"*The Art of Using Symbols Under Regulations*": Formalism, Mechanization, and the 19th-Century Transformation of Logic

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Abstract

The 19th-century mathematization of logic was, to a large extent, driven by philosophical responses to foundational concerns within mathematical practice. These concerns stemmed from the algebraic methods that were widely employed by Continental mathematicians, prompting philosophical attempts to justify these new techniques on formalist and psychologistic grounds. In this paper, I examine these attempts in the thought of two pivotal British mathematicians of the time, namely, George Peacock and Augustus De Morgan. This analysis offers insight into how philosophical reflection on algebraic practice contributed to the transformation of logic from a philosophical into a mathematical discipline, while also laying some groundwork for its subsequent mechanization.

1 Introduction

The 19^{th} century saw the transformation of logic from a primarily philosophical discipline, concerned with traditional syllogistic forms, into an essentially mathematical discipline, concerned with symbolic representations and formal methods of argumentation. Initiated in Great Britain by G. Boole and A. De Morgan, this gradual mathematization of logic was expanded by W.S. Jevons and J. Venn before spreading to Germany and North America via the works of E. Schröder and C.S. Peirce.[1] The aim of this paper is to bring attention to the fact that this transformation of logic was largely due to a formalist standpoint —strongly reminiscent of Hilbert's formalism almost 100 years later— that was put forward as a response to specific foundational concerns of the time.

The story, in broad brushstrokes, goes something like this. (a) Ingenious new methods and proof techniques in mathematics led to groundbreaking results, raising at the same time suspicions about the legitimacy and trustworthiness of both (methods and results). (b) A formalist standpoint was put forward by some thinkers as a foundational thesis to alleviate epistemic worries and ground the legitimacy of the strange new "tricks." The new formalist standpoint aimed to enhance rigor by means of emphasizing the *formal* (symbolic) aspect of mathematical arguments as opposed to their *contentual* aspect (meanings); thereby trying to secure against any mistaken steps or tacit assumptions sneaking in by overly relying on meanings or intuitions. (c) This shift in how certain reasoning moves are justified — i.e., based on symbolic form and independently of their semantic content— also provided fertile ground for the development of a further mechanistic/computational thinking within *mathematical* theories, in the sense of treating derivations as symbolic manipulations. (d) Additionally, this emphasis on symbolic manipulation had also great implications for the development of *logic* itself, for the rules governing these symbolic manipulations could be seen as emerging from a more fundamental and universal language (a content-free calculus, so to speak, whose rules are grounded in self-evident immediate moves).

Now, to the reader who is informed about the logical developments of the 20^{th} century, some echoes of the above story may sound familiar. For example, in relation to (a), one could think of *Cantor's results about infinite sets*; in relation to (b), *Hilbert's program*; in relation to (c), the development of *proof theory*, *recursion theory*, *computational mathematics*; in relation to (d), the further development

[1] Though, Schröder had started logical investigations from an abstract algebraic perspective independently and much earlier; see Peckhaus (2004).

of logical concepts (such as "recursive functions", "Turing machines", etc.) as well as of metatheoretical investigations about logical systems and axiomatics (e.g., *consistency, completeness, soundness*, etc.); i.e., what Dawson (2007) has aptly called "classical logic's coming of age."

