Model Constructions for $\mathbf{GBL}_{ewf}$

Andrew Lewis-Smith[0000−0001−9020−4055] and Zhiguang Zhao[0000−0001−5637−945X]

[1] Middlesex University, United Kingdom
[2] Taishan University, Tai'an, China

Abstract.

We develop model-theoretic constructions for fuzzy logics characterised by the poset product construction [11], particularly Generalised Basic Logic with Exchange, Weakening and Ex Falso Quodlibet ($\mathbf{GBL}_{ewf}$) and extensions, such as Hajek's Basic Logic (**BL**). This theory therefore generalises standard constructions from modal and intuitionistic logic to the fuzzy case. We build upon insights from [5,12,13,14,15,17].

1 Introduction

Hajek's basic logic (**BL**) and its constructive variant generalised basic logic (**GBL**) occupy a central place in contemporary research on fuzzy and substructural logics. **BL** is primarily studied algebraically. This is only natural: the logic is strongly algebraizable [8] and is the logic of t-norms [2]. **BL**, and therefore $\mathbf{GBL}_{ewf}$, can be seen as the 'logical core' of many of the mainstream fuzzy logics such as Łukasiewicz, Gödel or Product logic, each of which arise from **BL** by addition of an axiom to the standard Hilbert-sytle presentation [7].

The papers of [5,9,10,12,13] suggest an alternative view of the situation: fuzzy logics like $\mathbf{GBL}_{ewf}$ (i.e. **BL** sans pre-linearity) can be endowed with relational semantics extracted from poset product representations of the corresponding algebraic theories [13]. This suggests an analogy with classical normal modal and intuitionistic logics. One might ask whether this analogy extends to the classical modal model-theoretic constructions: It does.

Extending insights from recent work [5,11,13,14,15,16,17], the present paper gives fuzzy analogues of classic model-theoretic constructions for relational semantics, derived from poset product representations for $\mathbf{GBL}_{ewf}$; the relational semantics for the latter is presented [13] and some extensions (and subsystems) together with their Kripke semantics have also been considered: see $\mathbf{GBL}_{\mathbf{WEM}}$ [14], **BL** [12], Monadic **BL** [16], and **MTL** [17]. To our knowledge, this paper represents *the first attempt to develop model theory for fuzzy logics (with* $\mathbf{GBL}_{ewf}$ *as base system) along the lines of the canonical constructions from classical modal logic.*

Our goals here are several. First, there is the question of how far the analogy with classical modal logic extends. The answer (already anticipated in works such as [5,6], dealing with the question of frames and model completeness and modal translations respectively) may be surprising: Many-valued or fuzzy logics have been studied for over a century, but almost never systematically via model theory in the Tarskian or Kripkean (or really any) sense[3].

The present paper (and recent work cited above) allows this possibility: For example, Wesley Fussner has already precisely delineated the relationship between frame conditions and axiomatic extensions of $\mathbf{GBL}_{ewf}$ with conditions on the poset products of **MV**-chains, further mapping these into the substructural hierarchy of [1], and there's already an interesting premise for different translations (modal and temporal) [6]. But having canonical constructions available for fuzzy logics helps us to build structures that (we hope) *de-emphasise the algebraic technology*, or at least brings fuzzy logics (including and extending $\mathbf{GBL}_{ewf}$) closer to standard modal logics (with all the interconnections). This also helps to bring some sense of further unity to fuzzy logic research, in that ad hoc algebraic constructions for particular algebras or logics can be united, under the right perspective, via the structural abstraction of Kripkean model theory. Having analogues of canonical constructions helps towards this end.

Kripke models are *applicable*, bringing connections to automata and labelled transition systems, labelled calculi and semantic tableaux, as well as decidability and of course *other logical systems* characterised by Kripke models (e.g. modal logics, non-classical logics, etc.).

Traditionally, Kripke structures make for intuitive models and countermodels that serve as alternatives to e.g. matrices, and it is interesting to note that historically the algebraic models in modal logic appeared first, followed by topological models, with Kripke structures appearing some time later. This occasioned a large growth in the modal logic literature with the rich model theory for modal logics we have today. We hope that having these constructions available puts fuzzy logic on serious footing from a model-theoretic perspective, suggesting further completeness proofs, counterexamples, as well as results of a pure model-theoretic flavour familiar from first-order or modal logic.

Our paper proceeds as follows. We present $\mathbf{GBL}_{ewf}$'s Hilbert system, followed by suitable definitions of algebras, validity, and a *precis* of the relational semantics (as given in [13]). We then introduce the basic constructions with counterparts in standard modal logic, modifying according to our semantics. We

[3] Unless by 'model theory' we mean universal algebra, then fuzzy logic has been studied extensively in the algebraic literature. But this seems contentious overall to most logicians. We are not sure that classical model theorists working on stability theory or modal logicians working on Sahqvist theory *view themselves* as algebraists.

discuss bisimulation and bounded-morphisms in this setting, followed by disjoint unions, generated submodels, and tree-unfolding. We then present a generalisation of a construction introduced in [3] building top-models from standard models. We end with a new proof of soundness and completeness for $\mathbf{GBL}_{ewf}$ with respect to tree-structures.

2 Preliminaries

In this section, we provide preliminaries for the logic of commutative, integral, divisible and bounded residuated lattices i.e. the logic of $\mathbf{GBL}_{ewf}$. For further details, see [13].

