# Wadge Games for a Condition-Induced Topology

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**Abstract**

Wadge reducibility classically compares sets of infinite words via continuous reductions with respect to their topological complexity. Open sets in the Cantor topology $\tau$ correspond to satisfaction-monitorable properties, and Wadge reducibility can therefore be used to compare languages with respect to their monitorability. Motivated by runtime monitoring settings in which observed events may also include information about state properties, called conditions, we introduce a topology $\tau_C$ on $\Sigma^\omega$ that refines the Cantor topology, using an arbitrary language corresponding to a condition $c$. We then define an annotated Wadge game for $\tau_C$. Our main result shows that continuous reducibility from $\tau_C$ to $\tau$ is equivalent to the existence of a winning strategy for Player II in the associated annotated Wadge game. This yields a game-theoretic characterization of continuous reducibility and monitorability in the presence of both observable events and conditions.

## 1 Introduction

*Wadge reducibility* [9, 8, 7] is a fundamental notion in descriptive set theory for comparing the complexity of sets of infinite words via reductions by continuous functions. In its classical form, continuity is understood with respect to the *Cantor topology* on $\Sigma^\omega$, whose basic open sets are sets of the form $s \cdot \Sigma^\omega$, for $s \in \Sigma^*$, that is, the sets of all infinite words having $s$ as a prefix. This makes the Cantor topology a natural setting for studying properties that can be determined from finite information.

The relevance of this topological setting becomes particularly clear in the study of *monitorability*, a central notion in *runtime verification*. Runtime verification is a lightweight verification technique that analyses the current execution of a system with respect to a given specification [3]. Following the intuition of [2], a *monitor* is considered a computational entity which reads a trace, that is, a sequence of actions or letters, and attempts to determine, based on a finite prefix, whether the observed behavior satisfies or violates a given property. A property is *satisfaction-monitorable* (respectively, *violation-monitorable*) if its satisfaction (respectively, violation) by a trace can be concluded after observing a finite prefix of that trace [2].

The connection with topology arises from the fact that both monitorability and the Cantor topology are fundamentally based on finite information. In particular, existing work by Diekert and Leucker [5] and by Camerlo and Dagnino [4] shows that open sets in the Cantor topology correspond to satisfaction-monitorable properties, whereas closed sets correspond to violation-monitorable properties. These results provide a bridge between descriptive set theory and runtime verification and motivate the study of richer topological structures that capture more refined forms of finite observation, such as those considered in [1, 6]. From this perspective, a Wadge reduction may also be viewed as a form of monitorability reduction. Indeed, if a property is open in the Cantor topology, then it is satisfaction-monitorable, and a continuous reduction transfers this openness to the inverse image of the property. In this sense, such a reduction transfers monitorability from the classical setting to one enriched with condition information.

In light of these refined forms, we focus on monitors that detect *conditions*, as first introduced in [1]. Our formulation differs slightly from the original one, since we define conditions over sets of traces rather than over sets of processes. Monitors that detect conditions extend the classical monitoring setting by observing not only finite prefixes of a trace, but also whether given conditions hold for the corresponding suffixes. This enriches the observable information during monitoring and therefore calls for a corresponding refinement of the underlying topology.

To capture this enriched form of observation, we introduce a *condition-induced* topology on $\Sigma^\omega$. Given a non-empty set of conditions $C$ together with a valuation assigning to each condition the set of traces satisfying it, the topology $\tau_C$ refines the usual Cantor topology by adding information about the relevant suffixes. We restrict attention to the simplest non-trivial case in which $C$ is a singleton, allowing the main topological and game-theoretic ideas to be presented in a clean form, while the extension to finite sets of conditions is a natural extension of this work.

The main contribution of this paper is a game-theoretic characterization of continuous reducibility from the topology $\tau_C$ to the Cantor topology $\tau$. We introduce an annotated Wadge game in which traces are enriched with condition information, and show that the resulting game reflects the topology generated by these enriched observations. The key point is that honest annotated prefixes determine exactly the local neighborhoods of $\tau_C$. Using this correspondence, we prove that Wadge reducibility from $\tau_C$ to $\tau$ is equivalent to the existence of a winning strategy for Player II in the associated game. The main body of the paper presents the necessary definitions, the key neighborhood-basis lemma, and a proof outline, while the full proofs are given in the appendix.