Notwithstanding these well-known 20$^{\text{th}}$-century developments, the 19$^{\text{th}}$-century mathematization of logic emerged from some remarkably analogous concerns. Just as Hilbert advanced his formalist philosophy to "establish once and for all the certitude of mathematical methods" (Hilbert, 1926, 184) given the appearance of the set-theoretic paradoxes which threatened "even the most ordinary and fruitful concepts and the simplest and most important deductive methods of mathematics" (p.190[2]), almost 100 years earlier another formalist philosophy had been put forward. This time the epistemic dangers emanated from the new algebraic techniques employed by French and German analysts in the 18$^{\text{th}}$ century. For example, the application of operations like addition and multiplication to "strange new objects" such as $\sqrt{-1}$ or $\frac{d}{dx}$ or even subtracting a larger number from a lesser one had seemed preposterous to a number of British authors at the time.[3] And while again the new methods and results were much promising and desirable, a better foundation was needed to secure against "the extraordinary vagueness of the reasoning which [was] employed to establish these theorems"[4] (Peacock, 1834, 193). This created the sense of a need for better foundations and the British mathematicians did show sufficient sensitivity to such epistemological issues. Their foundational responses took, broadly speaking, two forms (often adopted simultaneously by some authors): one was to adopt and promote a formalist outlook on algebraic mathematics, while another was to retreat to some kind of psychological justifications that could ground the legitimacy of the new methods in fundamental operations of the mind ("laws of thought").

One of the first people to explicitly[5] put forward a formalist outlook as a direct response to these developments was George Peacock, a British algebraist and cofounder of the Analytical Society (in 1812, together with Charles Babbage and John Herschel). Peackock's formalism was not a "program", in any sense reminiscent of Hilbert's formalist program. Rather, it was an attempt to make explicit an emerging new outlook on the relation between mathematical symbols and mathematical objects that had already been in the air and tacitly assumed in practice. By expressing the implicit practice in the form of concrete *principles*, Peacock could show that there was nothing mysterious about the increasingly widening practice of manipulating symbols in ways that are not authorized by their original meanings; in fact, this practice could be well-defined, well-understood, and justified. Peacock's crystallization of formalist practice into concrete principles served as a crucial catalyst for the later

[2]Page numbers refer to the reprint by Benacerraf and Putnam (1984).

[3]Here are two striking examples of the prevalent attitude in Great Britain at the time. Francis Maseres wrote in his *Dissertation on The Use of The Negation Sign in Algebra* (1758):

> A single quantity can never be marked with either of these signs, or considered as either affirmative or negative; for if any single quantity, as b, is marked either with the sign $+$ or with the sign $-$ without affecting some other quantity, as a, to which it is to be added, or from which it is to be subtracted, the mark will have no meaning or signification: thus if it be said that the square of -5, or the product of -5 into -5, is equal to $+25$, such an assertion must either signify no more than 5 times 5 is equal to 25, without any regard to signs, or it must be mere nonsense or unintelligible jargon.

And this is William Frend, in the preface of his two-volume treatise on *The Principles of Algebra* (1796-9):

> A number may be greater or less than another number... You may put a mark before one, which it will obey; it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing, of multiplying a negative number into a negative number, and thus producing a positive number, of a number being imaginary ... they talk of solving an equation which requires two impossible roots to make it soluble: they can find out some impossible numbers, which being multiplied together produce unity.
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> From the age of Vieta ... to this of Maseres, ... many men of the greatest abilities have employed themselves in the pursuit of an idle hypothesis, and have laid down rules not founded in truth, nor of any sort of use in a science admitting in every step of the plainest principles of reasoning. ... Thus volumes upon volumes have been written on the stupid dreams of Athanasius, and on the impossible roots of an equation of n dimensions.

Both quotes can be found in Ewald (1996, 317-8).

[4]Peacock's comment here was a response to some passages in Cauchy (1821).

[5]Berkeley (1732) had also defended some kind of formalism with respect to reasoning and language in general.

mathematical turn in logic.

Let us then go back to the previous schematic story and fill in the details for the analogous 19^{th}-century developments. The radical new methods in relation to (a) refer to the use of algebraic techniques, employed in works like Laplace, Lagrange, Cauchy et al. The formalist attitude, in reference to (b), was explicitly articulated by Peacock, although, early traces of it can already be found in works like Berkeley (1732), Lambert (1786), and Servois (1814). In reference to (c), the implications of the formalist turn for *mathematics* can be discerned in the increased employment of symbolic methods for the solution of equations in the (mathematical) works of Gregory, De Morgan, Boole, and others. Finally, in reference to (d) and the implications of the same formalist turn for *logic* can be clearly seen in the radical transformation of logic into a symbolic discipline by the (logical) works of De Morgan, Boole, Jevons, Venn, MacFarlane, Bain, Huntington, and others. In the remaining of this paper, we will fill in some of the details to this story.

