Reductions between certain incidence problems and the Continuum Hypothesis

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Randomness and arbitrariness

This work investigates some problems which may be viewed as **thought experiments**. In order to state such thought experiments, we have chosen to proceed with a certain *ad hoc* **identification** (at principle, by stylistic reasons; a thought experiment has to be **compelling**, otherwise it would be ignored – but, of course, there are no naive choices in science).

We will identify **randomness** with **arbitrariness**.

In practice, such identification consists in the assumption that, when we declare that we are picking arbitrary objects, then there is **no pattern involved** and all **possible outcomes are unpredictable** since they obey an **equal probability distribution**.

Reductions of problems

In the field of Computational Complexity, one of the main techniques for relating and/or comparing the complexity of two problems is given by the notion of **reduction**.

If A and B are problems, we say that A **reduces to** B if there is a constructive transformation which maps any instance of A into an equivalent instance of B, meaning that a solution for the instance of B which is obtained via transformation provides a solution for the original instance of A.

One usually denotes "A reduces to B" by $A \leqslant B$; in such a case, it is usually said that "B **is at least as hard as** A", or "A **is at least as easy as** B" (or also "A **is at least as simple to be solved as** B").

The category $\mathcal{PV}$

A mathematical counterpart of the notion of reduction in Computational Complexity is given by the morphisms of the category $\mathcal{PV}$, which is a subcategory of the dual of the simplest case of the Dialectica Categories introduced by Valeria de Paiva.

Such morphisms are also known as **Galois-Tukey connections**, which is a terminology due to Peter Vojtáš.

Several connections between such category and Set Theory have been extensively studied by Andreas Blass in the 90's, and, more recently, have also been investigated by the speaker.

$\mathcal{PV}$ is the subcategory of the dual Dialectica category $\mathrm{Dial}_2(\mathbf{Sets})^{\mathrm{op}}$ whose objects – which, in general, are triples (A, B, E), where $E \subseteq A \times B$ – are those which satisfy the following conditions (which will be called the **MHD** conditions – where **MHD** stands for: Moore, Hrušák and Džamonja):

(1) $0 < |A|, |B| \leqslant 2^{\aleph_0}$.

(2) $(\forall a \in A)(\exists b \in B)[aEb]$

(3) $(\forall b \in B)(\exists a \in A)[\neg (aEb)]$

The morphisms between objects of $\mathcal{PV}$ are the very same morphisms of $\mathrm{Dial}_2(\mathbf{Sets})^{\mathrm{op}}$ – that is, a morphism from an object $o_2 = (A_2, B_2, E_2)$ to an object $o_1 = (A_1, B_1, E_1)$ is a pair of functions (φ, ψ), where $\varphi : A_1 \to A_2$ and $\psi : B_2 \to B_1$ are such that

$$(\forall a \in A_1)\ (\forall b \in B_2)\ [\varphi(a)\, E_2\, b \longrightarrow a\, E_1\, \psi(b)].$$

Morphisms of $\mathcal{PV}$ induce the so-called **Galois-Tukey pre-order**, which is defined in the following way: if $o_1 = (A_1, B_1, E_1)$ and $o_2 = (A_2, B_2, E_2)$ are objects of $\mathcal{PV}$, then we have

$$o_1 \leqslant_{GT} o_2 \iff \text{There is a morphism from } o_2 \text{ to } o_1.$$

The diagram below represents the situation where $o_1 \leqslant_{GT} o_2$:

$$\begin{array}{ccc} {}^{a}A_1 & E_1 & B_1{}^{\psi(b)} \\ \varphi \big\downarrow & & \big\uparrow \psi \\ {}_{\varphi(a)}A_2 & E_2 & B_2 \;\; {}_{b} \end{array}$$

Blass' interpretations: problems and reductions

Blass interprets each object $o = (A, B, E)$ in the following way: the object o represents a certain **problem** (or a type of problem); A is the particular set of **instances** of the problem represented by o; B is the set of **possible solutions** of such problem and E is the relation **"is solved by"** – that is, aEb says that "b solves a" .

In this sense, the Galois-Tukey order indeed measures complexity: if $o_1 \leqslant_{GT} o_2$ then the problems in o_1 are at least as easy to solve as problems in o_2, since the act of solving a problem in o_1 may be **reduced** to the act of solving of a (corresponding) problem in o_2.

$$
\begin{array}{ccccc}
{}^{a}A_1 & & E_1 & & B_1{}^{\psi(b)} \\
\varphi \downarrow & & & & \uparrow \psi \\
{}_{\varphi(a)}A_2 & & E_2 & & B_2 \;{}_{b}
\end{array}
$$

As you can see in the diagram: if $b \in B_2$ is a solution for the instance $\varphi(a)$ of o_2 then $\psi(b) \in B_1$ solves the instance a of o_1.