## 2 Preliminaries

Let $\Sigma$ be a finite alphabet. We equip $\Sigma^\omega$ with the Cantor topology, denoted by $\tau$, whose basic open sets are $s \cdot \Sigma^\omega$, $s \in \Sigma^*$. Thus, every open set is of the form $W \cdot \Sigma^\omega = \bigcup_{s \in W} s \cdot \Sigma^\omega$, for some $W \subseteq \Sigma^*$. A set $L \subseteq \Sigma^\omega$ is closed if and only if its complement $\Sigma^\omega \setminus L$ is open. We write $\mathcal{A}(\tau)$ for the least $\sigma$-algebra generated by the topology $\tau$ on $\Sigma^\omega$. A subset of $\Sigma^\omega$ is called *Borel* if it belongs to $\mathcal{A}(\tau)$.

For an infinite trace $t = t_0 t_1 t_2 \cdots \in \Sigma^\omega$ and $n \in \mathbb{N}$, we write $t{\restriction}n = t_0 t_1 \cdots t_{n-1} \in \Sigma^*$ and $t_{\geq n} = t_n t_{n+1} t_{n+2} \cdots \in \Sigma^\omega$. Thus, $t = (t{\restriction}n) \cdot t_{\geq n}$, for every $n \in \mathbb{N}$. For $s \in \Sigma^*$ and $t \in \Sigma^\omega$, we write $s \prec t$ when $s$ is a prefix of $t$, that is, when there exists $u \in \Sigma^\omega$ such that $t = s \cdot u$.

Let $L \subseteq \Sigma^\omega$. We say that $L$ is *satisfaction-monitorable* if, for every $t \in L$, there exists a finite prefix $s \prec t$ such that $s \cdot \Sigma^\omega \subseteq L$. Dually, $L$ is *violation-monitorable* if, for every $t \notin L$, there exists a finite prefix $s \prec t$ such that $s \cdot \Sigma^\omega \subseteq \Sigma^\omega \setminus L$. With respect to the Cantor topology $\tau$, a language $L \subseteq \Sigma^\omega$ is satisfaction-monitorable if and only if $L$ is open in $\tau$. Similarly, $L$ is violation-monitorable if and only if $L$ is closed in $\tau$.

We also use the annotated alphabet $\widehat{\Sigma} = \Sigma \times \{T, F\}$. The projection $\pi : \widehat{\Sigma}^\omega \to \Sigma^\omega$ is defined by $\pi\big((a_0, d_0)(a_1, d_1)(a_2, d_2) \cdots\big) = a_0 a_1 a_2 \cdots$. For $\widehat{t} = (a_0, d_0)(a_1, d_1) \cdots \in \widehat{\Sigma}^\omega$ and $n \in \mathbb{N}$, we write $\widehat{t}{\restriction}n = ((a_0, d_0), \ldots, (a_{n-1}, d_{n-1})) \in \widehat{\Sigma}^*$, where $\widehat{\Sigma}^*$ denotes the set of all finite words over $\widehat{\Sigma}$.

# 3 Contribution

## 3.1 A Condition-Induced Topology

To describe monitors that may use additional information about the future behavior of an infinite trace, we enrich the usual Cantor topology with condition information. This leads to a new topology in which finite observations record not only a prefix but also whether a specific condition holds for the relevant suffixes.

Given a non-empty set of conditions $C$, a pair $(C, v)$ is called a *condition framework*, where $v : C \to \mathcal{P}(\Sigma^\omega)$ is a valuation function. In this paper, we first restrict attention to the simplest non-trivial case in which $C$ is a singleton, namely $C = \{c\}$. Thus, the valuation assigns to the condition $c$ a set $v(c) \subseteq \Sigma^\omega$ of traces satisfying $c$.

We write $s \cdot v(c) = \{s \cdot t \mid t \in v(c)\}$, so a word belongs to $s \cdot v(c)$ exactly when it has a prefix $s$ and its suffix $t$ belongs to $v(c)$, hence satisfies the condition $c$.

In the following definition, we refine the Cantor topology by adding information about the behavior of suffixes with respect to the condition $c$.