2 From Peacock to De Morgan, and from *arithmetical* to *logical* algebra.

2.1 Peacock's formalism.

In trying to provide continental analysis with a rigorous justification, Peacock proposed a distinction between an *arithmetical algebra* (or *universal arithmetic*) and a *symbolical algebra*.

> There are, in fact, two distinct sciences, *arithmetical* and *symbolical algebra*, which are closely connected with each other ... The first ... would be, properly speaking, *universal arithmetic*: its general symbols would represent numbers only; its fundamental operations, and the signs used to denote them, would have the same meaning as in common arithmetic; it would reject the *independent* use of the signs + and −, though it would recognise the common rules for their incorporation, when they were preceded by other quantities or symbols: the operation of subtraction would be *impossible* when the subtrahend was greater than the quantity from which it was required to be taken, and therefore the proper *impossible* quantities of such a science would be the *negative* quantities of *symbolical* algebra; it would reject also the consideration of the multiple values of simple roots, as well as of the negative and impossible roots of equations of the second and higher degree: it is this species of algebra which alone can be legitimately founded upon arithmetic as its basis. (Peacock, 1834, 189; emphasis in original)

For example, expressions like '-3' or '$2 - 5$' would be meaningless within universal arithmetic, whereas they would be proper meaningful expressions (denoting a *negative number*) within symbolical algebra; and the same would hold (e.g.) for the expression denoting the complex root of a quadratic equation.

What is the relation between *arithmetical* and *symbolical algebra*? Says Peacock (1834):

> I know that it is the opinion of many persons, even amongst the masters of algebraical science, that arithmetic does supply a sufficient basis for symbolical algebra considered under its most general form ... In reply ... it ought to be remarked ... that there is nothing in the nature of the symbols of algebra which can essentially confine or limit their signification or value; that it is an abuse of the term generalization to apply it to designate the process of mind by which we pass from the meaning of $a - b$, when a is greater than b, to its meaning when a is less than b ...
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> But though the science of arithmetic, or of arithmetical algebra, does not furnish an adequate foundation for the science of symbolical algebra, it necessarily *suggests* its principles, or rather its laws of combination; for in as much as symbolical algebra, though arbitrary in the authority of its principles, is not arbitrary in their application, being required to include arithmetical algebra as well as other sciences, it is evident that their rules must be

> identical with each other, as far as those sciences proceed together in common: the real distinction between them will arise from the *supposition or assumption that the symbols in symbolical algebra are perfectly general and unlimited both in value and representation, and that the operations to which they are subject are equally general likewise.* (pp.192-5; emphasis in original)

What about the principles of symbolical algebra?

> It is more natural and philosophical ... to assume such principles as independent and *ultimate*, as far as the science itself is concerned, in whatever manner they may have been suggested, so that it may thus become essentially a science of symbols and their combinations, constructed upon its own rules, which may be applied to arithmetic and to all other sciences by interpretation: by this means, interpretation will *follow*, and not *precede*, the operations of algebra and their results...
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> In arithmetical algebra, the definitions of the operations determine the rules; in symbolical algebra, the rules determine the meaning of the operations, or more properly speaking, they furnish the means of interpreting them: but the rules of the former science are invariably the same as those of the latter, in as much as the rules of the latter are assumed with this view, and merely differ from the former in the universality of their applications ... We call those rules, or their equivalent symbolical consequences, assumptions, in as much as they are not deducible as conclusions from any previous knowledge of those operations which have corresponding names: and we might call them arbitrary assumptions, in as much as they are arbitrarily imposed upon a science of symbols and their combinations, which might be adapted to any other assumed system of consistent rules.