Two families of incidence problems

The described machinery of the $\mathcal{PV}$ category will be applied for two families of incidence problems (which may also be regarded as **challenges**), $\mathcal{C}_1$ and $\mathcal{C}_2$.

In both cases, to solve the problem (or win the challenge) one has to give an appropriate (but random) response to a certain initial data; notice that such procedure could also be interpreted as some one-round game between two players, were the first player gives the initial data and the second player wins if he gives a response which solves the problem.

Instances of problems of $\mathcal{C}_1$ are as follows: given a real number x, pick randomly a countably infinite set of reals A hoping that $x \in A$; , whereas instances of problems in $\mathcal{C}_2$ are as follows: given a countably infinite set of reals A, pick randomly a real number x hoping that $x \notin A$.

At the end of the talk, we argue that, in fact, there is no difference between starting with the real number or with the countable subset; we may even think of **simultaneous picks**.

The objects $(\mathbb{R}, [\mathbb{R}]^{\aleph_0}, \in)$ and $([\mathbb{R}]^{\aleph_0}, \mathbb{R}, \not\ni)$ – And what does our intuition tell us ?

Notice that, under Blass' interpretation, we are able to identify $\mathcal{C}_1$ with the object $(\mathbb{R}, [\mathbb{R}]^{\aleph_0}, \in)$: in this sense, each real number x **is** an instance of the problem, whose possible solutions are countably infinite subsets $A \in [\mathbb{R}]^{\aleph_0}$ such that $x \in A$.

And, in the precise same way, we identify $\mathcal{C}_2$ with the object $([\mathbb{R}]^{\aleph_0}, \mathbb{R}, \not\ni)$: in this sense, each countably infinite subset X **is** an instance of the problem, whose possible solutions are real numbers $x \in \mathbb{R}$ such that $x \notin A$.

One could arguably defend that, at least intuitively, problems of $\mathcal{C}_2$ are easier to solve than problems of $\mathcal{C}_1$ – if A is countable, then A is null and being a member of A is a very rare event, and so it should be (or, at least, one could plausibly believe that it should be) much more easy to pick randomly a real number which is not in A, rather than doing (essentially) the opposite.

Our results

The main aim of this research is to compare the complexities of $\mathcal{C}_1$ and $\mathcal{C}_2$, using GT-reductions to proceed with the measurements; such complexity comparisons relate, as will be shown, to both **CH** (**Continuum Hypothesis**) and **AC** (**Axiom of Choice**).

Our main results are the following:

- In the context of GT-reductions, we prove (within **ZFC**) that, on one hand, problems of $\mathcal{C}_2$ are **at least as easy to solve** as problems of $\mathcal{C}_1$.
- On the other hand, the statement "Problems of $\mathcal{C}_1$ have the **exact same complexity** of problems of $\mathcal{C}_2$" is shown to be an **equivalent of CH**.

$\mathcal{C}_2$ reducts to $\mathcal{C}_1$

Let us prove a proposition which confirms our intuition about problems of $\mathcal{C}_2$ being easier to solve than problems of $\mathcal{C}_1$.

Recall that we identify $\mathcal{C}_1$ with $o_1 = (\mathbb{R}, [\mathbb{R}]^{\aleph_0}, \in)$ and $\mathcal{C}_2$ with $o_2 = ([\mathbb{R}]^{\aleph_0}, \mathbb{R}, \not\ni)$.

Proposition – o_2 reducts to o_1

There is a morphism witnessing $o_2 \leqslant_{GT} o_1$.

In words: Problems of $\mathcal{C}_2$ are at least as simple to be solved as problems of $\mathcal{C}_1$.

The key of the proof is that, by the well-known König's Lemma, the cofinality of $2^{\aleph_0}$ is uncountable.

Proof of $o_2 \leqslant_{GT} o_1$

Enumerate $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$. For any $X \in [\mathbb{R}]^{\aleph_0}$, let $\gamma(X) < \mathfrak{c}$ be the ordinal

$$\gamma(X) = \sup\{\alpha < \mathfrak{c} : x_\alpha \in X\} + 1.$$

Let $\varphi : [\mathbb{R}]^{\aleph_0} \to \mathbb{R}$ be defined by putting, for every $X \in [\mathbb{R}]^{\aleph_0}$,

$$\varphi(X) = x_{\gamma(X)}.$$

Notice that, for any $X \in [\mathbb{R}]^{\aleph_0}$, every element of X has its ordinal index strictly smaller than the ordinal index of $\varphi(X)$.

