Characterisation of the big Ramsey degrees of 3-uniform hypergraphs

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Abstract

We characterise the big Ramsey degrees of countable universal 3-uniform hypergraphs. This is the first example of a relational structure with a relation of arity greater than 2 where big Ramsey degrees are fully understood. The proof is self-contained, based only on the Milliken tree theorem.

1 Introduction

A (3-uniform) hypergraph $\mathbf{H}$ is a pair $\mathbf{H} = (H, E_{\mathbf{H}})$ where H is a *vertex set* and $E_{\mathbf{H}} \subseteq \binom{H}{3}$ is a set of *hyper-edges*. We consider only finite and countable (that is, countably infinite) hypergraphs and all of them are 3-uniform.

Given hypergraphs $\mathbf{H}$ and $\mathbf{H}'$, we denote by $\mathrm{Emb}(\mathbf{H}, \mathbf{H}')$ the set of all embeddings $\mathbf{H} \to \mathbf{H}'$. Given a finite hypergraph $\mathbf{A}$ and an infinite hypergraph $\mathbf{K}$, the *big Ramsey degree* of $\mathbf{A}$ in $\mathbf{K}$ is the least $T \in \omega + 1$ such that for every finite r and coloring $\chi\colon \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to r$ there exists $f \in \mathrm{Emb}(\mathbf{K}, \mathbf{K})$ such that χ attains at most T values on $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{K}) = \{f \circ g : g \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})\}$. We call a countable hypergraph $\mathbf{K}$ *universal* if every other finite or countable hypergraph has an embedding into it.

In 2022, Balko, Chodounský, Hubička, Konečný, and Vena proved:

Theorem 1.1 ([4], see also [5])**.** *Let* $\mathbf{K}$ *be a universal countable 3-uniform hypergraph and* $\mathbf{A}$ *a finite hypergraph. Then the big Ramsey degree of* $\mathbf{A}$ *in* $\mathbf{K}$ *is finite.*

We characterise these big Ramsey degrees of the universal countable 3-uniform hypergraphs. This is a contribution to an ongoing and active line of research (see [8, 9, 7] for recent surveys) which started by Devlin and Laver's characterisation of big Ramsey degrees of the order of rationals [6] (see also [11]) followed by characterisation of big Ramsey degrees of the universal countable graph (a Rado graph) [10]. Recent progress has been made primarily on big Ramsey degrees of homogeneous structures in finite binary relational languages [**?**, 2]. Typically, first

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a finite upper bound on big Ramsey degrees is introduced and later the precise numbers are specified by means of counting special tree-like structures we call *diaries*. This work is not an exception; however, for the first time we obtain a characterisation of big Ramsey degrees of a structure that is not in a binary relational language.

To state this characterisation, we need to introduce notation and several definitions. As usual, we identify every $n \in \omega$ with the set $\{0, 1, \dots, n-1\}$. Hypergraph $\mathbf{H}$ is *ordered* if $H \subseteq \omega$ (where we will use the order $<$ of ω) and *enumerated* if $H \in \omega \cup \{\omega\}$. If $\mathbf{H}$ and $\mathbf{H}'$ are ordered hypergraphs and $f \colon \mathbf{H} \to \mathbf{H}'$, we say that f is an *ordered embedding* if it is *order-preserving*, that is, for every $u, v \in H$ satisfying $u < v$ we also have $f(u) < f(v)$. We call $\mathbf{H}$ and $\mathbf{H}'$ *order-isomorphic* if there exists a surjective ordered embedding $f \colon \mathbf{H} \to \mathbf{H}'$.