2.1 $\mathbf{GBL}_{ewf}$

The formulas of the logic of commutative, integral, bounded and divisible residuated lattices aka $\mathbf{GBL}_{ewf}$ (also Intuitionistic Łukasiewicz logic: again, see [13] for some detailed discussion[4] are defined as follows:

$$\phi := p|\bot|\phi \vee \phi|\phi \wedge \phi|\phi \otimes \phi|\phi \to \phi$$

We use $\mathcal{L}_{\otimes}$ to denote the language and $\mathsf{Form}(\mathcal{L}_{\otimes})$ to denote the set of all $\mathcal{L}_{\otimes}$-formulas. We use $\neg\phi$ to denote $\phi \to \bot$ and $\top$ to denote $\bot \to \bot$.

We present the Hilbert system for intuitionistic Łukasiewicz logic:

(A1) $\phi \to \phi$
(A2) $(\phi \to \psi) \to ((\psi \to \chi) \to (\phi \to \chi))$
(A3) $(\phi \otimes \psi) \to (\psi \otimes \phi)$
(A4) $(\phi \otimes \psi) \to \psi$
(A5) $(\phi \to (\psi \to \chi)) \to ((\phi \otimes \psi) \to \chi))$
(A6) $((\phi \otimes \psi) \to \chi)) \to (\phi \to (\psi \to \chi))$
(A7) $(\phi \otimes (\phi \to \psi)) \to (\phi \wedge \psi)$
(A8) $(\phi \wedge \psi) \to (\phi \otimes (\phi \to \psi))$
(A9) $(\phi \wedge \psi) \to (\psi \wedge \phi)$
(A10) $\phi \to (\phi \vee \psi)$
(A11) $\psi \to (\phi \vee \psi)$
(A12) $((\phi \to \psi) \wedge (\chi \to \psi)) \to ((\phi \vee \chi) \to \psi)$
(A13) $\bot \to \phi$
(MP) From $\phi, \phi \to \psi$ infer ψ.

We use $\vdash \phi$ to denote that ϕ is provable in this system.

[4] 'IntuitionisticŁukasiewicz logic' refers to the natural deduction presentation of $\mathbf{GBL}_{ewf}$, which is presented here in Hilbert-style: the equivalence between these two is shown in [13].

2.2 Algebraic Semantics

We situate the algebraic semantics characterising $\mathbf{GBL}_{ewf}$ and extensions in terms of the theory of residuated lattices.

Definition 21 (Commutative Residuated Lattice) *$\mathcal{A} = \langle A, \wedge, \vee, \otimes, 1, \rightarrow\rangle$ is called a* commutative residuated lattice *if*

- *$\langle A, \wedge, \vee, \otimes, 1\rangle$ is a commutative lattice-ordered monoid.*
- *$x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$.*

Definition 22 ($\mathbf{GBL}_{ewf}$-algebras and MV-algebras) *A* $\mathbf{GBL}_{ewf}$-algebra *is a commutative residuated lattice satisfying the following properties:*

- *Bounded from below by $\bot$: i.e. $\bot \leq x$ for all $x \in A$.*
- *Integral: 1 is the top element of the lattice, i.e. $x \leq 1$ for all $x \in A$. In this case we also denote 1 by $\top$.*
- *Divisibility property: if $x \leq y$ then $y \otimes (y \rightarrow x) = x$. This condition is equivalent to requiring that the residuated lattice satisfy the equation $x \otimes (x \rightarrow y) = y \otimes (y \rightarrow x) = x \wedge y$.*

An $\mathbf{MV}$-algebra is a $\mathbf{GBL}_{ewf}$-algebra satisfying the following properties:

- *Involutive: $\neg\neg x = x$.*
- *Pre-linearity $(x \rightarrow y) \vee (y \rightarrow x) = 1$.*

2.3 Semantics for $\mathbf{GBL}_{ewf}$

In this section, we review the semantics for $\mathbf{GBL}_{ewf}$, alias Intuitionistic Łukasiewicz logic. To interpret the classical Łukasiewicz logic, MV-algebras are enough. To interpret intuitionistic Łukasiewicz logic, we need an additional partial order such that each propositional variable is interpreted as the following *sloping function*:

Definition 23 (Sloping function) *Let $\mathcal{W} = \langle W, \preceq\rangle$ be a poset, and let $\{\mathcal{A}_w : w \in W\}$ be a collection of $\mathbf{MV}$-algebras indexed by the poset W. We assume that all $\mathcal{A}_w$ share the same top element $\top$ and bottom element $\bot$, and the rest elements are disjoint. We call $\mathcal{W} = \langle W, \preceq\rangle$ labelled with $\{\mathcal{A}_w : w \in W\}$* Kripke frames*. We use $v \prec w$ to indicate that $v \preceq w$ and $v \neq w$. A function $f : W \rightarrow \bigcup\{\mathcal{A}_w : w \in W\}$ is a* sloping function *if the following conditions are satisfied:*

- *f is* monotone*: for all $w, v \in W$, $v \preceq w$ implies $f(v) \leq f(w)$.*
- *For all $w \in W$, $f(w) > \bot$ implies $\forall v \succ w (f(v) = \top)$.*

The idea behind sloping functions is like the upsets in the Kripke semantics of intuitionistic propositional logic. Indeed, an upset of $\mathcal{W} = \langle W, \preceq\rangle$ is a subset $X \subseteq W$ such that for any $w \in X$ and $v \in W$, $w \preceq v$ implies $v \in X$, which corresponds to a sloping function $f_X : W \rightarrow \bigcup\{\mathcal{A}_w : w \in W\}$ such that $f_X(w) = 1$ if $w \in X$ and $f_X(w) = 0$ if $w \notin X$. In this sense, sloping functions generalize upsets in the semantics of intuitionistic Łukasiewicz logic.