Proof of $o_2 \leqslant_{GT} o_1$

Let us consider $\psi = \varphi$, that is, we take the pair of functions given by (φ, φ) as a candidate to be a morphism. We claim that such pair is, in fact, a morphism of $\mathcal{PV}$, from o_1 to o_2.

$$\begin{array}{ccccc} X & [\mathbb{R}]^{\aleph_0} & \not\ni & \mathbb{R} & x_{\gamma(Y)} \\ & \varphi \big\downarrow & & \big\uparrow \varphi & \\ x_{\gamma(X)} & \mathbb{R} & \in & [\mathbb{R}]^{\aleph_0} & Y \end{array}$$

Indeed: let X, Y be any countably infinite subsets of $\mathbb{R}$. If $\varphi(X) = x_{\gamma(X)} \in Y$, then $\gamma(X) < \gamma(Y)$, and therefore $x_{\gamma(Y)} = \varphi(Y) \notin X$.

Reverse GT-inequality $\equiv$ **CH**

As announced, whether the reverse GT-inequality between o_1 and o_2 holds or not is a question whose answer is independent of **ZFC**.

The Main Theorem – TFAE:

- The Continuum Hypothesis.
- $o_1 \leqslant_{GT} o_2$.

  In words: Problems of $\mathcal{C}_1$ are at least as simple to be solved as problems of $\mathcal{C}_2$.
- $o_1 \cong_{GT} o_2$.

  In words: Problems of $\mathcal{C}_1$ have the exact same complexity of problems of $\mathcal{C}_2$.

As we have proved that $o_2 \leqslant_{GT} o_1$ holds in **ZFC**, the equivalence between (*ii*) and (*iii*) is clear; so, it suffices to prove the equivalence between (*i*) and (*ii*).

Proof of $(i) \Rightarrow (ii)$:

Assuming **CH**, we enumerate $\mathbb{R} = \{x_\alpha : \alpha < \omega_1\}$.

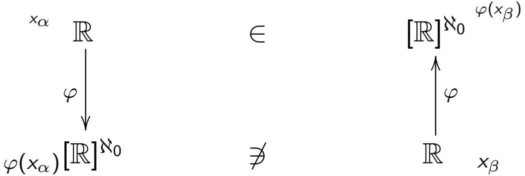
Let $\varphi : \mathbb{R} \to [\mathbb{R}]^{\aleph_0}$ be the function defined in the following way: for every $\alpha < \omega_1$,

$$\varphi(x_\alpha) = \begin{cases} \{x_\xi : \xi \leqslant \alpha\} & \text{if } \alpha \geqslant \omega \text{ ; and} \\ \{x_n : n < \omega\} & \text{otherwise.} \end{cases}$$

Notice that the reals indexed by finite ordinals (i.e., indexed by natural numbers) are elements of $\varphi(x)$ for any real number x.

Proof of $(i) \Rightarrow (ii)$:

Let us consider $\psi = \varphi$, that is, again we take a pair of functions of functions of the form (φ, φ) as a candidate to be a morphism. We claim that such pair is, indeed, a morphism of $\mathcal{PV}$, from o_2 to o_1.



Indeed: let $x = x_\alpha$ and $y = x_\beta$ be arbitrary real numbers. If $x_\beta \notin \varphi(x_\alpha)$ then β is an infinite ordinal, and also one has, necessarily, $\beta > \alpha$. It follows that $x_\alpha \in \varphi(x_\beta)$. This shows that (φ, φ) is a morphism of $\mathcal{PV}$ which witnesses $o_1 \leqslant_{GT} o_2$, as desired.

Proof of $(ii) \Rightarrow (i)$:

Arguing contrapositively, we assume $\neg$**CH** and show that **no** pair of functions (φ, ψ) is a morphism from o_2 to o_1.