Ramsey-type theorems for countable objects lead to fixing a specific enumeration. In the case of homogeneous structures, this enumeration naturally implies the following notion of a type over a finite initial part of the structure. Given an enumerated hypergraph $\mathbf{H}$, vertex $\ell \in H$, and sets $S, S' \subseteq H \setminus \ell$, we say that S and S' have the *same type* over ℓ, and write $S \simeq^\ell_{\mathbf{H}} S'$, if and only if $\mathbf{H} \restriction_{\ell \cup S}$ is order-isomorphic to $\mathbf{H} \restriction_{\ell \cup S'}$. If $S \simeq^0_{\mathbf{H}} S'$, we also put

$$S \wedge_{\mathbf{H}} S' = \max\{\ell \in H : \ell \leq \min(S \cup S') \text{ and } S \simeq^\ell_{\mathbf{H}} S'\}$$

and call it the *split* (or also the *meet*) of S and S' in $\mathbf{H}$. Given integer n, equivalence classes of $\simeq^\ell_{\mathbf{H}}$ on $\binom{H\setminus\ell}{n}$ are called *n-types* over ℓ. For brevity, given vertices $u, v \in H \setminus \ell$ we will also write $u \simeq^\ell_{\mathbf{H}} v$ instead of $\{u\} \simeq^\ell_{\mathbf{H}} \{v\}$ and $u \wedge_{\mathbf{H}} v$ instead of $\{u\} \wedge_{\mathbf{H}} \{v\}$.

We say that u is *lexicographically before* v, and write $u <^{\text{lex}}_{\mathbf{H}} v$, if either

1. $u < v$ and $u = u \wedge_{\mathbf{H}} v$, or
2. $i = u \wedge_{\mathbf{H}} v < \min(u, v)$ and $\{\{i, u\} \wedge_{\mathbf{H}} \{i, v\}, i, v\} \in E_{\mathbf{H}}$.

Example 1.2. Let $\mathbf{A}$ be a hypergraph with vertex set $A = \{0, 1, 2, 3\}$ and $E_{\mathbf{A}} = \{\{0, 1, 2\}\}$; then

1. $0 <^{\text{lex}}_{\mathbf{A}} 1 <^{\text{lex}}_{\mathbf{A}} 3 <^{\text{lex}}_{\mathbf{A}} 2$,
2. $2 \wedge_{\mathbf{A}} 3 = 1$,
3. $\{1, 2\} \wedge_{\mathbf{A}} \{1, 3\} = 0$.

All other meets are trivial.

Our characterisation is based on the following notion of diary.

Definition 1.3. A *3-uniform hypergraph diary* is an enumerated hypergraph $\mathbf{D} = (D, E_{\mathbf{D}})$ for which there exists a set $\mathsf{leaf}(\mathbf{D}) \subseteq D \setminus \{0\}$ such that every vertex $\ell \in D$ satisfies precisely one of the following roles:

(R1) **leaf**: $\ell \in \mathsf{leaf}(\mathbf{D})$.

(R2) **1-type split**: There exist $u, v \in \mathsf{leaf}(\mathbf{D}) \setminus \ell$ such that $\ell = u \wedge_{\mathbf{D}} v$. Moreover:

- (a) every hyper-edge containing ℓ contains also 0 and some vertex in $\mathsf{leaf}(\mathbf{D}) \setminus \ell$, and,
- (b) if $\{0, \ell, v\}, \{0, \ell, v'\} \in E_{\mathbf{D}}$ then $v \simeq^\ell_{\mathbf{D}} v'$.

(R3) **2-type split**: $\ell = 0$ or there exist $S, S' \in \binom{\mathsf{leaf}(\mathbf{D})\setminus\ell}{2}$ such that $\ell = S \wedge_{\mathbf{D}} S'$. Moreover:

- (a) every hyper-edge containing ℓ contains also two vertices in $\mathsf{leaf}(\mathbf{D}) \setminus \ell$, and,

(b) if $\{\ell, u_0, u_1\}, \{\ell, v_0, v_1\} \in E_{\mathbf{D}}$ then $\{u_0, u_1\} \simeq^{\ell}_{\mathbf{D}} \{v_0, v_1\}$.

Moreover, the following conditions are satisfied:

(S1) For every $u, v \in \mathsf{leaf}(\mathbf{D})$ it holds that $u \wedge_{\mathbf{D}} v$ is a 1-type split.

(S2) For every $S, S' \in \binom{\mathsf{leaf}(\mathbf{D})}{2}$ either $S \wedge_{\mathbf{D}} S' = \min S = \min S'$, or $S \wedge_{\mathbf{D}} S'$ is a 2-type split.