Remark 1. The definition implies that if $f : W \to \bigcup\{\mathcal{A}_w : w \in W\}$ is a sloping function and $f(w) < \top$ then $\forall v \prec w(f(v) = \bot)$. That is, along any increasing chain $w_1 \prec w_2 \prec \ldots \prec w_n$, there can only be at most one point w_i such that $\bot < f(w_i) < \top$, and for $j < i$ we must have $f(w_j) = \bot$, and for $j > i$ we must have $f(w_j) = \top$.

Lemma 1. *If $f, g : W \to \bigcup\{\mathcal{A}_w : w \in W\}$ are sloping functions, then so are $f \wedge g$, $f \vee g$ and $f \otimes g$:*

$$\begin{aligned} (f \wedge g)(w) &:= f(w) \wedge_{A_w} g(w) \\ (f \vee g)(w) &:= f(w) \vee_{A_w} g(w) \\ (f \otimes g)(w) &:= f(w) \otimes_{A_w} g(w). \end{aligned}$$

Proof. See [13, Lemma 3.2].

Definition 24 (Bova-Montagna structure) *A* Bova-Montagna structure *(*BM-*structure for short) is a pair $\mathcal{M} = \langle \mathcal{W}, V \rangle$ where $\mathcal{W} = \langle W, \preceq \rangle$ is a Kripke frame, and for any propositional variable p, the function $\lambda w.V(p, w) : W \to \bigcup\{\mathcal{A}_w : w \in W\}$ is a sloping function.*

Definition 25 *Given a function $f : W \to \bigcup\{\mathcal{A}_w : w \in W\}$, we define the function $\lfloor \inf \rfloor_{v \succeq w} f(v) : W \to \bigcup\{\mathcal{A}_w : w \in W\}$ as follows:*

$$\lfloor \inf \rfloor_{v \succeq w} f(v) := \begin{cases} f(w) & \text{if } \forall v \succ w(f(v) = \top) \\ \bot & \text{if } \exists v \succ w(f(v) < \top). \end{cases}$$

It is shown in [13, Lemma 3.5] that $\lfloor \inf \rfloor_{v \succeq w} f(v)$ is also a sloping function.

Definition 26 (Kripke semantics for $\mathcal{L}_\otimes$) *Given a BM-structure $\mathcal{M} = \langle \mathcal{W}, V \rangle$, the valuation function $V(p, w)$ on propositional variables p can be extended to all $\mathcal{L}_\otimes$-formulas as:*

$$\begin{aligned} V(\bot, w) &:= \bot \\ V(\phi \wedge \psi, w) &:= V(\phi, w) \wedge_{\mathcal{A}_w} V(\psi, w) \\ V(\phi \vee \psi, w) &:= V(\phi, w) \vee_{\mathcal{A}_w} V(\psi, w) \\ V(\phi \otimes \psi, w) &:= V(\phi, w) \otimes_{\mathcal{A}_w} V(\psi, w) \\ V(\phi \to \psi, w) &:= \lfloor \inf \rfloor_{v \succeq w} (V(\phi, v) \to V(\psi, v)). \end{aligned}$$

Lemma 2. *For any formula ϕ the function $\lambda w.V(\phi, w) : W \to \bigcup\{\mathcal{A}_w : w \in W\}$ is a sloping function.*

Proof. See [13, Lemma 3.7].

Definition 27 *Let $\Gamma = \psi_1, \ldots, \psi_n$. We say that:*

- *ϕ holds in a BM-structure $\mathcal{M}$ (notation: $\mathcal{M} \Vdash \phi$) if $V(\phi, w) = \top$ for all $w \in W$.*
- *ϕ is valid on a poset $\mathcal{W} = \langle W, \preceq \rangle$ (notation: $\mathcal{W} \Vdash \phi$) if $\mathcal{M} \Vdash \phi$ for all BM-structures based on $\mathcal{W}$.*

– *ϕ is valid (notation: $\Vdash \phi$), if $\mathcal{W} \Vdash \phi$ for all posets $\mathcal{W}$.*

We have the following soundness and completeness theorem for Intuitionistic Łukasiewicz logic:

Theorem 28 *$\vdash \phi$ iff $\Vdash \phi$.*

Proof. See [13, Theorems 3.13 and 3.14].

3 Model Constructions

In this section, we study some model constructions in $\mathbf{GBL}_{ewf}$, which are counterparts of the constructions in modal logic.

3.1 Bisimulation

Definition 31 *Let $\mathcal{M} = \langle W, \preceq, V\rangle$ and $\mathcal{M}' = \langle W', \preceq', V'\rangle$ be two* BM*-structures. A bisimulation between $\mathcal{M}, \mathcal{M}'$ is a non-empty relation $Z \subseteq W \times W'$ satisfying the following conditions for any wZw':*

– *Atom equivalence: If wZw' then $\mathcal{A}_w = \mathcal{A}_{w'}$ and $V(w, p) = V'(w', p)$ for all atomic propositions p, hereafter denoted by $w \simeq w$.*
– *Forth: If $w \prec u$ then there exists a $u' \in W'$ such that $w' \prec u'$ and uZu'.*
– *Back: If $w' \prec' u'$ then there exists a $u \in W$ such that $w \prec u$ and uZu'.*

We use $Z : \mathcal{M} \rightleftarrows \mathcal{M}'$ to denote that Z is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.

Two pointed BM*-structures $\langle\mathcal{M}, w\rangle$ and $\langle\mathcal{M}, w'\rangle$ are bisimilar or bisimulation equivalent, denoted $\langle\mathcal{M}, w\rangle \rightleftarrows \langle\mathcal{M}', w'\rangle$, if there is a bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that wZw'.*

Theorem 32 *If $\langle\mathcal{M}, w\rangle \rightleftarrows \langle\mathcal{M}', w'\rangle$, then $V(w, \phi) = V'(w', \phi)$.*

Proof. We prove by induction on the complexity of ϕ. For the atomic case as well as the $\wedge, \vee, \otimes$ cases, the proof is trivial. Now we consider the case where $\phi := \psi \to \chi$.

