The ∀∃∀-theory of the partial order of the enumeration degrees

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Joint work with S. Lempp and T. Slaman

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- To understand what $\forall\exists$-sentences are true in $\mathcal{D}$ we need to solve a slightly more complicated problem:

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We are given a finite partial order P and finite partial orders $Q_0, \ldots Q_n \supseteq P$. Does every embedding of P extend to an embedding of one of the Q_i?

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Theorem (Shore 78; Lerman 83)

That is the only obstacle.

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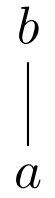
A degree $\mathbf{b}$ is a *strong minimal cover* of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and for every degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \leqslant \mathbf{a}$.

Theorem (Kent, Lewis-Pye, Sorbi 12)

There is a Δ^0_3 degree $\mathbf{a}$ and Π^0_2 enumeration degree $\mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$

The simplest lattice

Consider the lattice $\mathcal{L} = \{a < b\}$. What properties should possible extensions $Q_0, Q_1 \ldots Q_n$ have so that every embedding of $\mathcal{L}$ extends to Q_i for some i:



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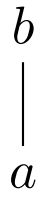

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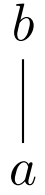

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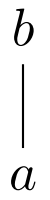

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Theorem (Slaman, Sorbi 14)

Every countable partial order can be embedded below any nonzero enumeration degree.

So these two conditions suffice.

A wild conjecture

Conjecture (Lempp, Slaman, Soskova)

Every finite lattice can be embedded into $\mathcal{D}_e$ as an interval of Π^0_2 enumeration degrees $[\mathbf{a}, \mathbf{b}]$ so that if $\mathbf{x} \leqslant \mathbf{b}$ then $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} < \mathbf{a}$.

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- Note! This would only solve the extension of embeddings problem: Every embedding of P would extend to an embedding of Q if Q satisfies the same two properties: have no new degree below any member of P and respect least upper bounds.
- If we allow more than one Q then we need a wilder conjecture, e.g.:

Conjecture

Every finite lattice can be embedded into $\mathcal{D}_e$ so that:

1. If $\mathbf{x} \leqslant \mathbf{b}$, where $\mathbf{b}$ is the image of the largest element then $\mathbf{x}$ is the image of an element from the lattice or bounded by an atom of the lattice.
2. Incomparable atoms and co-atoms form minimal pairs.

This implies the existence of strong minimal pairs.

Our results: Step 1

Slightly extend the Kent, Lewis-Pye, Sorbi result:

Theorem

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Construct Π^0_2 sets A and B so that:

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$A \leqslant_e \Psi(A, B)$ because A consists of (1) elements enumerated by higher priority strategies, (2) elements in the stream, (3) elements enumerated in A to code B at higher priority strategies.

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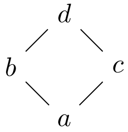

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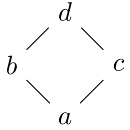

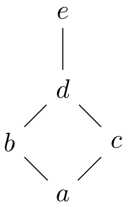

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Corollary

The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable.

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- This leaves $\bigvee B(q)$ as the only possible position for q.

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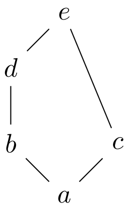
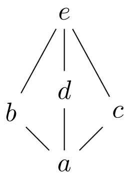
Can we embed all finite lattices in $\mathcal{D}_e$ as strong intervals?

Important test cases are N_5 and M_3:





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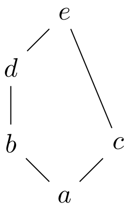

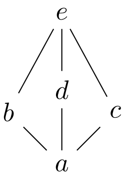

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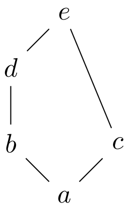

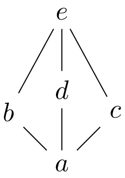

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Can we embed all countable (distributive) lattices into $\mathcal{D}_e$ as strong intervals?

Thank you!