(S3) For every $u < v \in \mathsf{leaf}(\mathbf{D})$ we have $\{0, u, v\} \in E_{\mathbf{D}}$ if and only if $u <^{\mathrm{lex}}_{\mathbf{D}} v$.

Notice that the set $\mathsf{leaf}(\mathbf{D})$ is uniquely defined. Let $\mathbf{A}$ be an ordered hypergraph. If $\mathbf{D}$ is a diary such that $\mathbf{D} \restriction_{\mathsf{leaf}(\mathbf{D})}$ is order-isomorphic to $\mathbf{A}$, then we call $\mathbf{D}$ a *diary of* $\mathbf{A}$. Our main result can be stated compactly as:

Theorem 1.4. *Let* $\mathbf{K}$ *be a universal countable 3-uniform hypergraph and* $\mathbf{A}$ *a finite 3-uniform hypergraph. Then the big Ramsey degree of* $\mathbf{A}$ *in* $\mathbf{K}$ *is the number of diaries of* $\mathbf{A}$ *multiplied by the size of the automorphism group of* $\mathbf{A}$*.*

It is interesting to observe hypergraph diaries serve as a direct extension of LSV-trees (the diaries of graphs) [10, 9]. An LSV-tree is defined analogously but consists of only two level types: leaf and split (where a split is a 1-type split). This simplification arises because, in a binary language, 2-types are fully determined by their constituent 1-types.

Observe that every vertex v of an enumerated 3-uniform hypergraph $\mathbf{K}$ gives rise to a graph $\mathbf{G}_v$ with vertex set v, where vertices $u, u' \in v$ form an edge if and only if $\{u, u', v\}$ forms a hyper-edge of $\mathbf{K}$. A hypergraph diary encodes LSV-trees of all graphs $\mathbf{G}_v$, $v \in \omega$. Split in LSV-tree then corresponds to 2-type splitting in a diary of $\mathbf{K}$. 1-type splits then represent splits between different LSV-trees of different graphs represented in $\mathbf{K}$.

2 Type-respecting embeddings and Ramsey-type theorems for hypergraphs

While existing upper-bound results are usually formulated in terms of trees or coding trees, they can be, equivalently, seen as Ramsey-type theorems for enumerated structures with a special kind of embeddings we call *type-respecting*. In the case of hypergraphs, these embeddings can be compactly defined as follows (for a more general definition, see the announcement [1]):

Definition 2.1 (Type-respecting embedding)**.** Given ordered hypergraphs $\mathbf{H}$ and $\mathbf{H}'$, we call an embedding $f\colon \mathbf{H} \to \mathbf{H}'$ *type-respecting* if it is order-preserving and for every $S, S' \subseteq H$ with $|S| = |S'| \leq 2$ it holds that $f(S \wedge_{\mathbf{H}} S') = f[S] \wedge_{\mathbf{H}'} f[S']$.

Given enumerated hypergraphs $\mathbf{H}$ and $\mathbf{H}'$ and $n < |H|$, we denote by $\mathrm{TREmb}_n(\mathbf{H}, \mathbf{H}')$ the set of all type-respecting embeddings that are the identity when restricted to n. The corresponding upper-bound Ramsey-type theorem (which we believe is of independent interest) can be formulated as follows:

Theorem 2.2. *Let* $\mathbf{G}$ *be an enumerated universal hypergraph. Then for every finite enumerated hypergraph* $\mathbf{A}$*, every* $n < |A|$ *and every coloring*

$$\chi : \mathrm{TREmb}_n(\mathbf{A}, \mathbf{G}) \to 2$$

there exists $f \in \mathrm{TREmb}_n(\mathbf{G}, \mathbf{G})$ *so that* χ *is constant on*

$$f \circ \mathrm{TREmb}_n(\mathbf{A}, \mathbf{G}).$$

This theorem can be obtained from a product form of the Milliken-tree theorem using similar method as in [4, 5]. However, as formulated here Theorem 2.2 is stronger than what directly follows form the main results of these papers.

