Categoricity of Shimura Varieties

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Let T be the complete first-order theory of V in this language. As $V(F)$ is bi-interpretable with F, then T has the same model-theoretic properties as ACF_0, in particular, it is uncountably categorical.

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Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half-plane. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ through Möbius transformations. The quotient $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is an algebraic variety isomorphic to $\mathbb{C}$.

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Quotient varieties can have more interesting structures than just being algebraic varieties. In order to witness this with model theory, it helps to expand the language and make it two-sorted: $q : U \to V(\mathbb{C})$

- V has the language of algebraic varieties,
- U at least has the structure of a Γ-action,
- q is a function symbol invariant under Γ.

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By some very deep theorems, Shimura varieties are canonically defined over a corresponding number field $E = E(G, X^+)$ called the *reflex field*.

Special Subvarieties

Suppose $V \subseteq S$ is a subvariety such that one can find a subdomain $X_V^+ \subseteq X^+$ and an algebraic subgroup $H < G$ defined over $\mathbb{Q}$, so that (H, X_V^+) also satisfies the axioms of Shimura data.

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A 0-dimensional special subvariety of S is called a *special point*.

Some Important Special Subvarieties

Let $p : X^+ \to S(\mathbb{C})$ be a Shimura variety. Choose $g_1, \ldots, g_n \in G(\mathbb{Q})^+$.
Define:

$$\begin{array}{rrcc} p_{\overline{g}} & : & X^+ & \to & S(\mathbb{C})^n \\ & & x & \mapsto & (p(g_1 x), \ldots, p(g_n x)). \end{array}$$

The image of $p_{\overline{g}}$ is a special subvariety of S^n which we denote $Z_{\overline{g}}$.

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Let $\mathrm{Th}(\mathbf{p})$ be the complete first-order theory of $p : X^+ \to S(\mathbb{C})$ in this language.

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Remark

Every special subvariety of S^m is definable over $E(\Sigma)$.

Standard Fibres

$\mathrm{Th}(\mathbf{p})$ is not uncountably categorical because there is no restriction on the sizes of the fibres of q. From construction, the fibres of p are all of size Γ, but as Γ is a countably infinite group, this condition cannot be stated in a first-order way.

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Let SF be the $\mathcal{L}_{\omega_1,\omega}$-sentence:

$$\forall x, y \in D \left(q(x) = q(y) \implies \bigvee_{\gamma \in \Gamma} x = \gamma y \right).$$

Let $\mathrm{Th}_{\mathrm{SF}}(\mathbf{p}) = \mathrm{Th}(\mathbf{p}) \cup \mathrm{SF}$.

Let Γ' be a normal finite-index subgroup of Γ. This induces a natural map:

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whose fibres have a simply transitive action of Γ/Γ'.

Galois Representations

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Let $z' \in \Gamma' \backslash X^+$ and $z \in \Gamma \backslash X^+$ be such that $\psi(z') = z$. Let L be a finitely generated field extension of $E(\Sigma)$ over which z is defined. THEN $\mathrm{Aut}(\mathbb{C}/L)$ also acts on $\psi^{-1}(z)$ in a way that is comptaible with the action of Γ/Γ'. Thus we get a homomorphism:

$$\rho_{z'} : \mathrm{Aut}(\mathbb{C}/L) \to \Gamma/\Gamma'.$$

Repeat the construction of the homomorphism $\rho_{z'}$ for all finite-index normal subgroups of Γ, and choose the z' in a compatible way. We then get a homomorphism:

$$\mathrm{Aut}(\mathbb{C}/L) \to \overline{\Gamma} := \varprojlim_{\Gamma'} \Gamma/\Gamma'.$$

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This last question is of great interest in number theory, and had already been independently considered by Richard Pink (2006), and in more specific cases, it is known as the Mumford-Tate conjecture (still, mostly open).