Consider $\langle\mathcal{M}, w\rangle$. We discuss in two cases:

– If $V(v, \psi) \leq V(v, \chi)$ for all $v \succ w$, then $V(w, \psi \to \chi) = V(w, \psi) \to V(w, \chi)$. By induction hypothesis, $V(w, \psi) = V'(w', \psi)$ and $V(w, \chi) = V'(w', \chi)$. To show that $V(w, \psi \to \chi) = V'(w', \psi \to \chi)$, it suffices to show that $V'(v', \psi) \leq V'(v', \chi)$ for all $v' \succ' w'$. Indeed, for all $v' \succ' w'$ there is a $v \succ w$ such that vZv', so $V'(v', \psi) = V(v, \psi) \leq V(v, \chi) = V'(v', \chi)$. Therefore we have $V(w, \psi \to \chi) = V'(w', \psi \to \chi)$.
– If there is a $v \succ w$ such that $V(v, \psi) > V(v, \chi)$, then $V(w, \psi \to \chi) = \bot$. Since for w' there is a $v' \succ' w$ such that vZv', we have $V'(v', \psi) = V(v, \psi) > V(v, \chi) = V'(v', \chi)$, therefore $V'(w', \psi \to \chi) = \bot = V(w, \psi \to \chi)$.

3.2 Bounded Morphisms

Definition 33 *Let $\mathcal{M} = \langle W, \preceq, V\rangle$ and $\mathcal{M}' = \langle W', \preceq', V'\rangle$ be* BM*-structures. A function $f : W \to W'$ is a bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$ if the following conditions are satisfied:*

- *Atom equivalence: $\mathcal{A}_w = \mathcal{A}_{f(w)}$ and $V(w, p) = V'(f(w), p)$ for all atomic propositions p.*
- *Forth: If $w \prec u$ then $f(w) \prec f(u)$.*
- *Back: If $f(w) \prec' u'$ then there exists a $u \in W$ such that $w \prec u$ and $f(u) = u'$.*
- *Antichain condition: If $X' \subseteq W'$ is an antichain, then $f^{-1}[X'] \subseteq W$ is also an antichain.*

Bounded morphisms between frames are similarly defined, by dropping the condition that $V(w, p) = V'(f(w), p)$ for all atomic propositions p. If f is surjective, then $\mathcal{M}'$ is a *bounded morphic image* of $\mathcal{M}$ (and similarly for frames).

Theorem 34 *Let $\mathcal{M} = (W, \preceq, V)$ and $\mathcal{M}' = (W', \preceq', V')$ be* BM*-structures and $f : \mathcal{M} \to \mathcal{M}'$ be a bounded morphism. Then:*

1. *For all $w \in W$, $V(w, \phi) = V'(f(w), \phi)$.*
2. *If f is surjective, then $\mathcal{M} \Vdash^{\mathrm{BM}} \phi$ implies $\mathcal{M}' \Vdash^{\mathrm{BM}} \phi$.*
3. *If f is surjective, then $\langle W, \preceq\rangle \Vdash^{\mathrm{BM}} \phi$ implies $\langle W', \preceq'\rangle \Vdash^{\mathrm{BM}} \phi$.*

Proof. For 1, we prove by induction on the formula structure of ϕ, which is similar to the proof of Theorem 32. Item 2 follows immediately from item 1. For item 3, suppose that $\langle W', \preceq'\rangle \nVdash^{\mathrm{BM}} \phi$, then there is a valuation V'_0 on $\langle W', \preceq'\rangle$ such that $V'_0(w', \phi) < \top$ for some $w' \in W'$. Then we can define a valuation V_0 on $\langle W, \preceq\rangle$ such that $V_0(w, p) = V'_0(f(w), p)$ for all atomic propositions p. Then by the antichain condition, we have that the valuation V_0 is well-defined. By item 1, we have that $V_0(w, \phi) < \top$ for some $w \in W$ such that $f(w) = w'$.

3.3 Disjoint-Unions

Definition 35 *We define the disjoint union of an arbitrary family W_i for $i \in I$ of (not necessarily disjoint) sets as $\uplus_{i\in I} W_i := \bigcup_{w\in W}(W_i \times \{i\})$. We have the natural embedding $\epsilon_i : W_i \to \uplus_{i\in I} W_i$ such that $\epsilon_i(w) = (w, i)$.*

Definition 36 *Consider a family of Kripke frames $\{\mathcal{W}_i = \langle W_i, \preceq_i\rangle\}_{i\in I}$ and a family of* BM*-structures $\{\mathcal{M}_i = \langle \mathcal{W}_i, V_i\rangle\}_{i\in I}$ over these, with each $w \in W_i$ labelled with an algebra $\mathcal{A}_w$.*

1. *The disjoint union of $\{\mathcal{W}_i\}_{i\in I}$ is the frame $\uplus_{i\in I}\mathcal{W}_i = \langle \uplus_{i\in I} W_i, \preceq\rangle$, where $(w_0, i_0) \preceq (w_1, i_1)$ iff $i_0 = i_1 = i$ and $w_0 \preceq w_1$, and each (w, i) is labelled with the algebra $\mathcal{A}_w$ for $w \in W_i$.*
2. *The disjoint union of $\{\mathcal{M}_i\}_{i\in I}$ is the* BM*-structure $\uplus_{i\in I}\mathcal{M}_i = \langle \uplus_{i\in I}\mathcal{W}^i, V\rangle$ where $V(p, (w, i)) = V_i(p, w)$ for all $w \in W_i$ and all atomic propositions.*

