of the conjecture should be, essentially, vacuous.
- There are good reasons to believe that the proof would go through to ACVF.

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This reduces Peterzil's conjecture to strongly minimal structures interpretable in o-minimal theories:

Theorem (H-Onshuus-Peterzil)

If the universe of the interpretation is 1-dimensional, then the strongly minimal set is locally modular.

Remark

As in the case of ACF, if Peterzil's conjecture is true then a strongly minimal structure interpretable in an o-minimal structure is either locally modular, or 2-dimensional.

Peterzil's conjecture – cont.

Theorem (Eleftheriou-H.-Peterzil)

Assume that $\mathcal{G}$ is a strongly minimal group interpretable in an o-minimal expansion of a field, with 2-dimensional universe. If $\mathcal{G}$ is not locally modular then $\mathcal{G}$ is an algebraic group over an algebraically closed field.

Peterzil's conjecture – cont.

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- As in the case of algerbaically closed fields nothing is known in case the universe of the strongly minimal group is of dimension greater than 2.
- The topology on $\mathcal{G}$ is not the affine topology, but the *group topology*.
- What would be the right topology in case a group is not given?

Some final words

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- Can one formulate such a conjecture withstanding Hrushovski's and other counter examples?

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- Is there are model theoretic theory of tangency that would give a framework for Zilber's Trichotomy.
- Zilber's full-fledged Trichotomy conjecture also suggested a complete classification of geometries of strongly minimal sets.
- Can one formulate such a conjecture withstanding Hrushovski's and other counter examples?
- There are some promising attempts (due mostly to Mermelstein). It seems that reducts may have an important role to play.

Thank you!