Thus, we have to check that, under $2^{\aleph_0} > \aleph_1$, the following formula holds:

$$(\forall \varphi, \psi : \mathbb{R} \to [\mathbb{R}]^{\aleph_0})(\exists x, y \in \mathbb{R})[\,\neg(y \notin \varphi(x) \to x \in \psi(y))\,],$$

or, equivalently,

$$(\forall \varphi, \psi : \mathbb{R} \to [\mathbb{R}]^{\aleph_0})(\exists x, y \in \mathbb{R})[y \notin \varphi(x) \wedge x \notin \psi(y)].$$

Proof of $(ii) \Rightarrow (i)$:

Fix $A \subseteq \mathbb{R}$ with $|A| = \aleph_1$. It follows that

$$\left| \bigcup_{x \in A} \varphi(x) \right| \leqslant \aleph_1$$

and, as we are assuming $2^{\aleph_0} > \aleph_1$, we may pick some $y \notin \bigcup_{x \in A} \varphi(x)$. To get done, notice that, as $\psi(y)$ is a countable subset of the reals, we necessarily have $A \setminus \psi(y) \neq \emptyset$ – and therefore we may pick some $x \in A$ such that $x \notin \psi(y)$.

For this particular pair $\{x, y\}$ of reals we have what we want, that is, $y \notin \varphi(x)$ and $x \notin \psi(y)$.

Reformulating using ideals

It is possible to reformulate our main result in the language of **combinatorics of ideals**.

As it is probably known by the audience, a family $\mathcal{I}$ of subsets of a non-empty set X is said to be an **ideal of subsets of X** if it is a proper, non-empty subset of $\mathcal{P}(X)$ which is closed under taking subsets and under taking finite unions.

The following are some of certain well-known cardinal invariants which may be defined for any ideal:

- $\mathrm{cov}(\mathcal{I})$ (the **covering number** of $\mathcal{I}$) is the smallest size of a subfamily of $\mathcal{I}$ which covers X – that is,

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}.$$

- $\mathrm{non}(\mathcal{I})$ (the **uniformity** of $\mathcal{I}$) is the smallest size of a subset of X which is not in $\mathcal{I}$ – that is,

$$\mathrm{non}(\mathcal{I}) = \min\{|A| : A \subseteq X \text{ and } A \notin \mathcal{I}\}.$$

Norms and the method of morphisms

In the cases where $\mathcal{I}$ is an ideal of subsets of $\mathbb{R}$ with (at least a base of) size not larger than the continuum, those cardinals invariants are examples of the so-called **norms** (or **evaluations**) of objects of $\mathcal{PV}$.

Norms of $\mathcal{PV}$ objects

Let $o = (A, B, E)$ be an object of $\mathcal{PV}$. Its **norm** is the cardinal number $||o|| = ||(A, B, E)||$ given by

$$||o|| = \min\{|Y| : Y \subseteq B \text{ and } (\forall a \in A)(\exists b \in Y)[aEb]\}.$$

The main application of $\mathcal{PV}$ (and of he GT-ordering) to Set Theory is the so-called **method of morphisms** in the proof of inequalities between cardinal invariants of the continuum – which was once declared by Blass as being an **empirical fact**.

"Folklore" – The method of morphisms

Let o_1 and o_2 be objects of $\mathcal{PV}$.

$$\text{If } o_1 \leqslant_{GT} o_2 \text{ then } ||o_1|| \leqslant ||o_2||.$$

Thus, clearly one has that if $o_1 \cong_{GT} o_2$ then $||o_1|| = ||o_2||$.

What we have done is ...

One can easily check, indeed, that $\mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ are norms – namely, $\mathrm{cov}(\mathcal{I}) = ||(X, \mathcal{I}, \in)||$ and $\mathrm{non}(\mathcal{I}) = ||(\mathcal{I}, X, \not\ni)||$.

Also, all of our arguments could have been done for the objects $(\mathbb{R}, \mathcal{I}_C, \in)$ and $(\mathcal{I}_C, \mathbb{R}, \not\ni)$, where $\mathcal{I}_C$ denotes the ideal $[\mathbb{R}]^{\leqslant \aleph_0}$ of all countable subsets of $\mathbb{R}$ (including the finite ones).