Although there definitely exist non-trivial differences, this distinction between a core part of algebra that "can be legitimately founded upon arithmetic as its basis" and another one that is "essentially a science of symbols and their combinations, constructed upon its own rules, which may be applied to arithmetic ... by interpretation" is very much reminiscent of Hilbert's distinction between contentual (*real*) mathematics and (symbolic) metamathematics.[6]

2.2 De Morgan's combination of formalism with psychologism.

In traditional historiography of logic,[7] De Morgan is almost exclusively mentioned for his *direct* contributions to logic; for instance, his efforts to go beyond the simple subject-predicate form and investigate a logic of relations and of quantified predicates, his introduction of the notion of *universe of discourse*, and his broader works on syllogisms (see De Morgan 1847 as well as his series of articles "On the Syllogism", collected in De Morgan 1966). But, as I argue, De Morgan also exerted a hitherto underappreciated yet crucial *indirect* influence on logic's transformation into a mathematical discipline. This influence operated through his active participation in the foundational debates of the time, his adoption of Peacock's formalist philosophy, and his integration of that framework with a psychologistic account of symbolic operations as fundamental operations of the mind (hence, as primarily logical). I further argue that this integration itself exerted an underappreciated influence on the development of recursion theory.

Initially, De Morgan, too, was suspicious of the unrestricted use of negative and imaginary numbers. In 1831, he would write in a textbook aimed for students of mathematics:[8]

[6] The differences between the two approaches are many, though, and investigating them would require a full paper on its own. For one, there is no concern in Peacock with solely finitary methods and proofs, even though finitism is a central element in Hilbert's program.

[7] See, e.g., Kneale and Kneale (1966), Grattan-Guinness (2000), George and van Evra (2002), Peckhaus (2009).

[8] Page numbers here refer to the 1898 edition.

> We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility. It depends upon the fact, which must be verified by experience, that the common rules of algebra may be applied to these expressions without leading to any false results. An appeal to experience of this nature appears to be contrary to the first principles laid down at the beginning of this work. (1831, 152)
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> The imaginary expression $\sqrt{-a}$ and the negative expression $-b$ have this resemblance, that either of them occurring as the solution of a problem indicates some inconsistency or absurdity. As far as real meaning is concerned, both are equally imaginary, since $0 - a$ is as inconceivable as $\sqrt{-a}$. (1831, 156)

But in a later work on the foundations of algebra (1842), De Morgan would already adopt a (formalist) distinction between *two kinds* of algebra, along quite similar lines with Peacock's:[9]

> [3] Algebra now consists of two parts, the technical, and the logical. Technical algebra is *the art of using symbols under regulations* which ... are prescribed as the definitions of the symbols. Logical algebra is the *science* which investigates the method of giving meaning to the primary symbols, and of interpreting all subsequent symbolic results. ... [A] symbol is *defined* when such rules are laid down for its use as will enable us to accept or reject any proposed transformation of it, or by means of it. A simple symbol is *explained* when such a meaning is given to it as will enable us to accept or reject the application of its definition, as a consequence of that meaning. (1842, §3; emphasis added.)

Nevertheless, in a similar fashion to other British algebraists that had a taste for foundational issues (most notably, Boole), De Morgan's formalism quickly becomes intermingled with a good portion of *psychologism*:

> [4] A symbol is not the representation of an external object absolutely, but *of a state of the mind* in regard to that object; of a *conception* formed ... Those who do not remember this, *the real use of a symbol*, are apt to dogmatize, declaring one or another [signification] of a symbol ... by [the] impression produced on their own minds, to be real, true, natural, or necessary: it being neither one nor the other, except with reference to *the particular mind* in question. (1842, §4; emphasis added.)

