On axioms for multiverses of set theory

Abstract

Gitman and Hamkins [5] showed that the collection of countable, recursively saturated models of set theory satisfy Hamkins's multiverse axioms [6]. Among these axioms is the Well-Foundedness Mirage axiom, which asserts that every universe is seen to be ω-nonstandard from the perspective of some larger universe. Any multiverse satisfying this axiom must have that all of its worlds are recursively saturated. We inquired as to whether this forced recursive saturation could be avoided by weakening the Well-Foundedness Mirage axiom. The main results of this article answer this in the positive. We give two different versions of the Well-Foundedness Mirage axiom—what we call Weak Well-Foundedness Mirage and Covering Well-Foundedness Mirage—and construct two multiverses satisfying these two axioms, none of whose worlds are recursively saturated.

The multiversist position in philosophy of set theory holds that rather than there being a single concept of set, there are many universes of sets, each as legitimate as the others, giving rise to many different concepts of set.[1] Among multiversists, there is disagreement on just what universes of sets ought be. While these debates are at a level beyond the subject matter of set theory, they admit formalization within set theory, thus allowing the tools of mathematics to be applied. Let us adopt the terminology of referring to a nonempty collection of models of set theory as a *multiverse*. Members of a multiverse will be called *universes* or *worlds*.

Hamkins [6] gave various axioms describing properties the set theoretic multiverse should in his view satisfy. Gitman and Hamkins [5] showed that the countable, recursively saturated models of ZFC form a natural model of the multiverse axioms of [6]. A consequential (and controversial!) axiom here is the Well-Foundedness Mirage axiom, which asserts that every world is seen by some bigger world to be ω-nonstandard. Standard techniques in the model theory of set theory yield that any world which is an element of an ω-nonstandard model must be recursively saturated. So any multiverse satisfying the Well-Foundedness Mirage axiom must consist only of recursively saturated worlds.

This article is an attempt to better understand the connection between the Well-Foundedness Mirage axiom and recursive saturation in Hamkinsian multiverses. We formulate two possible ways to weaken this axiom and show that they admit multiverses in which worlds needn't be recursively saturated.

Let us catalog some multiverses axioms—some directly from [5] and others being variants—before we explicate recursive saturation and state the main results of this article. Our first multiverse axioms express that the multiverse contains universes of different width.

- *Closure Under Set Forcing*. If M is a world and $\mathbb{P} \in M$ is a poset, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over M.
- *Closure Under Class Forcing*. If M is a world and $\mathbb{P} \subseteq M$ is a tame class forcing notion, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over M.[2]
- *Closure Under Inner Models*. If M is a world and $W \subseteq M$ is an inner model (= transitive class model) of ZFC, then W is a world.
- *Closure Under Grounds*. If M is a world and $W \subseteq M$ is a ground—meaning that $M = W[G]$ for $G \subseteq \mathbb{P} \in W$ a W-generic—then W is a world.

[1] For the reader who is not familiar with the universist versus multiversist debate, we recommend [8] for an overview.

[2] See [4, Section 2.2] for a definition of tameness, which is equivalent to the preservation of ZFC.

Note that these axioms are already enough to describe a robust multiverse notion. For instance, if M is any countable model of ZFC then the *generic multiverse of* M, the closure of M under taking (set) forcing extensions and grounds, will satisfy the Closure Under Set Forcing and Closure Under Grounds axioms. More broadly, if we fix a countable ordinal α so that there is a transitive model of ZFC of height α, then the collection of transitive models of set theory of height α form a multiverse satisfying the above four closure axioms.

We can weaken Closure Under Set Forcing or Closure Under Class Forcing by restricting what forcing notions are allowed. One case of this will be of interest in this paper. A class forcing notion $\mathbb{P}$ is said to have the Ord-*cc* if every antichain of $\mathbb{P}$ is a set. Every class forcing with the Ord-cc is tame.

- *Closure Under* Ord-*cc Class Forcing* If M is a world and $\mathbb{P} \subseteq M$ is forcing with the Ord-cc, then there is a world $M[G]$ where $G \subseteq \mathbb{P}$ is generic over M.

We might also think that universes can have different heights, and formulate axioms which express this. The following axioms express that the multiverse contains shorter worlds.

- *Closure Under Rank-Initial Segments*. If M is a world and θ is an ordinal in M so that ${V_\theta}^M \models$ ZFC, then ${V_\theta}^M$ is a world.

- *Closure Under* $\in$*-Initial Segments*. If M is a world and $N \in M$ is a transitive set so that $N \models$ ZFC, then N is a world.

The following axioms can be seen as strengthenings of the previous two axioms, as well as the Closure Under Inner Models and Closure Under Grounds axioms. The latter two of the following axioms are the first we will see which force the multiverse to contain nonstandard models.[3]

- *Standard Realizability*. If M is a world and $N \subseteq M$ is a definable transitive class model of ZFC, then N is a world.

