ON SEMIRINGS OF FIRST-ORDER THEORIES

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Abstract. We consider the set $T(\sigma)$ of all first-order theories in a language σ, equipped with two binary operations $\cdot$ and $\cap$, defined via direct products of models and intersection of theories, respectively. We show that the algebraic structure $\langle T(\sigma); \cdot, \cap\rangle$ is a semiring.

We study subsemirings of $\langle T(\sigma); \cdot, \cap\rangle$ defined by absorption formulas (ADF-semirings), i.e., semirings consisting of theories that are absorbed by a fixed theory. We establish some structural properties of ADF-semirings as well as axiomatic properties of theories that formula define ADF-semirings

Introduction. Let σ be a first-order language and let $T(\sigma)$ denote the set of all complete first-order theories of language σ. For structures A and B of language σ, let $A \times B$ be their direct product. We define a binary operation $\{\cdot\}$ on $T(\sigma)$ by

$$T \cdot S \;=\; \mathrm{Th}\big(\{A \times B \mid A \models T \text{ and } B \models S\}\big),$$

for any theories $T, S \in T(\sigma)$. By the well-known Feferman–Vaught Theorem [1, Theorem 5.1], this operation is well defined; that is, it does not depend on the particular choice of structures A and B realizing the theories T and S.

It is straightforward to verify that $\langle T(\sigma), \cdot\rangle$ is a commutative semigroup. In fact, $\langle T(\sigma), \cdot\rangle$ is a monoid whose identity element is the theory of a trivial structure. A subsemigroup of $\langle T(\sigma), \cdot\rangle$ is called a *semigroup of theories*.

We introduce the notion of *absorption formula definable* semigroups of theories and show that each such semigroup, when equipped with the additional binary operation $\cap$ given by set-theoretic intersection of theories, forms a commutative semiring. Such semirings are called *AFD-semirings of theories*.

We investigate structural properties of AFD-semirings and of the theories that define them.

For basic definitions and background results from model theory, semiring theory, and universal algebra used throughout this paper, we refer the reader to [2, 3, 4], respectively.

Preliminaries. For any class $\mathcal{K}$ of algebraic structures, we denote by $\mathrm{Th}(\mathcal{K})$ the set of all sentences that are true in every structure of $\mathcal{K}$. For a set of sentences Σ, we denote by $\mathrm{Mod}(\Sigma)$ the class of all structures satisfying every sentence in Σ. For brevity, we often write $T(\mathcal{K})$ instead of $\mathrm{Th}(\mathcal{K})$, and likewise $T(A)$ for $\mathrm{Th}(\{A\})$, and $\mathrm{Mod}(\varphi)$ for $\mathrm{Mod}(\{\varphi\})$.

The following lemma was established in [5, Lemma 2.2].

Lemma 1. [5] *Let* $\{T\} \cup \{T_i \mid i \in I\}$ *be a set of theories in the language* σ*. Then*

$$T \cdot \Big(\bigcap_{i\in I} T_i\Big) \;=\; \bigcap_{i\in I}(T \cdot T_i).$$

As a consequence, $\langle T(\sigma); \cdot, \cap\rangle$ is a semiring. A subsemiring of $\langle T(\sigma); \cdot, \cap\rangle$ is called a *semiring of theories*.

AFD-semirings of theories. A semigroup of theories S is called an *absorption formula definable semigroup* (AFD-semigroup) if there exists a theory $T \in T(\sigma)$ such that

$$S \;=\; \langle\{\, X \in T(\sigma) \mid X \cdot T = T \,\};\, \cdot\rangle.$$

In this event, we say that T *formula-defines* S. A theory T is called *idempotent* if $T \cdot T = T$.

Theorem 2. [6] *For every absorption formula definable semigroup of theories* S*, there exists a unique idempotent complete theory* $T \in S$ *that formula-defines* S*.*

For a nonempty set of theories S, we denote by $\mathrm{Mod}(S)$ the class of all structures A such that $A \models K$ for some $K \in S$; that is,

$$\mathrm{Mod}(S) = \bigcup\{\mathrm{Mod}(K) \mid K \in S\}.$$

Let now T be an idempotent complete theory, and let

$$S_T = \langle\{\, X \in T(\sigma) \mid X \cdot T = T \,\};\, \cdot\rangle$$

be the AFD-semigroup formula-defined by T.

Let $\{T_i \mid i \in I\} \subseteq S_T$. By Lemma 1 and the absorption condition $T \cdot T_i = T$ for all $i \in I$, we have

$$T \cdot \Big(\bigcap_{i\in I} T_i\Big) = \bigcap_{i\in I}(T \cdot T_i) = T.$$

Hence $\bigcap_{i\in I} T_i \in S_T$, so S_T is closed under $\cap$. Thus:

Proposition 3. *The structure* $\langle S_T; \cdot, \cap\rangle$ *is a semiring.*

We say that a semiring of theories is an *absorption formula definable semiring* (AFD-semiring) if its $(\cdot)$-reduct is an AFD-semigroup.

