Σ^1_3 GANDY-HARRINGTON FORCING

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While most natural questions about the first level of the projective hierarchy—Σ^1_1, Π^1_1, and Δ^1_1 sets—can be decided in ZFC, many natural questions at the higher levels are independent. However, large cardinal axioms imply Projective Determinacy (PD), which seems to give a complete picture of the projective sets. Moschovakis, Martin, and others discovered that, under PD, the structural properties of the projective sets have a periodic nature, with some properties (like Uniformization), oscillating between the Π^1_n sets for odd n and the Σ^1_n sets for even n. Because of this periodicity, it is the odd projective levels that have the structural properties which are most analogous to those of the first projective level.

Gandy-Harrington forcing is a basic technique in descriptive set theory for studying the first level of the projective hierarchy. For example, Harrington's celebrated proof of Silver's Dichotomy uses this technique. The forcing notion is simple to describe: one forces with nonempty Σ^1_1 sets, ordered by inclusion. In [1], we have looked at natural analogues and applications of Gandy-Harrington forcing at the next odd level, the Σ^1_3 sets. A version of this forcing had already been considered before, establishing an analogue of Silver's Dichotomy but for $\mathbf{\Pi}^1_3$ equivalence relations [3].

A new phenomenon that appears at the Σ^1_3 level, and beyond, is the need to consider codes for (infinite) ordinals. A partial surjection $\pi : \mathbb{R} \rightharpoonup \alpha$ allows us to consider the descriptive set-theoretic complexity of relations on α in the codes given by π, i.e. via their π-preimages. Sharps for reals give rise to a natural map onto u_ω, the supremum of the first ω uniform indiscernibles (itself a uniform indiscernible). This partial surjection onto u_ω comes from the evaluation of Skolem terms using the first ω uniform indiscernibles inside models of the form $L[x]$ for reals x.

We will now discuss this coding in a bit more detail. First, for $x \in \mathbb{R}$, $x^\#$ is the theory of $L[x]$ in indiscernible parameters, coded as a real. Let U be the set of sharps, that is reals of the form $x^\#$ for some real x. Also fix a recursive enumeration $\langle \tau_n \mid n < \omega \rangle$ of all Skolem terms for ordinals in the language of $L[x]$ (with constants for indiscernibles). Let WO_ω be the set of pairs of the form (u, n) for $u \in U$ and $n \in \omega$. We say $w \in \mathrm{WO}_\omega$ is a *sharp code* for an ordinal α if, letting $w = (x^\#, n)$, $\tau_n^{L[x]}(u_1, \dots, u_k) = \alpha$, where k is the arity of τ_n and $u_1, \dots, u_k$ are the first k uniform indiscernibles. Indiscernability implies that $\alpha < u_{k+1}$ and a result of Solovay shows that every ordinal less than u_{k+1} is coded in this way. Given $w \in \mathrm{WO}_\omega$, we let $|w|$ be the ordinal coded by α. Solovay showed U and WO_ω are Π^1_2 sets of reals and the relations $|w| \le |z|$, $|w| < |z|$, and $|w| = |z|$ on WO_ω are Δ^1_3.

Definition 0.1. A relation R on $\mathbb{R} \times u_\omega$ is Σ^1_3 if there is a Σ^1_3 relation R^* such that for any $\alpha < u_\omega$, $R(x, \alpha)$ iff $R^*(x, z)$ for any sharp code z for α. Similarly, for $\beta < u_\omega$, R is on $\mathbb{R} \times u_\omega$ is $\Sigma^1_3(\alpha)$ iff there is a Σ^1_3 relation Q on $\mathbb{R} \times u_\omega \times u_\omega$ such that $R(x, \alpha)$ iff $Q(x, \alpha, \beta)$.

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In all versions of Σ^1_3 Gandy-Harrington forcing, old and new, one considers forcing with nonempty $\Sigma^1_3(s)$ sets of reals for certain parameters $s \in [u_\omega]^{<\omega}$. By carefully choosing the allowed set of parameters, we were able to find a higher version of the Harrington-Shore-Slaman theorem [2].

Theorem 0.2. *Assume $\boldsymbol{\Delta}^1_2$-Determinacy. There is a countable set $H \subseteq u_\omega$ such that if $X \subseteq \mathbb{R}$ is a nonempty Σ^1_3 set of reals and $A \subseteq u_\omega$ is $\Sigma^1_3(x)$ for any $x \in X$, then A is $\Sigma^1_3(s)$ for some $s \in H^{<\omega}$.*

We also established some connections with inner model theory, building on work of Zhu [5]. For any real x, $M_1(x)$ is the minimal inner model with one Woodin cardinal containing x. We let $\delta^{M_1(x)}$ denote the Woodin cardinal of $M_1(x)$. Using our Σ^1_3 Gandy-Harrington forcing, we showed the following inner model theoretic basis theorem, an analogue of the Gandy basis theorem.

Theorem 0.3. *Assume $\boldsymbol{\Delta}^1_2$-Determinacy. Let X be a nonempty Σ^1_3 set of reals. Then there is an $x \in X$ such that $\delta^{M_1(x)} = \delta^{M_1}$ and $M_1|\delta^{M_1} \subseteq M_1(x)|\delta^{M_1(x)}$.*

Finally, we used a product version of Σ^1_3 Gandy Harrington forcing to establish a reverse mathematics result. The real $M_1^\#$ is an analogue of $0^\#$ corresponding to indiscernibles for M_1. Woodin showed that $\boldsymbol{\Delta}^1_2$-Determinacy implies $M_1^\#$ exists over ZFC [4]. In fact, this can be carried out in full second-order arithmetic. However, we showed that this is not provable from Σ^1_3-Comprehension.

Theorem 0.4. $\boldsymbol{\Sigma}^1_3$*-Comprehension +* $\boldsymbol{\Delta}^1_2$*-Determinacy does not prove "$M_1^\#$ exists".*

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