Thus, the equivalence **CH** $\iff (\mathbb{R}, \mathcal{I}_C, \in) \cong_{GT} (\mathcal{I}_C, \mathbb{R}, \not\ni)$ implies (by the method of morphisms) the following statement: **"If CH holds, then $||(\mathbb{R}, \mathcal{I}_C, \in)|| = ||(\mathcal{I}_C, \mathbb{R}, \not\ni)||$"** – that is,

$$\textbf{CH} \Rightarrow \mathrm{cov}(\mathcal{I}_C) = \mathrm{non}(\mathcal{I}_C).$$

However, the **absolute, ZFC** values of $\mathrm{cov}(\mathcal{I}_C)$ and $\mathrm{non}(\mathcal{I}_C)$ are easily seen to be, respectively, $\mathfrak{c}$ and $\aleph_1$!!! So, what we have been done, at the end of the day, is to show (using morphisms) that

$$\textbf{CH} \iff \underbrace{\mathrm{cov}(\mathcal{I}_C)}_{\mathfrak{c}} = \underbrace{\mathrm{non}(\mathcal{I}_C)}_{\aleph_1}$$

Decision problems, independence and simultaneity

Our families of problems, $\mathcal{C}_1$ and $\mathcal{C}_2$, could have been presented as **decision problems**. A decision problem is a problem which has only two possible solutions, "yes" or "no" . Formally, a decision problem Π consists of a set D_Π of **instances**, and a subset $Y_\Pi \subseteq D_\Pi$ of **yes-instances**. Decision problems $\Pi_1(A)$ of $\mathcal{C}_1$ and $\Pi_2(A)$ of $\mathcal{C}_2$ would be like this:

$D_{\Pi_1(A)} = \mathbb{R}$
$Y_{\Pi_1(A)} = A$
INPUT: x
Y-N QUESTION: Is it true that $x \in A$?

$D_{\Pi_2(A)} = \mathbb{R}$
$Y_{\Pi_2(A)} = \mathbb{R} \setminus A$
INPUT: x
Y-N QUESTION: Is it true that $x \notin A$?

Notice also that, within this presentation, we are assuming that x and A may be given **simultaneously**, either for the problems of $\mathcal{C}_1$ or for the problems of $\mathcal{C}_2$. Such assumption is justified by the fact that we are dealing with **two independent events** – meaning, to pick randomly **a real number** and **a countable set of reals**.

All things considered (regarding the described modelling of $\mathcal{C}_1$ and $\mathcal{C}_2$ as families of decision problems, and also assuming that any sequential order of picking is possible, including the possibility of simultaneity), it seems that the results of this paper provide information for the discussion of the following question:

The Main Question – A "hit or miss" game

Before being given a countable set A of reals and a real number x, both to be **randomly** taken, should one say that it will be **easier** (or it will be **more likely**) that, eventually, this real number x **will miss** the countable set A ? Or should one say that, under the very same conditions and interpretations, **it will hit it** ?

Notice that, in the preceding question, "being easier" refers to **complexity**, and "being more likely" refers to **probability**.

To give a full answer is an undecidable problem

So, if one accepts the exhibiting-reductions approach to discuss the complexity of the problems $\mathcal{C}_1$ and $\mathcal{C}_2$, then a full answer to the "complexity part" of the Main Question depends on the Continuum Hypothesis and therefore we would have faced **an undecidable problem** (in the sense that the statement **"Problems of $\mathcal{C}_1$ have the exact same complexity of problems of $\mathcal{C}_2$"** was shown to be independent of the usual axioms of Set Theory).

A disclaimer

The author refrains from going much further on the philosophical discussion of the pure **ZFC** theorems presented in this work. However, he believes that those results will be appealing not only for mathematicians, but for mathematical philosophers as well – since our Main Question may raise a number of inquiries and issues on themes such as complexity, probability, randomness and arbitrariness, applicability of reductions between problems, simultaneity and symmetry, etc. . . . not to mention **CH** itself . . .

Speaking out more explicitly, the author will not defend the thesis that the results of this work should be regarded as some kind of **evidence against the Continuum Hypothesis**. (eppur, lo è !)

On the probability part of the Main Question: Freiling's paper on **throwing darts at the real line**

The (proof of) the equivalence of **CH** presented in this paper is quite similar to one among a number of equivalences presented by Freiling in his famous paper on **throwing darts at the real line**.

The main point of Freiling, however, was to present a **"philosophical proof" of the negation of CH** (as explicitly declared in the very first phrase of the abstract of his paper), based on certain probability reasonings (elegantly exposed in terms of the throw of darts); in fact, the mathematical content of Freiling's proof is due to Sierpiński.

After the "philosophical justification", Freiling proceeds to convince the reader that the following formula should be valid:

$$\mathbf{A}_{\aleph_0} \equiv (\forall f : \mathbb{R} \to [\mathbb{R}]^{\aleph_0})(\exists x, y \in \mathbb{R})[y \notin f(x) \wedge x \notin f(y)].$$

And finally, he has shown that $\mathbf{A}_{\aleph_0} \iff \neg\mathbf{CH}$.

