Asymptotic Theories of Classes Defined by Forbidden Homomorphisms

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Abstract

We study the first-order almost-sure theories for classes of finite structures that are specified by homomorphically forbidding a set $\mathcal{F}$ of finite structures. If $\mathcal{F}$ consists of undirected graphs, a full description of these theories can be derived from the Kolaitis-Prömel-Rothschild theorem, which treats the special case where $\mathcal{F} = \{K_n\}$. The corresponding question for finite sets $\mathcal{F}$ of finite directed graphs is wide open. We present a full description of the almost-sure theories of classes described by homomorphically forbidding finite sets $\mathcal{F}$ of oriented trees; all of them are ω-categorical. In our proof, we establish a result of independent interest, namely that every constraint satisfaction problem for a finite digraph has first-order convergence, and that the corresponding asymptotic theory can be described as a finite linear combination of ω-categorical theories.

Keywords. Almost-sure theory, random digraph, graph homomorphism, oriented tree, constraint satisfaction problem

1 Introduction

A famous result of Glebskii, Kogan, Ligons'kii, and Talanov [9] and Fagin [7] states that the almost-sure theory $T_{\mathcal{G}}$ of the class $\mathcal{G}$ of all finite undirected graphs is *complete*, i.e., every first-order property of graphs holds asymptotically almost surely, or its negation holds asymptotically almost surely (this is sometimes referred to as a *first-order 0-1 law*). Moreover, $T_{\mathcal{G}}$ is *ω-categorical*, i.e., it has exactly one countable model up to isomorphism, the so-called *Rado graph* [16], which is of fundamental importance in model theory and can be described as the Fraïssé-limit of the class of all finite graphs [11].

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If instead of $\mathcal{G}$ we consider the class $\mathcal{G}_3$ of all finite *triangle-free* graphs, i.e., finite graphs that do not contain a K_3 subgraph, we still get a first-order 0-1 law, and the resulting almost-sure theory again has a single countably infinite model M_3. However, somewhat surprisingly, this model is *not* the Fraïssé-limit of the class of all finite undirected triangle-free graphs: Erdős, Kleitman, and Rothschild [5] proved that if n tends to infinity, almost all triangle-free graphs with n vertices are *bipartite*, i.e., have a homomorphism to K_2. Clearly, there are triangle-free graphs G without a homomorphism to K_2 (e.g., the cycle with five vertices, C_5). The structure M_3 does not contain copies of G, and hence M_3 is not the Fraïssé-limit of the class of all finite triangle-free graphs.

The mentioned result about triangle-free graphs has been generalised by Kolaitis, Prömel, and Rothschild [12] to the class of finite graphs without a copy of K_k, for every $k \geq 3$.

Theorem 1 (Kolaitis, Prömel, and Rothschild [12])**.** *For every $k \geq 3$, a random K_k-free graph is asymptotically almost surely $(k-1)$-partite.*

In fact, Kolaitis, Prömel, and Rothschild determined the almost-sure theory of the class of K_k-free graphs. In subsequent work, Prömel and Steger [15] studied classes of finite graphs that that are given by forbidding some graph H as a *(weak) subgraph*. In this article, we study graph classes that are specified by forbidding a set $\mathcal{F}$ of finite graphs *homomorphically*, rather than forbidding them as (induced or weak) subgraphs.[1] In particular, we will be interested in the almost-sure theory of the class

$$\text{Hom-Forb}(\mathcal{F}) := \{G \in \mathcal{G} \mid \text{no } F \in \mathcal{F} \text{ has a homomorphism to } G\}.$$

The difference between subgraphs and homomorphisms is irrelevant if $\mathcal{F} = \{K_n\}$, because K_n has a homomorphism to a graph G if and only if K_n is isomorphic to an (induced) subgraph of G. Quite remarkably, one can obtain a complete description of the almost-sure theory of Hom-Forb($\mathcal{F}$) for every set $\mathcal{F}$ of finite graphs (even if $\mathcal{F}$ is infinite; see Theorem 2). Let $\chi(G)$ be the *chromatic number* of G, i.e., the smallest n such that G has a homomorphism to K_n. If $\mathcal{F} = \emptyset$ then Hom-Forb($\mathcal{F}$) $= \mathcal{G}$ and the almost-sure theory is the theory of the Rado graph [16]. For non-empty $\mathcal{F}$, define $k_{\mathcal{F}} := \min_{G \in \mathcal{F}} \chi(G)$. We show that the graphs in Hom-Forb($\mathcal{F}$) are asymptotically almost surely $(k_{\mathcal{F}} - 1)$-partite, that Hom-Forb($\mathcal{F}$) satisfies a first-order 0-1 law, and that the almost-sure theory of Hom-Forb($\mathcal{F}$) is ω-categorical, i.e., has an up to isomorphism unique countable model.

The class Hom-Forb($\mathcal{F}$) has also been studied in model theory. The *age* of a relational structure H is the class of all finite structures that embed into H. Cherlin, Shelah, and Shi [2] showed that there exists an (up to isomorphism unique) countable model-complete structure $M_{\mathcal{F}}$ whose age equals Hom-Forb($\mathcal{F}$). However, if $k_{\mathcal{F}} > 2$ then Hom-Forb($\mathcal{F}$)

[1] Yet another different setting is forbidding sets of graphs as *minors*; see [10] for recent results in this setting.

contains graphs that are not $(k_{\mathcal{F}} - 1)$-colorable, and hence this model does *not* satisfy the almost-sure theory of Hom-Forb($\mathcal{F}$) (unless $\mathcal{F} = \emptyset$).