This element of psychologism allows De Morgan to distinguish between three different mental images that a symbolic expression like '$a + b$' might present to one's mind: either the *magnitude* (e.g., the final sum, if a and b represent numbers) or the *operation* itself (the act of summation, i.e., a process in time) or just *the shift* of one's attention from an initial to a final state (j.e., the shift from the addends to the result).

> A symbol may thus denote either *magnitude*, *operation*, by which magnitude is attained, or the *conception* of one extreme arrived at, the other having been the previous object of contemplation ... The earlier algebraists most certainly dwelt on the first notion; $a + b$ is with them the result of an operation, in which the method of obtaining it is completely forgotten, that the result $a + b$ is actually obtained by a distinct operation.
>
> ...
>
> The modern algebraists usually dwell on the second notion, namely that of operation; and this I shall adopt ... (1842, §5,§7; emphasis added.)

This was a key insight. Besides anticipating a distinction that would also be drawn a century later by Church (1941), between functions as rules of correspondence (functions in intension) and functions as graphs (functions in extension), De Morgan's attachment to psychological justifications allowed him to take two significant steps. First, it enabled him to conceptualize operations as reversible *processes*:

[9] In a footnote to §9, De Morgan explicitly acknowledges that his distinction mirrors closely the one that Peacock had proposed a few years earlier.

> Every process by which we can pass from one object of contemplation to another, involves a second by which we can reinstate the first object in its position: or every direct *process* has another which is its inverse. (1842, §8; emphasis added)

Second, it led him to formulate one of the earliest (proto-)recursive characterizations of numerical operations:

> [3] Let the object of contemplation be simple magnitude of any one kind, as in the arithmetic of concrete quantity. The process which must precede all others is what we call *selecting one magnitude for consideration.* Previously to this step, we have no object under our perceptions, and may write 0 as the representative of this preceding state, and as the recognition of its existence. This first magnitude we may call 1, and the *operation of transition* from one state to the other we may denote by $0 + 1$. ... If we represent our present state by $(0+1)$, we may consider that with respect to any other possible magnitude our position is what it was when we denoted it by 0. If we now denote it by $0'$, we may, as, before, make the transition from $0'$ to $0' + 1$, which implies that we have further taken into consideration a new magnitude of the same amount.
>
> [4] This result, $(0 + 1) + 1$, we may, if we please to consider it as attained by one operation, signify by $0 + 2$: and so on. Using the symbol $-$ to denote the process by which we retrace our steps, we have all that is necessary to express addition and subtraction. The principle which I wish here to enforce is, that addition is connected with the symbol 0 in a manner which requires us to imagine that we start from one magnitude, as it were from a new 0, and renew* the process by which we passed from the first 0 to that magnitude. (1842, §§13-14; added emphasis)

That is, '0' may be understood as signifying the initial state in an iterative process that concerns successive "contemplations" of magnitudes. The way De Morgan describes the process in the passage is as if he is just thinking about iteration (something along the lines of a successor function). Nevertheless, the footnote to the word 'renew' shows that, instead of simple iteration, he clearly has in mind a *recursive* process: "Any one who doubts the justness of this fundamental position should add six to four on his fingers, *having previously refreshed his notions of six and four by the same process*" (emphasis added).

Thus, De Morgan essentially anticipates[10] the recursive approach to arithmetical operations that would later be extensively deployed by Grassmann (1861), Peirce (1933), Dedekind (1888), Peano (1889), Poincaré (1905, ch.1), Skolem (1923), and others.[11] Note, however, that this recursive conception cannot arise if one understands an expression like '$0 + 2$' in just the first way described by De Morgan (the way of the "earlier algebraists"); i.e., if "the method of obtaining it is completely forgotten" (recall the quote from above). It was, therefore, crucial that De Morgan chose to think of symbols like '+' and '−' as signifying processes/operations. And this was clearly a result of his psychologist outlook on symbolic algebra and logic.