**Definition 1** (Topology $\tau_C$)**.** Let $\mathcal{F} = \{\, s \cdot \Sigma^\omega \mid s \in \Sigma^* \,\} \cup \{\, s \cdot v(c),\ s \cdot (\Sigma^\omega \setminus v(c)) \mid s \in \Sigma^* \,\}$. We denote by $\tau_C$ the topology on $\Sigma^\omega$ generated by the subbasis $\mathcal{F}$. Equivalently, the finite intersections of elements of $\mathcal{F}$ form a basis for $\tau_C$.

To associate the topology $\tau_C$ with infinite games, we encode traces together with the truth values of the condition. This allows us to construct annotated traces over the alphabet $\widehat{\Sigma} = \Sigma \times \{T, F\}$.

**Definition 2** (Honest annotation set)**.** Define

$$A(v) = \left\{ \widehat{t} = ((a_0, d_0), (a_1, d_1), \dots) \in \widehat{\Sigma}^\omega \;\middle|\; \forall n \in \mathbb{N},\ \big(d_n = T \iff (\pi(\widehat{t}))_{\geq n} \in v(c)\big) \right\}.$$

The set $A(v)$ is called the set of *honest annotated plays*.

Therefore, an element of $A(v)$ determines, at each position, whether the corresponding suffix satisfies the condition $c$.

## 3.2 Wadge Reducibility and the Wadge Game for $\tau_C$

Wadge reducibility can be defined more generally with respect to arbitrary topologies. Accordingly, we introduce the corresponding notions of Wadge reducibility and Wadge game for the topology $\tau_C$. In the following, $L_1$ and $L_2$ denote Borel subsets of $\Sigma^\omega$.

**Definition 3** (Wadge reducibility for $\tau_C$)**.** Let $L_1, L_2 \subseteq \Sigma^\omega$. We write $L_1 \leq_W^{\tau_C} L_2$ if there exists a continuous function $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$, such that $L_1 = f^{-1}(L_2)$.

*Remark* 1. A function $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ is continuous if the preimage of every open set in $\tau$ is open in $\tau_C$. The choice of the function $f$ reflects the idea that the domain is enriched by condition information, while the codomain remains in the classical Cantor setting. In particular, if a set is open in $(\Sigma^\omega, \tau)$, then by continuity its inverse image under $f$ is open in $(\Sigma^\omega, \tau_C)$. Thus, if a property $L_2$ is satisfaction-monitorable in the Cantor topology $\tau$, the existence of such a continuous reduction implies that $L_1 = f^{-1}(L_2)$ is satisfaction-monitorable in $\tau_C$, that is, satisfaction-monitorable with conditions. In this sense, such a reduction may be viewed as transferring monitorability from the classical setting to the condition-based one.

**Example 1** (A simple condition-based reduction)**.** Let $\Sigma = \{a, b\}$ and let $C = \{c\}$, where $v(c) = \mathrm{Inf}_b = \{t \in \Sigma^\omega \mid t \text{ contains infinitely many occurrences of } b\}$. Consider the condition-based language $L_1 = \bigcup_{s \in \Sigma^*} s \cdot a \cdot v(c)$. Equivalently, $L_1$ contains exactly those traces in which some occurrence of $a$ is followed by a suffix satisfying $c$, i.e., by infinitely many later occurrences of $b$. Let $L_2 = \Sigma^* \cdot b \cdot \Sigma^\omega$, the Cantor-open language of traces in which a $b$ eventually appears. We define a function $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ as follows. For $t = t_0 t_1 t_2 \cdots$, let $f(t) = y_0 y_1 y_2 \cdots$, where

$$y_n = \begin{cases} b, & \text{if } t_n = a \text{ and } t_{\geq n+1} \in v(c), \\ a, & \text{otherwise.} \end{cases}$$

Therefore, for every $t \in \Sigma^\omega$,

$$\begin{aligned} t \in L_1 &\iff \exists n \in \mathbb{N} \text{ such that } t_n = a \text{ and } t_{\geq n+1} \in v(c) \\ &\iff \exists n \in \mathbb{N} \text{ such that } f(t)_n = b \\ &\iff f(t) \in L_2. \end{aligned}$$

Thus, $L_1 = f^{-1}(L_2)$. Moreover, $f$ is continuous because the preimage of a basic Cantor-open set can be written using finitely many subbasic open sets of $\tau_C$. Consequently, $L_1 \leq_W^{\tau_C} L_2$.

