COMPUTABILITY OF CLASSES OF COUNTABLE GRAPHS

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Of all the structures encountered in mathematics and computer science, graphs are among the most frequently used. In practice, they have proven highly effective across diverse fields like biology, the social sciences, physics, engineering, and information technology. In mathematical logic, graphs are often used to establish general results or to construct specific examples and counterexamples. They are, in a sense, the algorithmically "universal" structures: if an arbitrary structure possesses a particular computability-theoretic property, there is likely a graph that shares that same property. However, while the computability of the class of all countable graphs is well understood, much less is known about specific subclasses $\mathcal{G}$ defined by particular properties.

Given a collection $\mathcal{S}$ of finite graphs, let

$$\mathsf{Free}(\mathcal{S}) = \{\mathcal{H} : \text{no } \mathcal{G} \in \mathcal{S} \text{ is an induced subgraph of } \mathcal{H}\}$$

be the set of $\mathcal{S}$-free graphs. Such classes of graphs are well-studied in combinatorics and many natural graph classes arise in this way. For example, the bipartite graphs are the $\mathcal{S}$-free graphs where $\mathcal{S}$ is the collection of odd cycles and the chordal graphs are the $\mathcal{S}$-free graphs where $\mathcal{S}$ is the collection of cycles of length four or more.

In this talk we first survey what has been known about computability properties of graphs in general and then proceed to a restricted case of a single forbidden induced subgraph F. We will write $\mathsf{Free}(F)$ for such F-free graphs. For finite graphs, this case is well-understood with a structural dichotomy centered around the graph $\mathcal{P}_4$, the path consisting of four vertices. We expand this line of research to infinite countable graphs using the tools of computable structure theory and descriptive set theory.

As in the finite case, in the infinite case we establish $\mathsf{Free}(\mathcal{P}_4)$ as the main character of structural dichotomies for classes of graphs of the form $\mathsf{Free}(F)$. We show the following results:

Theorem 0.1. *Let $\mathcal{G} = \mathsf{Free}(F)$ for some finite graph F. Then $\mathcal{G}$ has the following properties if and only if $\mathcal{G} \subseteq \mathsf{Free}(\mathcal{P}_4)$:*

- *$\mathcal{G}$ is Σ-small, which means that there are at most countably many existential types.*
- *$\mathcal{G}$ satisfies the computable embeddability condition, which means that each existential type is computable.*
- *$\mathcal{G}$ is not on top for bi-interpretability, which means that there is a graph which is not bi-interpretable with a F-free graph. (In particular, there is an automorphism group of a graph which is not the automorphism group of a F-free graph.)*

We do find a new phenomenon for infinite graphs which does not appear for finite graphs. If a class of structures is on top for bi-interpretability, then it is on top for Borel reducibility, but not necessarily vice versa. Well-known examples of classes which are on top for Borel reducibility, but not for bi-interpretability, are trees and linear orders. We show that $\mathcal{P}_4$-free graphs can be added to this list.

Theorem 0.2. $\mathsf{Free}(\mathcal{P}_4)$ *is on top for Borel reducibility but not for effective bi-interpretability.*

In particular, out of all classes $\mathsf{Free}(F)$, only $\mathsf{Free}(\mathcal{P}_4)$ is on top for Borel reducibility but not on top for effective bi-interpretability.

This is joint work with Vittorio Cipriani, Matthew Harrison-Trainor, Liling Ko and Dino Rossegger.

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