- *Set-Like Realizability*. If M is a world and $N \subseteq M$ is a definable set-like class model of ZFC, then N is a world. Here, by saying N is set-like we mean that M thinks each element of N has set-many $\in^N$-predecessors.

- *Realizability*. If M is a world and $N \subseteq M$ is a definable class model of ZFC, then N is a world.

We hope that the names of these axioms make clear the distinction between them. Standard Realizability does not allow for N to be nonstandard; Set-Like Realizability does, but requires N to be set-like; and Realizability is the most permissive, only requiring that N satisfy the right axioms. We summarize the relationship between M and N by saying, respectively, that N is *standard realizable*, *set-like realizable*, or *realizable* from M.

There is a subtlety here which we should address. Namely, if N is realizable from M then are we asking that $N \models$ ZFC or that $M \models$ "$N \models$ ZFC"? These can be different for omega-nonstandard models. (By overspill any omega-nonstandard model M of ZFC has a set which externally is seen to be a model of ZFC. But of course if M thinks ZFC is inconsistent, then that set in M won't internally be seen to be a model of ZFC.) Following [5] we will go with the latter interpretation, and ask for the internal certificate.

In the other direction, we might think that there is no tallest universe, that every world is an element of a larger world. We might think that this larger universe is so much larger that it sees that the smaller universe is countable.

[3] Our convention when talking about models of set theory will be to use the same letter for both the model and its universe. When we refer to nonstandard models we will usually suppress explicitly writing the membership relation for nonstandard models, writing M rather than $(M, \in^M)$. On occasion we will wish to emphasize that M is nonstandard and will write it out fully to make this emphasis.

- *Countability*. If M is a world then there is a world N so that $M \in N$ and $N \models M$ is countable.

Observe that if the multiverse satisfies Closure Under Set Forcing and if every world is an element of another world, then Countability is satisfied for free.

S.D. Friedman proposed what he called the hyperverse program [1], which involves looking at the multiverse of countable, transitive models of set theory. It is easy to check this this multiverse satisfies Standard Realizability, Closure Under Set and Class Forcing, and Countability.

A more radical multiverse is one where every world is seen to be not only be countable by some larger world but also seen to be ω-nonstandard. This is the content of the Well-Foundedness Mirage axiom of.

- *Well-Foundedness Mirage*. If M is a world then there is a world N so that $M \in N$ and N thinks that M is ω-nonstandard.

Having seen this axiom, let's explicate its connection to recursive saturation. This notion, introduced by Barwise and Schlipf [2] is robust and has proved to be important for the study of nonstandard models.

Definition 1. A model M of set theory is *recursively saturated* if it realizes every finitely consistent, computable type. That is, if $p(x) = \{\varphi_n(x, a) : n \in \omega\}$ is a type with a parameter $a \in M$ so that $\{\varphi_n : n \in \omega\}$ is a computable subset of ω and for each $n \in \omega$ there is $x \in M$ so that $M \models \varphi_i(x, a)$ for each $i < n$, then there is a $x \in M$ so that $M \models \varphi_n(x, a)$ for each $n < \omega$.

Observe that if M is recursively saturated then it must be ω-nonstandard—consider the type $\{x > n : n \in \omega\} \cup \{x \in \omega\}$. From the other direction, not every ω-nonstandard model will be recursively saturated. One way to see this goes through the observation that if M is recursively saturated then the definable ordinals of M must be bounded, because the type asserting that x has higher rank than every definable ordinal is computable. But there are models all of whose ordinals are definable.

Definition 2 (Paris [9])**.** A model of set theory is a *Paris model* if each of its ordinals is definable without parameters.[4]

Theorem 3 (Paris [9])**.** *If T is a consistent extension of* ZFC *then T has a countable ω-nonstandard Paris model.*

One way in which a model of may be forced to be recursively saturated is if it's an element of an ω-nonstandard model of set theory.

Proposition 4. *Suppose $M \models$ ZFC is ω-nonstandard and $N \in M$ is a model. Then N is recursively saturated.*[5]

Proof. This is an immediate consequence of Lachlan's work [7] on satisfaction classes over ω-nonstandard models. For the benefit of the reader, we also give a direct argument. Let $p(x) = \{\varphi_n(x) : n \in \omega\}$ be a computable type with a parameter from N which is finitely consistent over N. We may assume without loss that $p(x)$ is increasing, meaning that if $n > m$ then $\varphi_n(x)$ implies $\varphi_m(x)$. By assumption, $\varphi_n(x)$ is realized for each standard n. Being a model of set theory, M may carry out the definition of $p(x)$ internally. Of course, this will produce formulae $\varphi_n(x)$ for all $n \in \omega^M$. But because $p(x)$ is computable, its standard elements are absolute to any model, and so M correctly defines $\varphi_n(x)$ and correctly sees that $\varphi_n(x)$ is realized for each standard n. By overspill applied in M there must be a nonstandard e so that M thinks N has an element x satisfying $\varphi_e(x)$. So also $N \models \varphi_n(x)$ for each standard n, as desired. □

[4] For the reader who wishes to know more about Paris models we recommend [3], which contains a thorough investigation thereof.