The full proof that $f$ is continuous, and the details of this reduction are given in Appendix A.2.

We now extend the classical Wadge game to the present setting, where finite observations are enriched by condition information.

**Definition 4** (Wadge game for $\tau_C$)**.** Let $L_1, L_2 \subseteq \Sigma^\omega$. The Wadge game for $\tau_C$, denoted by $WG_{\tau_C}(L_1, L_2)$, is a game of perfect information between Player I and Player II, whom we refer to as *he* and *she*, respectively.

Player I produces an annotated $\omega$-word $\widehat{t_1} = (a_0, d_0)(a_1, d_1)(a_2, d_2) \cdots \in \widehat{\Sigma}^\omega$, while Player II plays symbols from the alphabet $\Sigma \cup \{\mathsf{skip}\}$.

Thus, at each step, Player II may either output a letter from $\Sigma$ or play skip. The outcome of Player II is the infinite word $t_2 \in \Sigma^\omega$ obtained by deleting all occurrences of skip. If Player II produces only finitely many non-skip moves, then she loses.

*Strategies.* The functions $\sigma_I : (\Sigma \cup \{\mathsf{skip}\})^* \to \widehat{\Sigma}$ and $\sigma_{II} : \widehat{\Sigma}^+ \to \Sigma \cup \{\mathsf{skip}\}$ denote the strategies of Player I and Player II, respectively.

*Winning condition.* Player II wins a play if and only if one of the following holds:

1. $\widehat{t_1} \notin A(v)$, or
2. $\widehat{t_1} \in A(v)$, and $\pi(\widehat{t_1}) \in L_1 \iff t_2 \in L_2$.

The topology $\tau_C$ and the game $WG_{\tau_C}(L_1, L_2)$ are both based on the same finite observable information, namely annotated prefixes. Theorem 1 below shows that they coincide, in the sense that continuous reducibility with respect to $\tau_C$ is equivalent to the existence of a winning strategy for Player II.

Given a finite annotated prefix, the next definition associates with it the set of all infinite traces that are compatible with the information recorded so far. These sets will serve as the basic neighborhoods linking the topology $\tau_C$ with the game-theoretic setting.

**Definition 5** (Neighborhood determined by a finite annotated prefix)**.** Let

$$\widehat{s} = ((a_0, d_0), \ldots, (a_{n-1}, d_{n-1})) \in \widehat{\Sigma}^+$$

and let $s = a_0 a_1 \cdots a_{n-1} \in \Sigma^*$. Define

$$U_{\widehat{s}} = s \cdot \Sigma^\omega \cap \bigcap_{\substack{i<n \\ d_i = T}} \big((s{\restriction}i) \cdot v(c)\big) \cap \bigcap_{\substack{i<n \\ d_i = F}} \big((s{\restriction}i) \cdot (\Sigma^\omega \setminus v(c))\big).$$

The sets $U_{\widehat{s}}$ are the central link between annotated traces and the topology $\tau_C$. Along an honest annotated trace, they form a decreasing family of neighborhoods. The following lemma shows that they in fact form a neighborhood basis, and therefore provide the key connection between the topology and the game.

**Lemma 1** (Neighborhood basis). *Let $\widehat{t} \in A(v)$, and let $t = \pi(\widehat{t})$. Then, the family $\big(U_{\widehat{t}\restriction n}\big)_{n \geq 1}$ is a neighborhood basis at $t$ in $(\Sigma^\omega, \tau_C)$. Equivalently, if $O \in \tau_C$ and $t \in O$, then there exists $n \geq 1$ such that $U_{\widehat{t}\restriction n} \subseteq O$.*

**Theorem 1** (Game-reduction equivalence for $\tau_C$). *Let $L_1, L_2 \subseteq \Sigma^\omega$ be Borel. Then,*

$$L_1 \leq_W^{\tau_C} L_2 \quad \Longleftrightarrow \quad \textit{Player II has a winning strategy in } WG_{\tau_C}(L_1, L_2).$$

The proof of the theorem is based on the fact that the sets $U_{\widehat{s}}$ encode exactly the finite annotated information relevant to the topology $\tau_C$. In particular, along an honest annotated trace, they form a neighborhood basis. We now outline the main idea of the argument, while the full proof is given in the appendix.