For AFD-semirings of theories we have the following.

Proposition 4. *Let* T *be a theory that formula-defines the semiring* S_T*, and let*

$$\mathcal{S}_T^+ = \{\, X \cap T \mid X \in S_T \,\} = \{\, X + T \mid X \in S_T \,\}.$$

Then $\langle \mathcal{S}_T^+; +, \cdot\rangle$ *is a semiring with zero.*

Proof. Let $X, Y \in S_T$. By Lemma 1 and the absorption property of T, we have $(X \cap T) \cdot T = X \cdot T \cap T \cdot T = T$. Thus every $A \in \mathcal{S}_T^+$ satisfies $A \cdot T = T$.

Using Lemma 1 again, we compute:

$$\begin{aligned}(X \cap T) \cdot (Y \cap T) &= ((X \cap T) \cdot Y) \cap ((X \cap T) \cdot T)\\ &= (X \cdot Y) \cap (T \cdot Y) \cap (X \cdot T) \cap (T \cdot T) = (X \cdot Y) \cap T.\end{aligned}$$

Thus $\mathcal{S}_T^+$ is closed under $\cdot$. Since $A \cdot T = T$ and $A \cap T = A$ for all $A \in \mathcal{S}_T^+$, it follows that T is a zero element in $\mathcal{S}_T^+$. □

By Theorem 2, we obtain:

Theorem 5. *For every absorption formula definable semiring of theories S, there exists a unique idempotent complete theory $T \in S$ that formula-defines S.*

Rees-like representation of AFD-semirings. A nonempty subset I of a semiring S is called an *ideal* if (i) $a, b \in I$ implies $a \cap b \in I$ (additive closure), and (ii) $s \in S$, $a \in I$ imply $s \cdot a \in I$ and $a \cdot s \in I$ (absorption).

Let I be an ideal of S. Denote by $\theta(I)$ the least congruence on S containing $I \times I$. The congruence $\theta(I)$ is called a *Rees congruence* if

$$\theta(I) = \{(x, y) \in S \times S \mid x = y \text{ or } x, y \in I\}.$$

Equivalently, $\theta(I)$ is a Rees congruence when the quotient $S/\theta(I)$ consists of the single block I together with the singleton classes $\{\{x\} \mid x \in S \setminus I\}$.

In this case the quotient semiring $S/\theta(I)$ is called the *Rees factor semiring* of S modulo I. It is isomorphic to the semiring with underlying set $(S \setminus I) \cup \{0\}$, where 0 is a new element, and the operations (denoted by the same symbols $\cdot$ and $\cap$) are defined by

$$s \cdot t = \begin{cases} s \cdot t, & \text{if } s, t, s \cdot t \notin I, \\ 0, & \text{otherwise,} \end{cases} \qquad s \cap t = \begin{cases} s \cap t, & \text{if } s, t, s \cap t \notin I, \\ 0, & \text{otherwise.} \end{cases}$$

A semiring S is called an *ideal extension* of a semiring A by a semiring B if A is an ideal of S and the Rees factor semiring $S/\theta(A)$ is isomorphic to B.

For any two theories $M, T \in T(\sigma)$, put

$$S_M = \{\, X \in T(\sigma) \mid X \cdot M = M \,\},$$

and

$$S_T(M) = \{\, X \in S_M \mid X \cdot T = T \,\}, \qquad S_T^*(M) = S_M \setminus S_T(M).$$

One easily checks that S_M is a commutative subsemigroup of $\langle T(\sigma); \cdot \rangle$, and that $S_T(M)$ is a commutative subsemigroup of S_M. Moreover, S_M is an AFD-semigroup.

Since

$$M \cdot (X \cap Y) = (M \cdot X) \cap (M \cdot Y) = M \qquad \text{for all } X, Y \in S_M,$$

and likewise

$$T \cdot (X \cap Y) = (T \cdot X) \cap (T \cdot Y) = T \qquad \text{for all } X, Y \in S_T(M),$$

it follows that S_M, together with the operation $\cap$, forms an AFD-semiring, and $S_T(M)$ forms an AFD-semiring relative to S_M.

By Theorem 5, we may assume that M is an idempotent complete theory and thus $M \in S_M$. Repeating the argument used in the proof of Theorem 2, we obtain

Theorem 6. *For any theories M, T, and AED-semiring S_M there exists an idempotent complete theory K such that*

$$S_T(M) = \{\, X \in S_M \mid X \cdot K = K \,\}.$$

Thus we may assume that T in $S_T(M)$ itself is idempotent and complete.

Theorem 7. *Let $M, T \in T(\sigma)$ be theories and suppose $S^*_T(M)$ is nonempty. Then:*

(i) *$S^*_T(M)$ is an ideal of the semiring $\langle S_M;\ \cdot, \cap\rangle$;*
(ii) *$\langle S_M;\ \cdot, \cap\rangle$ is an ideal extension of $\langle S^*_T(M); \cdot, \cap\rangle$ by the semiring $\langle S_T(M) \cup \{0\}; \cdot, \cap\rangle$.*

Proof. (*Sketch of proof.*) By Theorems 5 and 6, we may assume that both M and T are complete theories.