In 1849 De Morgan published the textbook *Trigonometry and Double Algebra*, whose Book II was partly devoted to an effort to found the use of negative and complex numbers in algebraic problems along the lines that these are formal, syntactic manipulations that are authorized by the rules of a purely syntactic calculus.[12] The idea was, of course, not original to him; it had already appeared in the works of Peacock and Gregory and could be traced even further back. It is found, for instance, in a letter by Servois (1814) to Argand regarding the geometric interpretation of complex numbers, as well as in an essay by Lambert (1786). In the latter, Lambert suggested that the axioms of geometry be

[10] Notwithstanding his early anticipation of a recursive approach to addition and subtraction, De Morgan's characterization remains incomplete, as it lacks the standard two-clause structure (comprising a base case and an inductive step). He also seems to have overlooked the fact that multiplication could be similarly defined by recursion.

[11] De Morgan's anticipation of recursion has been overlooked even in the otherwise authoritative history of recursive functions by Adams (2011).

[12] The term 'double algebra' in the title refers to what we today would call 'complex analysis'.

treated as a symbolic calculus in an attempt to justify Euclid's fifth axiom in a purely formal manner; that is, by syntactically deriving it from the other axioms.[13] Nevertheless, De Morgan provided one of the clearest and most rigorous expositions of the idea of a purely syntactic calculus. In the words of Ewald (1996, 349), he was "the first mathematician to try to give a precise, purely syntactic description of a symbolic algebra (in essence, a field)— explicitly gathering together and describing, in a single place, its symbols and its rules of syntax."

3 Conclusions

It has generally been held that psychologism was rather an impediment than aid to the development of logic.[14]. This is a justified view, of course, in light of the astonishing degree to which Frege's anti-psychologism helped him break new ground in logic. Indeed, the search for fundamental laws of thought sometimes kept logicians too much focused keeping them too restricted to a view of syllogisms as exclusively concerning classes instead of propositions. But psychologism has surely played an under-appreciated role in the development of certain areas of logic, such as recursion theory. For example, as explained already, it was only through his view of symbolic expressions, like '+' and '−', as expressing fundamental processes of thought —i.e., by his integration of formalism with psychologism— that De Morgan reached one of the earliest recursive characterizations of arithmetic. And, indeed, it was precisely such considerations —whether of numbers as "free *creations of the human mind*"[15] (Dedekind, 1888), of number theory as grounded in a "purely *intuitive* basis of concrete signs" (Hilbert, 1926), or of a "recursive *mode of thought*" (Skolem, 1923)— that also underpinned the later seminal works that paved the way for a logic conducive to recursion and mechanization. That is, works that were consistently combining formalist and psychologistic stances.

Before closing a final comment is in order. In his authoritative sourcebook in logic, van Heijenoort (1967) wrote about Frege's *Begriffsschrift*:

> What Frege does is to construct logic as a language that need not be supplemented by any intuitive reasoning. Thus he is very careful to describe his system in purely formal terms. ... He is fully aware that any system requires rules that ... are void of any intuitive logic; they are "rules for the use of our signs"... This is one of the great lessons of Frege's book. It was a new one in 1879, and it did not at once pervade the world of logic. (1967, 4)

I hope that the above analysis of De Morgan's views shows that this (great indeed) lesson, in fact, was neither a new one in 1879 nor one that needed much longer to pervade the world of logic. It was a lesson already well-understood by De Morgan and the other British algebraists of the 19th century, as demonstrated even by the quoted sentence in the title of this paper.

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[13] See Ewald (1996, 314-5).

[14] See, e.g., Kneale and Kneale (1966, 407)) or Coffa (1991, chs.1-2).

[15] "In speaking of arithmetic (algebra, analysis) as merely a part of logic I mean to imply that I consider the number-concept ... an immediate product of the pure *laws of thought*. ... [N]umbers are *free creations of the human mind*." (Preface to the first edition, 1888; emphasis added)

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