If $\mathcal{F}$ is a set of finite *directed* graphs, then much less is known about the almost-sure theory of Hom-Forb($\mathcal{F}$), even if $\mathcal{F}$ is finite. Some results have been obtained by Kühn, Osthus, Townsend, and Zhao [14] for Hom-Forb(T_3) and Hom-Forb(C_3), where T_3 is the transitive tournament with three vertices and $\vec{C}_3$ is the directed cycle with three vertices, answering questions that have been raised in model theory [3]. In particular, they proved that asymptotically almost surely the graphs in Hom-Forb(T_3) are bipartite. It is easy to come up with infinite sets $\mathcal{F}$ of finite digraphs such that Hom-Forb($\mathcal{F}$) does not have a 0-1 law.

1.1 Our Results

We show that every class of undirected graphs defined by forbidden homomorphisms has a 0-1 law and an ω-categorical almost-sure theory.

Theorem 2. *Let $\mathcal{F}$ be a non-empty set of finite undirected graphs. Then the almost-sure theory of* Hom-Forb($\mathcal{F}$) *equals the first-order theory of the generic $(k_{\mathcal{F}} - 1)$-partite graph.*

For classes of directed graphs, we give a full description of the almost-sure theories of classes of digraphs described by forbidding finitely many orientations of finite trees; again, all of these classes have a 0-1 law and an ω-categorical almost-sure theory:

Theorem 3. *Let $\mathcal{F}$ be a finite set of finite oriented trees. Then the almost-sure theory of* Hom-Forb($\mathcal{F}$) *is (pseudo-finite and) ω-categorical; In particular,* Hom-Forb($\mathcal{F}$) *has a first-order 0-1 law.*

To prove this, we show a result that is of independent interest: for every finite digraph G the CSP of G satisfies a first-order convergence law, and the asymptotic theory of the CSP is linearly composed from finitely many ω-categorical theories in a sense that will be made precise. This result and some others generalise from digraphs to general relational structures. For concreteness and clarity of presentation, we focus on the case of graphs and digraphs in this article.

1.2 Connection with Random Graphs

With the same proof technique as used for oriented graphs to prove our result, we can also prove a result about random (simple undirected) graphs with a homomorphism to a fixed finite graph, which we believe is of independent interest.

Theorem 4. *Let H be a finite simple undirected graph. Then asymptotically almost surely, a graph with vertices $\{1, \ldots, n\}$ and a homomorphism to H is ℓ-partite, where ℓ is the size of the largest clique contained in H, and each color class asymptotically almost surely has at least $\lfloor n/\ell \rfloor$ many elements.*

A weaker result (with $\lfloor n/\ell \rfloor$ replaced by $n/2\ell$) for the special case where $H = K_\ell$ has been shown by Turner [17]. Our strengthening relies on the proof technique that we use for studying the almost sure theory of CSP(D) for oriented graphs D. Our results also imply that for a graph G with vertices $\{1, \dots, n\}$ and a homomorphism to another fixed finite graph H asymptotically almost surely there exists a unique partition of the vertices into ℓ parts such that all edges of G are between distinct parts of the partition; here, ℓ is the size of the largest clique in H. This generalises a result of Dyer and Frieze [4, Theorem 3.1] from $H = K_\ell$ to arbitrary finite graphs H, and already follows from our results for undirected graphs.

1.3 Connection with Model Theory

The ω-categorical theories that arise in our main result, Theorem 3, satisfy a property that is of interest in model theory: they are *pseudo finite*, i.e., every finite subset of the theory has a finite model. An example of a pseudo-finite theory is the theory of the Rado graph, which follows from the fact that its first-order theory equals the almost-sure theory of finite undirected graphs. A class of theories whose pseudo-finiteness can be shown similarly was isolated by Kruckman [13]. Showing that a theory is pseudo finite can be quite challenging. It is a major open question whether the Fraïssé-limit of the class of all finite triangle-free graphs has a pseudo-finite theory (see, e.g., [6]). The mentioned result of Erdős, Kleitman, and Rothschild shows that this problem cannot be approached via almost sure theories of finite structures in the same way as for the Rado graph. More generally, Cherlin [3, Page 16] asked for which amalgamation classes of finite structures described by forbidding finitely many substructures have a 0-1 law, and if so, what the limit theory looks like.

1.4 Connection with Database Theory

Note that sets $\mathcal{F}$ of finite structures that are homomorphically forbidden can be interpreted from a database perspective: each structure $\mathfrak{F} \in \mathcal{F}$ can be viewed as a so-called *conjunctive query* ϕ, by turning the elements of the structure into existentially quantified variables and the relations of the structure into a conjunction of atomic formulas. Conjunctive queries are the most important class of queries encountered in database theory. It is well-known and easy to see that a finite structure $\mathfrak{B}$ (i.e., a *database*) satisfies ϕ if and only if $\mathfrak{F}$ has a homomorphism into $\mathfrak{B}$; see [1]. Results about Hom-Forb($\mathcal{F}$) can then be interpreted as statements about the uniform distribution on finite structures that do *not* satisfy the given set of conjunctive queries. An important subclass of conjunctive queries are those that are *Berge acyclic*; this is precisely the notion of acyclicity that we need for our results. For other acyclicity notions, see [8].

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