Proposition 37 *Given a family of frames* $\mathcal{W}_i = \langle W_i, \preceq_i \rangle$ *(for* $i \in I$*), a family of* BM*-structures* $\mathcal{M}_i = \langle \mathcal{W}_i, V_i \rangle$ *(for* $i \in I$*), we have:*

1. *For every* $i \in I$ *and* $w \in W_i$*,* $V_i(w, \phi) = V((w, i), \phi)$*.*
2. $\uplus_{i \in I} \mathcal{M}_i \Vdash^{\mathrm{BM}} \phi$ *iff for every* $i \in I$*,* $\mathcal{M}_i \Vdash^{\mathrm{BM}} \phi$*.*
3. $\uplus_{i \in I} \mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$ *iff for every* $i \in I$*,* $\mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$*.*

Proof. 1. For item 1, just notice that $Z_i := \{(w, (w, i)) \mid w \in W_i\}$ forms a bisimulation between $\mathcal{M}_i$ and $\uplus_{i \in I} \mathcal{M}_i$.

2. For item 2, it follows from item 1.
3. For item 3, for the left to right direction, to show that $\mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$ for every $i \in I$, consider any valuation V_i on $\mathcal{W}_i$, define the valuation V on $\uplus_{i \in I} \mathcal{W}_i$ such that $V(p, (w, i)) = V_i(p, w)$ for all $w \in W_i$ and $V(p, w') = \top$ for all $w' \notin W_i$. Then it is easy to see that V is a well-defined valuation on $\uplus_{i \in I} \mathcal{W}_i$ and $Z := \{(w, (w, i)) \mid w \in W_i\}$ forms a bisimulation between $\langle \mathcal{W}_i, V_i \rangle$ and $\langle \uplus_{i \in I} \mathcal{W}_i, V \rangle$, therefore from Theorem 32 and $\uplus_{i \in I} \mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$ we have that $\mathcal{W}_i, V_i \Vdash^{\mathrm{BM}} \phi$, therefore $\mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$.
   For the other direction, suppose $\mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$ for every $i \in I$. Consider any valuation V on $\uplus_{i \in I} \mathcal{W}_i$, by defining $V_i(w, p) := V((w, i), p)$ for all $i \in I$, $w \in W_i$ and atomic propositions p, we can get well-defined valuations V_i on $\mathcal{W}_i$ and $Z := \{(w, (w, i)) \mid w \in W_i\}$ forms a bisimulation between $\langle \mathcal{W}_i, V_i \rangle$ and $\langle \uplus_{i \in I} \mathcal{W}_i, V \rangle$, therefore from Theorem 32 and $\mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$ we have that $(\uplus_{i \in I} \mathcal{W}_i, V, (w, i) \Vdash^{\mathrm{BM}} \phi) = \top$, therefore $\uplus_{i \in I} \mathcal{W}_i \Vdash^{\mathrm{BM}} \phi$.

3.4 Generated Submodels

If $\preceq \subseteq W \times W$ is any binary relation over W, and $W' \subseteq W$, we write $\preceq\restriction_{W'} := \preceq \cap (W' \times W')$ for the restriction of $\preceq$ to W'. Similarly for a valuation V on W, $V \restriction_{W'}$ stands for its restriction to W'.

Definition 38 *Let* $\mathcal{W} = \langle W, \preceq \rangle$ *be a frame,* $\mathcal{M} = \langle \mathcal{W}, V \rangle$ *a* BM*-structure over* $\mathcal{W}$*, and* $W' \subseteq W$*.*

1. $\mathcal{W}' = \langle W', \preceq\restriction_{W'} \rangle$ *is a* generated subframe *of* $\mathcal{W}$ *over* W' *(notation:* $\mathcal{W}' = \mathcal{W} \restriction_{W'}$*), if* W' *is upward closed, i.e.* $w \in W'$ *and* $w \preceq u$ *implies* $u \in W'$*.*
2. $\mathcal{M}' = \langle \mathcal{W} \restriction_{W'}, V \restriction_{W'} \rangle$ *is a* generated substructure *of* $\mathcal{M}$ *over* W' *(notation:* $\mathcal{M}' = \mathcal{M} \restriction_{W'}$*), if* $\mathcal{W}'$ *is a generated subframe of* $\mathcal{W}$ *over* W'*.*

Proposition 39 *Let* $\mathcal{M} = \langle W, \preceq \rangle$ *be a* BM*-structure and* $\mathcal{M}' = \mathcal{M} \restriction_{W'}$ *be the generated substructure of* $\mathcal{M}$*.*

1. *For every* $w \in W'$*,* $V(w, \phi) = V \restriction_{W'} (w, \phi)$*.*
2. *If* $\mathcal{M} \Vdash^{\mathrm{BM}} \phi$ *then* $\mathcal{M}' \Vdash^{\mathrm{BM}} \phi$*.*
3. *If* $\langle W, \preceq \rangle \Vdash^{\mathrm{BM}} \phi$ *then* $\langle W', \preceq\restriction_{W'} \rangle \Vdash^{\mathrm{BM}} \phi$*.*

Proof. 1. For item 1, the proof is similar to Proposition 37, just notice that $Z := \{(w, w) \mid w \in W'\}$ forms a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.