*Proof idea.* In the forward direction, a continuous reduction $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ induces a strategy for Player II by letting her follow the longest common prefix of the image of the set $U_{\widehat{s}}$ determined by the current annotated history $\widehat{s}$. The neighborhood basis lemma guarantees that, along an honest play, the sets $U_{\widehat{s}}$ form a decreasing family of neighborhoods around the trace, and continuity ensures that the corresponding prefixes converge to $f(t)$.

In the backward direction, a winning strategy for Player II defines a function $f$ by assigning to each trace the output produced against its honest annotation. The winning condition yields $L_1 = f^{-1}(L_2)$, while continuity follows from the fact that a strategy depends only on finite history, that is, once a finite output prefix has been produced, every trace with the same honest annotated history up to that point yields the same prefix. □

# 4 Conclusion and Future Work

The present paper introduces a condition-induced topology $\tau_C$ on $\Sigma^\omega$ for the topological characterization of monitors that detect conditions, namely monitors that observe not only finite prefixes of a trace but also whether a particular condition holds along the relevant suffixes. Within this setting, we defined the corresponding notions of Wadge reducibility and Wadge game, and showed that they coincide through a game-theoretic characterization of continuous reducibility. The key insight was that honest finite annotated prefixes determine exactly the local neighborhoods of $\tau_C$.

Beyond this specific setting, these results contribute to the development of a topological theory of monitorability across different abstract monitoring settings. This perspective also suggests several directions for future work. One natural continuation is to define the corresponding topology over sets of processes rather than sets of traces, and to examine how the resulting process-based topology relates to $\tau_C$. Another direction is to study the case where $L_1$ and $L_2$ are non-Borel sets, and to determine whether the equivalence between reducibility and winning strategies continues to hold in that setting.

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# A Appendix

This appendix contains the technical proofs omitted from the main body. We first discuss the basic topological properties of $\tau_C$, then prove the neighborhood-basis lemma, and finally establish the game-reduction equivalence theorem.

## A.1 Topological Properties of $\tau_C$

We begin by proving the standard topological properties for $\tau_C$, namely that the family generated by the subbasis $\mathcal{F}$ is indeed a topology and that the finite intersections of elements of $\mathcal{F}$ form a basis.

Let $\mathcal{F}$ be the subbasis for $\tau_C$, as it is defined in Definition 1 and let

$$\mathcal{B} = \left\{ \bigcap_{i=1}^{n} F_i \;\middle|\; n \geq 1,\ \{F_i\}_{i=1}^{n} \subseteq \mathcal{F} \right\}$$

be the corresponding basis. In the following, we show that $\tau_C$ is a topology generated by unions of elements of $\mathcal{B}$.

**Proposition 1.** *The family $\mathcal{B}$ is a basis for $\tau_C$.*

*Proof. (i)* ($\mathcal{B}$ covers $X$.) We have $X = \Sigma^\omega = \varepsilon \cdot \Sigma^\omega \in \mathcal{F}$, and therefore $X \in \mathcal{B}$. Hence, $\bigcup \mathcal{B} = X$.

*(ii)* (For every $B, B' \in \mathcal{B}$, $B \cap B'$ is a union of members of $\mathcal{B}$.) Let $B, B' \in \mathcal{B}$. Then, there exist $F_1, \ldots, F_n \in \mathcal{F}$ and $F'_1, \ldots, F'_m \in \mathcal{F}$ such that

$$B = \bigcap_{i=1}^{n} F_i, \qquad B' = \bigcap_{j=1}^{m} F'_j.$$

Therefore,

$$B \cap B' = \left( \bigcap_{i=1}^{n} F_i \right) \cap \left( \bigcap_{j=1}^{m} F'_j \right) = \bigcap_{k=1}^{n+m} F''_k,$$

where $F''_1, \ldots, F''_{n+m}$ are the sets $F_1, \ldots, F_n, F'_1, \ldots, F'_m$. Since each $F''_k \in \mathcal{F}$, it follows that $B \cap B' \in \mathcal{B}$. In particular, $B \cap B' = \bigcup \{B \cap B'\}$, so $B \cap B'$ is a union of members of $\mathcal{B}$.