(i) We must show that $X \cap Y \in S^*_T(M)$ for all $X, Y \in S^*_T(M)$, and that $X \cdot Y \in S^*_T(M)$ for all $X \in S_M$ and $Y \in S^*_T(M)$.

Assume toward contradiction that $X \cap Y \notin S^*_T(M)$ for some $X, Y \in S^*_T(M)$. Then $X \cap Y \in S_T(M)$, hence $T = T \cdot (X \cap Y)$. Using Lemma 1, $T \cdot (X \cap Y) = (T \cdot X) \cap (T \cdot Y)$, and therefore $T = (T \cdot X) \cap (T \cdot Y)$. Since T is complete and $T \cdot X$, $T \cdot Y$ are consistent theories, we obtain $T = T \cdot X = T \cdot Y$. Thus $X, Y \in S_T(M)$, contradicting $X, Y \in S^*_T(M)$. Hence $X, Y \in S^*_T(M)$ implies $X \cap Y \in S^*_T(M)$. Moreover, if $X \in S^*_T(M)$ and $Y \in S_M$, then $X \cap Y \in S^*_T(M)$ by the same reasoning.

The fact that $X \cdot Y \in S^*_T(M)$ for all $X \in S_M$ and $Y \in S^*_T(M)$ is exactly statement (i) of Theorem 3.3 in [5].

(ii) Since $X \cap Y \in S^*_T(M)$ and $X \cdot Y \in S^*_T(M)$ for all $X \in S^*_T(M)$ and $Y \in S_M$, it follows that $S^*_T(M)$ is an ideal of the semiring $\langle S_M; \cdot, \cap\rangle$. Using these closure properties, one checks that the congruence $\theta(S^*_T(M))$ is a Rees congruence. Thus the Rees factor semiring $S_M/\theta(S^*_T(M))$ is isomorphic to $\langle S_T(M) \cup \{0\}; \cdot, \cap\rangle$, which establishes the ideal extension. □

Axiomatizability of semirings of theories defined by absorption formulas. Lemma 1 allows us to define a unary operation * on the semiring $\langle T(\sigma), \cdot, \cap\rangle$ by

$$S^* \ = \ \bigcap\{X \in T(\sigma) \mid S \cdot X = S \cdot S\},$$

for any $S \in T(\sigma)$.

The main result of this section is the following theorem.

Theorem 8. *Let a complete theory T formula define a semiring of theories S_T. Then:*

(1) *$\operatorname{Mod} S_T$ is axiomatizable.*
(2) *If T is finitely based, then T^* is finitely based.*
(3) *If T is a universal theory, then T^* is a strong universal Horn theory.*
(4) *If T is a positive universal theory, then T^* is a positive universal Horn theory.*
(5) *If T is an inductive theory, then T^* is an inductive theory.*

Proof. (*Sketch of proof.*) (1) By definition, $\operatorname{Mod} S_T \ = \ \bigcup\{\operatorname{Mod}(K) \mid K \in S_T\}$. One checks directly that $\operatorname{Mod}(T^*) \ \subseteq \ \bigcup\{\operatorname{Mod}(K) \mid K \in S_T\}$.

To complete the proof of items (1)–(3), consider the set S^c_T of all complete theories in S_T, that is,

$$S^c_T = \{C \in S_T \mid C \text{ is a complete theory}\}.$$

One can check that $\operatorname{Mod} S_T = \operatorname{Mod} S^c_T$.

Item (2) now follows from [6, Theorem 3.1]; item (3) from [6, Corollary 4.2]; and item (4) from [6, Theorem 4.4].

(5) We show that any directed union of structures from $\operatorname{Mod} S_T$ again belongs to $\operatorname{Mod} S_T$. Let $\{A_i \mid i \in I\}$ be a directed system of structures from $\operatorname{Mod} S_T$, and let

$$B = \bigcup_{i \in I} A_i.$$

We must prove that $B \in \operatorname{Mod} S_T$, i.e. that $T(B) \cdot T = T$. Let $A \in \operatorname{Mod} T$ and $T(A) = T$.

By definition, $T(B) \cdot T = T$ if and only if $B \times A \equiv A$.

Observe that

$$B \times A = A \times \Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} (A \times A_i).$$

The family $\{A \times A_i \mid i \in I\}$ is directed since $X \leq Y$ implies $A \times X \leq A \times Y$ for any structures X, Y.

Because $A \times A_i \in \operatorname{Mod} T$ for all $i \in I$ and T is inductive, we have

$$\bigcup_{i \in I} (A \times A_i) \in \operatorname{Mod} T.$$

Hence $B \times A \equiv A$, and therefore $T(B) \cdot T = T$. This shows that $B \in \operatorname{Mod} S_T$, completing the proof. ☐

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