2. For item 2, it follows immediately from item 1.

3. For item 3, suppose that $\langle W, \preceq\rangle \Vdash^{\mathrm{BM}} \phi$ and consider any valuation V' on $\langle W', \preceq\restriction_{W'}\rangle \Vdash^{\mathrm{BM}} \phi$. By defining V on $\langle W, \preceq\rangle$ such that $V(w,p) = V'(w,p)$ for all $w \in W'$ and $V(w,p) = \top$ for all $w \notin W'$, we have that $Z := \{(w,w) \mid w \in W'\}$ forms a bisimulation between $\langle W, \preceq, V\rangle$ and $\langle W', \preceq\restriction_{W'}, V'\rangle$. Therefore from $\langle W, \preceq\rangle \Vdash^{\mathrm{BM}} \phi$ we get $\langle W', \preceq\restriction_{W'}, V'\rangle \Vdash^{\mathrm{BM}} \phi$. Thus $\langle W', \preceq\restriction_{W'}\rangle \Vdash^{\mathrm{BM}} \phi$.

3.5 Tree-unravelling

Definition 310 (Tree-unravelling of frames) *We define* $W[u] := \{(u = w_0, w_1, \ldots, w_n) \mid n \in \mathbb{N}, w_i \in W$ *and* $w_i \preceq w_j$ *for* $i < j\}$. *We define the tree-unravelling of the Kripke frame* $\mathcal{W} = \langle W, \preceq\rangle$ *at* $u \in W$ *as* $\mathcal{W}[u] := \langle W[u], \preceq'\rangle$ *with root* $u = (u)$, $\preceq' := \{(\alpha, \beta) \mid \alpha$ *is an initial fragment of* $\beta\}$ *and* $\mathcal{A}_{(w_0,\ldots,w_n)} = \mathcal{A}_{w_n}$.

Definition 311 (Tree-unravelling of models) *The tree unravelling of* $\mathcal{M} = \langle W, \preceq, V\rangle$ *from* $u \in W$ *is the rooted Kripke structure* $\mathcal{M}[u] := \langle W[u], \preceq', V'\rangle$ *with root* $u = (u)$, *where* $\preceq' := \{(\alpha, \beta) \mid \alpha$ *is an initial fragment of* $\beta\}$, $V'((w_0, \ldots, w_n), p) = V(w_n, p)$ *and* $\mathcal{A}_{(w_0,\ldots,w_n)} = \mathcal{A}_{w_n}$.

Corollary 312 *Given any pointed* BM*-structure* $\langle \mathcal{M}, w\rangle$, *we have that* $V(w, \phi) = V'(w, \phi)$ *where* V' *is the valuation for* $\mathcal{M}[w]$.

Proof. It suffices to see that $f : W[w] \to W$ where $f(w, w_1, \ldots, w_n) = w_n$ is a bounded morphism.

3.6 Adding Top Node

Definition 313 (Positive formula) *A formula in the language* $\mathcal{L}_\otimes$ *is a positive formula, if it is built from atomic formulas including* $\top$ *and excluding* $\bot$ *by applying* $\wedge, \vee, \otimes$ *and* $\to$.

Definition 314 (Top-model) *Given any BM-structure* $\mathcal{M} = \langle \mathcal{W}, V\rangle$, *its corresponding top-model* $\mathcal{M}^+ = \langle \mathcal{W}^+, V'\rangle$ *is obtained by adding a top node in* $\mathcal{W}$ *where all propositional variables are assigned value* $\top$ *there.*

Lemma 3. *Let* t *be the top of any top model* $\mathcal{M}^+$, *and let* ϕ *be a positive formula without free variables. Then* $t \Vdash^{\mathrm{BM}} \phi$.

Proof. By induction on the length of ϕ.

Proposition 315 *Every positive formula as well as* $\bot$ *is invariant under the top-model construction, i.e. for all* $w \in W$, $V'(w, \phi) = V(w, \phi)$.

Proof. We prove by induction on the complexity of ϕ. For the basic cases and the cases of $\wedge, \vee, \otimes$ the proof is easy. We only consider the $\to$-case where $\phi = \psi \to \chi$.

– If for all $w \prec v \in W$ we have $V(v, \psi) \leq V(v, \chi)$, then by induction hypothesis, $V'(v, \psi) \leq V'(v, \chi)$ for all $w \prec v \in W$ and $V'(t, \psi) = V'(t, \chi) = \top$, therefore $V(w, \phi) = V(w, \psi) \to V(w, \chi) = V'(w, \psi) \to V'(w, \chi) = V'(w, \phi)$.
– If there is a successor $v \succ w$ in W such that $V(v, \psi) > V(v, \chi)$, then by induction hypothesis, $V'(v, \psi) > V'(v, \chi)$, so both $V(v, \psi \to \chi)$ and $V'(v, \psi \to \chi)$ are equal to $\bot$.

Proposition 316 *For any $\mathcal{L}_\otimes$-formula ϕ, there is a procedure transforming ϕ into a positive formula ϕ^+ or $\phi^+ = \bot$ such that ϕ is equivalent to ϕ^+ in all top-models.*

Proof. We construct the formula ϕ^+ using the following algorithm:

1. We first remove all occurrences of $\top$ and $\bot$ using the following equivalences:
   – Remove $\bot$: $\bot \wedge \phi \equiv \phi \wedge \bot \equiv \bot$, $\bot \otimes \phi \equiv \phi \otimes \bot \equiv \bot$, $\bot \vee \phi \equiv \phi \vee \bot \equiv \phi$, $\bot \to \phi \equiv \top$, $\phi \to \bot \equiv \neg\phi$, $\neg\bot \equiv \top$.
   – Remove $\top$: $\top \wedge \phi \equiv \phi \wedge \top \equiv \phi$, $\top \otimes \phi \equiv \phi \otimes \top \equiv \phi$, $\top \vee \phi \equiv \phi \vee \top \equiv \top$, $\top \to \phi \equiv \phi$, $\phi \to \top \equiv \top$, $\neg\top \equiv \bot$.