[5] ZFC is vast overkill here. For instance, it suffices that $M \models$ KP.

A natural weakening of the Well-Foundedness Mirage axiom is to not require that worlds be seen to be ω-nonstandard but merely be seen to be nonstandard.

- *Weak Well-Foundedness Mirage*. If M is a world then there is a world N so that $M \in N$ and N thinks that M is nonstandard.[6]

The first main result of this article is that weakening the Well-Foundedness Mirage axiom in this way does indeed free us from forced recursive saturation.

Main Theorem 5. *Assume that every real is in a transitive model of* ZFC*. Then the collection of countable, nonstandard models of* ZFC *form a multiverse satisfying the Closure Under Set and Class Forcing, Realizability, Countability, and Weak Well-Foundedness Mirage axioms. Necessarily, this multiverse contains worlds which are not recursively saturated.*

Moreover, under the same assumption the collection of countable, nonstandard but ω-standard models of ZFC *form a multiverse satisfying the Closure Under Set and Class Forcing, Standard Realizability, Countability, and Weak Well-Foundedness Mirage axioms. No world of this multiverse may be recursively saturated.*

This result amounts to combining a few standard techniques in the study of nonstandard models. It is a different way to weaken Well-Foundedness Mirage which provides a more substantive problem.

Before we state this new multiverse axiom, let us give some definitions. Given models of set theory $M \subseteq N$ we say that N *end-extends* M if $a \in^N b \in M$ implies $a \in M$. End-extensions are ubiquitous within set theory. Here are two examples of different flavor: (1) if M and N are transitive models of set theory and $M \in N$ then N end-extends M, and (2) any forcing extension $M[G]$ end-extends M. Given an end-extension N of M, we say that M is *topped* by N if, intuitively, M is an element of N. More formally, M is topped by N if there is $m \in N$ so that $M = \{a \in N : a \in^N m\}$. If N is a transitive model, then M is topped in N if and only if M is covered by N, where M is *covered* by N if there is $m \in N$ so that $M \subseteq \{a \in N : a \in^N m\}$. For nonstandard end-extensions, however, the two notions diverge and being covered is strictly weaker than being topped.

Let us remark on an alternative characterization of not being topped.

Observation 6. *Suppose M is end-extended by N, where both are models of* ZFC*. Then M is not topped by N if and only if there is an infinite descending coinitial sequence in $\mathrm{Ord}^N \setminus \mathrm{Ord}^M$.*

The above discussion about recursive saturation can be rephrased as saying that if M is topped by an ω-nonstandard model N, then M is recursively saturated. However, this need not be the case if M is merely covered by N. This is how we will weaken the Well-Foundedness Mirage axiom. We must weaken the Countability axiom for the same reason.

- *Covering Countability*. If M is a world then there is a world N with $(m, e) \in N$ which end-extends M so that N thinks (m, e) is countable.
- *Covering Well-Foundedness Mirage*. If M is a world then there is a world N with $(m, e) \in N$ which end-extends M so that N thinks (m, e) is ω-nonstandard.

The second main theorem of this article gives a multiverse which satisfies these two axioms which contains non-recursively saturated worlds.

Main Theorem 7. *Assume that* ZFC *is consistent. Then there is a multiverse, some of whose worlds are not recursively saturated, which satisfies Closure Under Set Forcing and* Ord*-cc Class Forcing, Set-Like Realizability, Covering Countability, and Covering Well-Foundedness Mirage.*

[6] One could argue that this axiom would be better termed the Well-Foundedness Mirage axiom while the other should more properly be called the ω-Standardness Mirage. We will dismiss such quibbles and not proliferate terminology.

The remainder of this article is organized as follows. First we have a section reviewing some classical theorems about extensions of models of set theory which we will make use of. We recast these theorems as multiverse axioms. Next is a section where we prove Main Theorem 5. Following is a section where we prove Main Theorem 7. We conclude with a short section with some open questions.

Open questions There are two ways in which the multiverse constructed in the proof of Theorem 7 fell short of our hopes. It only satisfies Closure Under Ord-cc Class Forcing and Set-Like Realizability. We would like a covering multiverse which satisfies Closure Under Class Forcing and Realizability, but that did not work out for our construction. We used Ord-cc-ness, and we used the set-like part of Set-Like Realizability essentially in the proofs of Lemmas leading to our main theorems.

Question 8. *Is there a multiverse which satisfies Realizability, Closure Under Class Forcing, Covering Countability, and Covering Well-Foundedness Mirage?*

We also want to know whether there is a natural collection of models which satisfies Covering Well-Foundedness Mirage, similar to the Gitman and Hamkins model [5] of the Well-Foundedness Mirage axiom and our model of the Weak Well-Foundedness Mirage axiom.

Question 9. *Is there a natural model of the Covering Well-Foundedness Mirage axiom?*

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