Advantages of our approach

We believe that our approach has some advantages, in the comparison with the one of Freiling's, mainly by three reasons:

- Freiling has never formalized his probability argument; his philosophical justification was only intuitive.
- After his intuitive argument, Freiling came to his formula $\mathbf{A}_{\aleph_0}$ and has shown its equivalence with $\neg\mathbf{CH}$. In our work, a very similar formula, which is

$$(\forall\varphi, \psi : \mathbb{R} \to [\mathbb{R}]^{\aleph_0})(\exists x, y \in \mathbb{R})[y \notin \varphi(x) \wedge x \notin \psi(y)],$$

  has **naturally appeared** under a formal and well-defined mathematical context: the one of comparing complexities. So, we believe our results were stated under a very specified, clear and motivated mathematical context.
- Our final formulation (namely, the modelling using decision problems) avoids discussing **symmetry**. In fact, **there are no arguments of symmetry in the proofs of this paper**. Nevertheless, it is a fact that in our work we embrace (by presupposing) concepts as **independence** and **simultaneity**.

A case against the Axiom of Choice ?

A number of objections which have been made to Freiling's darts could also apply to our approach based on reductions.

For instance, natural generalizations of Freiling's statement $\mathbf{A}_{\aleph_0}$ contradict **AC** as well – that is, Freiling's case against **CH** may be, in fact, a case against **AC**.

Indeed: if one considers that "being a member of a set whose cardinality is **less than** $\mathfrak{c}$" is also a **very rare** event, then one could proceed with a similar **"darts argument"** to conclude that the following formula should be valid:

$$\mathbf{A}_{<\mathfrak{c}} \equiv (\forall f : \mathbb{R} \to [\mathbb{R}]^{<\mathfrak{c}})(\exists x, y \in \mathbb{R})[y \notin f(x) \wedge x \notin f(y)].$$

Nonetheless, it is easily seen that $\mathbf{A}_{<\mathfrak{c}}$ is false if we assume that there is a well-ordering of $\mathbb{R}$.

On some precise deductive strengths

Working similarly as in this work, one could design the less-than-$\mathfrak{c}$ versions of our problems and show that the existence of morphisms witnessing $(\mathbb{R}, [\mathbb{R}]^{<\mathfrak{c}}, \in) \leqslant_{GT} ([\mathbb{R}]^{<\mathfrak{c}}, \mathbb{R}, \not\ni)$ follows, in **ZF**, from the statement **"$\mathbb{R}$ can be well ordered"** .

Regarding our absolute result (that is, **ZFC** proves $o_2 \leqslant_{GT} o_1$), if one tries to adapt its proof to get $([\mathbb{R}]^{<\mathfrak{c}}, \mathbb{R}, \not\ni) \leqslant_{GT} (\mathbb{R}, [\mathbb{R}]^{<\mathfrak{c}}, \in)$, the conclusion is that such GT-inequality follows, within **ZFC**, from the statement **"$2^{\aleph_0}$ is a regular cardinal"** (which holds, for instance, under Martin's Axiom).

The preceding paragraph justifies the following problem:

Problem

Determine the precise deductive strengths of
(i) $(\mathbb{R}, [\mathbb{R}]^{<\mathfrak{c}}, \in) \leqslant_{GT} ([\mathbb{R}]^{<\mathfrak{c}}, \mathbb{R}, \not\ni)$, relatively to **ZF**; and
(ii) $([\mathbb{R}]^{<\mathfrak{c}}, \mathbb{R}, \not\ni) \leqslant_{GT} (\mathbb{R}, [\mathbb{R}]^{<\mathfrak{c}}, \in)$, relatively to **ZFC**.

What is the precise role of the Axiom of Choice ?

Even the absolute result $o_2 \leqslant_{GT} o_1$ encompasses critical uses of the Axiom of Choice.

First of all, and more obviously, the function φ used in its proof is defined using an enumeration (essentially, a well-ordering) of $\mathbb{R}$.

Second, and less obviously, one should recall that König's Lemma (which says that for all infinite cardinal κ, the cofinality of 2^κ is larger than κ) is a corollary of the so-called **König's Theorem**, and such theorem may be rephrased into an equivalent of **AC**.

Our final question

What happens if we consider the families of problems $\mathcal{C}_1$ and $\mathcal{C}_2$ in a choiceless Set Theory ? What is the precise role of the Axiom of Choice in all results of this paper ? How much of the Axiom of Choice is needed in order to get to the very same conclusions ?

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Thanks and I hope see you soon in Salvador !

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