3 Main ingredients of the proof of optimality

After establishing the existence of a diary for every 3-uniform hypergraph, we use Theorem 2.2 to obtain a Ramsey-type theorem for diaries equipped with a more restricted form of type-respecting embeddings.

Definition 3.1 (Diary-embedding)**.** Given diaries $\mathbf{D}$ and $\mathbf{E}$, we call an embedding $f\colon \mathbf{D}\to\mathbf{E}$ a *diary-embedding* if it is order-preserving, $f(0)=0$, $f[\mathsf{leaf}(\mathbf{D})]\subseteq\mathsf{leaf}(\mathbf{E})$ and for every $S,S'\subseteq \mathsf{leaf}(\mathbf{D})$ with $|S|=|S'|\leq 2$ it holds that $f(S\wedge_{\mathbf{D}} S')=f[S]\wedge_{\mathbf{E}} f[S']$.

Given diares $\mathbf{D}$ and $\mathbf{E}$, we write $\mathrm{DEmb}(\mathbf{D},\mathbf{E})$ for the set of all diary-embeddings $\mathbf{D}\to\mathbf{E}$.

Theorem 3.2 (Diaries are Ramsey)**.** *Let $\mathbf{E}$ be a diary such that $\mathbf{E}\restriction_{\mathsf{leaf}(\mathbf{E})}$ is a universal hypergraph, and let $\mathbf{D}$ be a finite diary. Then for every 2-coloring $\chi\colon \mathrm{DEmb}(\mathbf{D},\mathbf{E})\to 2$ there exists a diary embedding $f\colon \mathbf{E}\to\mathbf{E}$ such that χ is constant on*

$$f\circ \mathrm{DEmb}(\mathbf{D},\mathbf{E}).$$

This results in a refined upper bound that is in fact optimal in the following sense.

Theorem 3.3 (Diaries are unavoidable)**.** *Let $\mathbf{D}$ and $\mathbf{E}$ be diaries, $\mathbf{K}$ a universal hypergraph, and $f\colon \mathbf{K}\to\mathbf{E}\restriction_{\mathsf{leaf}(\mathbf{E})}$ an embedding. Then there exists a diary-embedding $g\colon \mathbf{D}\to\mathbf{E}$ such that $g[\mathsf{leaf}(\mathbf{D})]\subseteq f[H]$.*

Given an enumerated hypergraph $\mathbf{H}$, vertex $\ell\in H$ and set $S\subseteq H\setminus\ell$, we denote by $\mathbf{Env}^{\mathbf{H}}_{\ell}(S)$ its *minimal envelope at level ℓ*. This is the smallest enumerated hypergraph for which there exists $f\in \mathrm{TREmb}_{\ell}(\mathbf{Env}^{\mathbf{H}}_{\ell}(S),\mathbf{H})$ such that $S\subseteq f[\mathrm{Env}^{\mathbf{H}}_{\ell}(S)]$. Envelopes can be colored and Theorem 2.2 can be applied. The main step in proving Theorem 3.3 is the iterated use of Theorem 2.2 which gives the following.

Lemma 3.4 (Type stabilization lemma)**.** *Let $\mathbf{G}$ be an enumerated generic (universal and homogeneous) hypergraph and $\mathbf{K}$ an enumerated hypergraph. For every embedding $f:\mathbf{G}\to\mathbf{K}$ there exists an embedding $g:\mathbf{G}\to\mathbf{G}$ such that*

(I1) *$f\circ g$ is an order-preserving embedding $\mathbf{G}\to\mathbf{K}$.*

(I2) *For every $\ell\in G\setminus\{0\}$, $u,v\in G\setminus\ell$,*

$$u\simeq^{\ell}_{\mathbf{G}} v \implies f(g(u))\simeq^{f(g(\ell-1))+1}_{\mathbf{K}} f(g(v)).$$