   If the resulting formula is $\top$ or $\bot$ or a formula without $\top, \bot, \neg$, then we define ϕ^+ to be the resulting formula. Otherwise, we get a formula ψ which does not contain any $\top$ or $\bot$ but contains at least one $\neg$, and go to the next stage. Notice that all the equivalences hold in all models, therefore also in all top-models.
2. Since ψ does not contain any $\top$ or $\bot$ but contains at least one $\neg$, then we consider the first occurrence of $\neg$ in ψ such that $\neg\theta$ is a subformula of ψ and θ is positive. Then we replace $\neg\theta$ by $\bot$ and get the formula γ.
   We can show that ψ is equivalent to γ in all top-models: indeed, it suffices to show that $\neg\theta$ is equivalent to $\bot$ in all top-models. Indeed, $V'(w, \neg\theta) = \bot = V'(w, \bot)$ because there is a $t \succeq w$ such that $V'(t, \theta) = \top > V'(t, \bot)$.
   Then we go to the previous stage and work on the new formula γ.

Since the algorithm reduces the number of symbols, it will terminate and output a formula either positive or is $\bot$ itself.

Proposition 317 *For any $\mathcal{L}_\otimes$-formula ϕ, it is invariant under the top-model construction iff it is equivalent to ϕ^+.*

Proof. For formulas ϕ equivalent to ϕ^+, it is invariant under the top-model construction since positive formulas and $\bot$ are invariant under top-model construction. If a formula ϕ is invariant under top-model construction, then $V(w, \phi) = V'(w, \phi) = V'(w, \phi^+) = V(w, \phi^+)$.

Lemma 4. *If $\vdash_{\mathbf{GBL}_{ewf}} \phi \to \psi$, then $\vdash_{\mathbf{GBL}_{ewf}} \phi^+ \to \psi^+$.*

Proof. We use completeness with respect to relational semantics. If $\nvdash_{\mathbf{GBL}_{ewf}}$ $\phi^+ \to \psi^+$, then there is a pointed BM-structure $(\mathcal{M}, w \Vdash^{\mathrm{BM}} \phi^+ \to \psi^+) < \top$. Then by the top-model invariance we have $V(w, \phi^+ \to \psi^+) = V'(w, \phi^+ \to \psi^+) = V'(w, \phi \to \psi)$, refuting $\phi \to \psi$, therefore $\nvdash_{\mathbf{GBL}_{ewf}} \phi \to \psi$.

Theorem 318 *If ϕ is positive, then $\vdash_{\mathbf{GBL}_{ewf}} \phi \Leftrightarrow \vdash_{\mathbf{GBL_{WEM}}} \phi$, where $\mathbf{GBL_{WEM}}$ is $\mathbf{GBL}_{ewf}$ extended with the weak exluded middle $\neg\phi \vee \neg\neg\phi$.*

Proof. For the left-to-right direction, it is trivial. For the other direction, notice that from the construction of ϕ^+, if we have a formula $\phi(\psi_1/p_1, \ldots, \psi_n/p_n)$, then $(\phi(\psi_1/p_1, \ldots, \psi_n/p_n))^+ = \phi^+(\psi_1^+/p_1, \ldots, \psi_n^+/p_n)$. Therefore, consider any derivation of ϕ in $\mathbf{GBL_{WEM}}$, we have instances of the weak excluded middle $\psi_1, \ldots, \psi_m$ such that $\vdash_{\mathbf{GBL}_{ewf}} \psi_1 \wedge \ldots \wedge \psi_m \to \phi$. By the previous lemma, we have $\vdash_{\mathbf{GBL}_{ewf}} (\psi_1 \wedge \ldots \wedge \psi_m)^+ \to \phi^+$, while ψ_i is of the form $\neg\theta \vee \neg\neg\theta$, therefore ψ_i^+ is equivalent to $\bot \vee \neg\bot$ which is equivalent to $\top$, so we have $\vdash_{\mathbf{GBL}_{ewf}} \phi^+$, which is $\vdash_{\mathbf{GBL}_{ewf}} \phi$.

4 Another Completeness Argument for $\mathbf{GBL}_{ewf}$

In the preceding sections, we have reviewed various model-constructions for $\mathbf{GBL}_{ewf}$. Now we are in a position to give a new completeness proof for $\mathbf{GBL}_{ewf}$, employing bisimilarity (as is done in classical modal logic).

Theorem 41 (Adequacy for Tree-structures) *The logic $\mathbf{GBL}_{ewf}$ is sound and complete with respect to tree-shaped Kripke frames.*

Proof. Soundness is easy to see. For completeness, suppose that ϕ is not provable in $\mathbf{GBL}_{ewf}$. Then ϕ fails in a $\mathbf{GBL}_{ewf}$-algebra $\mathcal{A}$ by algebraic completeness, and since every $\mathbf{GBL}_{ewf}$-algebra is embeddable into a poset product $\mathcal{B}$ whose factors are **MV**-chains, we have that $\mathcal{B} \nVdash^{\mathrm{BM}} \phi$. By Proposition 3.12 [13], we have that there is a BM-structure $\mathcal{M}^{\mathcal{A}}$ such that $\mathcal{M}^{\mathcal{A}}, w \nVdash^{\mathrm{BM}} \phi$. Now we can take the generated submodel at w', $\mathcal{M}' = \mathcal{M}^{\mathcal{A}} \restriction_{W'}$ be the generated substructure of $\mathcal{M}$. We unravel this submodel $\mathcal{M}'$ to a tree model $\mathcal{M}'[w']$. This model $\mathcal{M}'[w']$ is such that $V(w, \phi) = V'(w', \phi)$, where V' is the valuation for $\mathcal{M}'[w']$ i.e. $\mathcal{M}'[w']$ will be bisimilar to $\mathcal{M}$, the original pointed model, so $\mathcal{M}'[w'] \nVdash^{\mathrm{BM}} \phi$.

5 Conclusion

In the preceding sections we have developed fuzzy analogues of standard model constructions from modal logic, appropriate for $\mathbf{GBL}_{ewf}$ and extensions thereof, along with a "top model construction" generalised from Jankov's logic (sometimes denoted **KC**) as considered in [3]. We have also given purely semantic argument for adquacy based on the constructions given above.

There are various possibilities for future research. One direction converts the model constructions above into games of various types. Such games could help to identify the algorithmic content of model constructions in the fuzzy arena, analogous to game-based semantic constructions in the intermediate, constructive and modal logic literature. We also imagine *bisimulation games* [21,4,18,20,22] for $\mathbf{GBL}_{ewf}$ and related systems (especially **BL**) could be useful for identifying suitable fuzzy automata [23,19] or transition systems.

Another possibility is to pursue model constructions found in the fuzzy logic literature that do not have obvious analogues in modal logic. These betray algebraic origin. For example, one might consider *ordinal sum-based* constructions – perhaps there are model constructions that can involve merging different algebras of interest (e.g. two different kinds of residuated lattice). This direction is promising, as there is no shortage of ordinal sum representations and embedding results in fuzzy logic with exotic presentations (occasions where e.g. L-groups and MV-chains interact in a poset structure). But converting these same into relational semantics and seeking poset product alternatives where they exist [16,17], and finally comparing such constructions for a range of systems may inspire novel model constructions quite different from what we see in the foregoing. Thus fuzzy logics may have a richer variety of examples in Kripke structures, departing from standard constructions originating in modal logic.

References

1. A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory: Hypersequents and hypercompletions. *Annals of Pure and Applied Logic*, 168(3):693 – 737, 2017.
2. A. Ciabattoni and F. Montagna. Proof theory for locally finite many-valued logics: Semi-projective logics. *Theoretical computer science*, 480:26–42, 04 2013.
3. D. de Jongh and Z. Zhao. Positive formulas in intuitionistic and minimal logic. In M. Aher, D. Hole, E. Jerábek, and C. Kupke, editors, *Logic, Language, and Computation-10th International Tbilisi Symposium on Logic, Language, and Computation, TbiLLC 2013*, volume 8984 of *LNCS*, pages 175–189. Springer, 2013.
4. T.-F. Fan. Fuzzy bisimulation for gödel modal logic. *IEEE Transactions on Fuzzy Systems*, 23(6):2387–2396, 2015.
5. W. Fussner. Poset products as relational models. *Studia Logica*, 110, June 2021.
6. W. Fussner and W. Zuluaga Botero. Some modal and temporal translations of generalized basic logic. In U. Fahrenberg, M. Gehrke, L. Santocanale, and M. Winter, editors, *Relational and Algebraic Methods in Computer Science*, pages 176–191, Cham, 2021. Springer International Publishing.
7. P. Hájek. Basic fuzzy logic and BL-algebras. *Soft Computing*, 2(3):124–128, Sep 1998.
8. P. Hajek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 1998.
9. P. Jipsen and F. Montagna. On the structure of generalized BL-algebras. *algebra universalis*, 55(2):227–238, Aug 2006.
10. P. Jipsen and F. Montagna. The blok–ferreirim theorem for normal GBL-algebras and its application. *Algebra universalis*, 60(4):381–404, May 2009.
11. P. Jipsen and F. Montagna. Embedding theorems for classes of GBL-algebras. *Journal of Pure and Applied Algebra*, 214(9):1559 – 1575, 2010.
12. A. Lewis-Smith. A Kripke Semantics for Hajek's BL. *arXiv e-prints*, pages arXiv–2308, 2023.
13. A. Lewis-Smith, P. Oliva, and E. Robinson. Kripke semantics for intuitionistic Łukasiewicz logic. *Studia Logica*, 109:313–339, 2021.
14. A. Lewis-Smith and Z. Zhao. A Kripke Semantics for Intuitionistic Łukasiewicz Logic with Weak Excluded Middle. *Submitted*, 2025.

15. A. Lewis-Smith and Z. Zhao. Correspondence Theory for Intuitionistic Łukasiewicz Logic. *Submitted*, 2025.
16. A. Lewis-Smith and Z. Zhao. A Kripke Semantics for Monadic BL Chains. *Submitted*, 2025.
17. A. Lewis-Smith and Z. Zhao. Two Kripke semantics for MTL. *Submitted*, 2025.
18. L. A. Nguyen. Logical characterizations of fuzzy bisimulations in fuzzy modal logics over residuated lattices. *Fuzzy Sets and Systems*, 431:70–93, 2022. Logic and Related Topics.
19. L. A. Nguyen. Fuzzy simulations and bisimulations between fuzzy automata. *International Journal of Approximate Reasoning*, 155:113–131, 2023.
20. L. A. Nguyen, I. Micić, N.-T. Nguyen, and S. S. and. Depth-bounded fuzzy bisimulation for fuzzy modal logic. *Cybernetics and Systems*, 0(0):1–18, 2024.
21. S. Qiao, P. Zhu, and W. Pedrycz. Approximate bisimulations for fuzzy-transition systems. *Fuzzy Sets and Systems*, 472:108533, 2023.
22. M. S. Stanković. Bisimulations for kripke models of fuzzy multimodal logics. *Faculty of Science and Mathematics,University of Nis*, 2022.
23. M. Ćirić, J. Ignjatović, N. Damljanović, and M. Bašić. Bisimulations for fuzzy automata. *Fuzzy Sets and Systems*, 186(1):100–139, 2012. Theme: Logic and computer science.