(I3) *For every $\ell\in G\setminus\{0\}$, $S=\{u_0,u_1\},S'=\{v_0,v_1\}\in\binom{G\setminus\ell}{2}$ with $u_0<u_1$ and $v_0<v_1$, if*

$$\mathbf{Env}^{\mathbf{G}}_{\ell}(S)=\mathbf{Env}^{\mathbf{G}}_{\ell}(S') \text{ and } \left|\mathrm{Env}^{\mathbf{G}}_{\ell}(S)\right|\leq \ell+3$$

then

$$f[g[S]]\simeq^{f(g(\ell-1))+1}_{\mathbf{K}} f[g[S']]$$

and

$$f(g(u_0))<^{\mathrm{lex}}_{\mathbf{K}} f(g(u_1)) \iff f(g(v_0))<^{\mathrm{lex}}_{\mathbf{K}} f(g(v_1)).$$

By this lemma, one can find for every embedding $f:\mathbf{K}\to\mathbf{K}$ an embedding $g:\mathbf{K}\to\mathbf{K}$ such that $f\circ g$ is close to being type-respecting. By careful analysis of the structure of the diaries and their interplay with envelopes, the unavoidability follows.

4 Future directions

The techniques presented extend naturally to arity higher than 3 as well as to random relational structures in finite relational languages. The main structure of the proof, by defining the upper-bound theorem on (a restriction of) type-respecting embeddings, showing that diaries are Ramsey, and obtaining the lower bound by iterating the upper bound, has been recently applied for partial orders [2]. With a more general tree-type theorem, such as [3], we expect this to lead to a general method of characterising big Ramsey degrees which also yields a systematic approach to Ramsey expansions. However, there are multiple open problems related to situations where upper bound theorem for type-respecting embeddings does not give envelopes of bounded size; perhaps the most striking one is the question whether a homogeneous 3-uniform hypergraph omitting an irreducible hypergraph with 4 vertices and 3 edges has finite big Ramsey degrees. See [8, 9] for lists of open problems.

References

[1] Andrés Aranda, Samuel Braunfeld, David Chodounský, Jan Hubička, Matěj Konečný, Jaroslav Nešetřil, and Andy Zucker. Type-respecting amalgamation and big Ramsey degrees. In Daniel Král̆ and Jaroslav Nešetřil, editors, *Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications EUROCOMB'23*, pages 57–65. MUNI Press, 2023.

[2] Martin Balko, David Chodounský, Natasha Dobrinen, Jan Hubička, Matěj Konečný, Lluís Vena, and Andy Zucker. Characterisation of the big Ramsey degrees of the generic partial order. *Transactions of the American Mathematical Society*, 2025.

[3] Martin Balko, David Chodounský, Natasha Dobrinen, Jan Hubička, Jaroslav Nešetřil, Matěj Konečný, and Andy Zucker. Ramsey theorem for trees with successor operation. Preprint, arXiv:2311.06872, 2023.

[4] Martin Balko, David Chodounský, Jan Hubička, Matěj Konečný, and Lluís Vena. Big Ramsey degrees of 3-uniform hypergraphs are finite. *Combinatorica*, 42(2):659–672, 2022.

[5] Samuel Braunfeld, David Chodounský, Noé de Rancourt, Jan Hubička, Jamal Kawach, and Matěj Konečný. Big Ramsey degrees and infinite languages. *Advances in Combinatorics*, 2024:4, 2024. 26pp.

[6] Denis Devlin. *Some partition theorems and ultrafilters on ω*. PhD thesis, Dartmouth College, 1979.

[7] Natasha Dobrinen. Ramsey theory of homogeneous structures: current trends and open problems. In *International Congress of Mathematicians*, pages 1462–1486. EMS Press, 2023.

[8] Jan Hubička and Matěj Konečný. Twenty years of Nešetřil's classification programme of Ramsey classes. *Computer Science Review*, 59:100814, 2026.

[9] Jan Hubička and Andy Zucker. A survey on big Ramsey structures. *Zbornik Radova (Beograd)*, 22(30):317–355, 2025. Selected topics in combinatorial analysis II.

[10] Claude Laflamme, Norbert W. Sauer, and Vojkan Vuksanovic. Canonical partitions of universal structures. *Combinatorica*, 26(2):183–205, 2006.

[11] Stevo Todorcevic. *Introduction to Ramsey spaces*, volume 174. Princeton University Press, 2010.