Thus, both basis conditions are satisfied and hence $\mathcal{B}$ is a basis for a topology on $X$. By definition, $\tau_C = \{\bigcup A \mid A \subseteq \mathcal{B}\}$, so this topology is exactly $\tau_C$. Therefore, $\mathcal{B}$ is a basis for $\tau_C$. □

**Proposition 2.** *The collection $\tau_C = \{\bigcup A \mid A \subseteq \mathcal{B}\}$ is a topology on $\Sigma^\omega$.*

*Proof. (i)* ($\varnothing, X \in \tau_C$.) Take $A = \varnothing \subseteq \mathcal{B}$. Then, $\bigcup A = \bigcup \varnothing = \varnothing \in \tau_C$. To show that $X \in \tau_C$, note that $X = \Sigma^\omega = \varepsilon \cdot \Sigma^\omega$, and since $\varepsilon \in \Sigma^*$, it follows that $X \in \mathcal{F}$. Hence, $X \in \mathcal{B}$. Therefore, taking $A = \{X\} \subseteq \mathcal{B}$, we obtain

$$\bigcup A = \bigcup \{X\} = X \in \tau_C.$$

*(ii) (Closure under arbitrary unions.)* Let $I$ be a non-empty set and let $\{U_i\}_{i\in I} \subseteq \tau_C$ be an arbitrary family of open subsets of $X$. Since each $U_i \in \tau_C$, there exists a family $A_i \subseteq \mathcal{B}$ such that $U_i = \bigcup A_i$, for every $i \in I$. Hence,

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} \bigcup A_i = \bigcup \left( \bigcup_{i\in I} A_i \right).$$

Moreover, since each $A_i \subseteq \mathcal{B}$, we have $\bigcup_{i\in I} A_i \subseteq \mathcal{B}$. Therefore, $\bigcup_{i\in I} U_i$ is a union of basis elements, and thus $\bigcup_{i\in I} U_i \in \tau_C$.

*(iii) (Closure under finite intersections.)* Let $U_1, \ldots, U_m \in \tau_C$ with $m \geq 1$. We show that $\bigcap_{i=1}^m U_i \in \tau_C$. Since $\tau_C$ consists of unions of basis elements and $\mathcal{B}$ is a basis, for each $i \in \{1, \ldots, m\}$ there exists a family $A_i \subseteq \mathcal{B}$ such that $U_i = \bigcup_{B\in A_i} B$. Hence,

$$\bigcap_{i=1}^m U_i = \bigcap_{i=1}^m \bigcup_{B\in A_i} B.$$

We show that this intersection can be written as

$$\bigcap_{i=1}^m U_i = \bigcup_{(B_1,\ldots,B_m)\in A_1\times\cdots\times A_m} \left( \bigcap_{i=1}^m B_i \right). \tag{1}$$

($\subseteq$) Let $x \in \bigcap_{i=1}^m U_i$. Then, $x \in U_i$ for every $i$, and since $U_i = \bigcup_{B\in A_i} B$, there exists $B_i \in A_i$ such that $x \in B_i$. Thus, $x \in \bigcap_{i=1}^m B_i$, and therefore $x$ belongs to

$$\bigcup_{(B_1,\ldots,B_m)\in A_1\times\cdots\times A_m} \left( \bigcap_{i=1}^m B_i \right).$$

($\supseteq$) Conversely, let $x \in \bigcap_{i=1}^m U_i$. Then, there exist $B_i \in A_i$ such that $x \in \bigcap_{i=1}^m B_i$. Hence, $x \in B_i \subseteq U_i$ for every $i$, so $x \in \bigcap_{i=1}^m U_i$. Thus, equality (1) holds.

Finally, each set $\bigcap_{i=1}^m B_i$ is again a basis element, since every $B_i \in \mathcal{B}$ is a finite intersection of subbasic sets from $\mathcal{F}$, and the intersection of finitely many such sets is still a finite intersection of elements of $\mathcal{F}$. Hence, $\bigcap_{i=1}^m B_i \in \mathcal{B}$. Therefore, the right-hand side of (1) is a union of basis elements, so it belongs to $\tau_C$. Thus, $\bigcap_{i=1}^m U_i \in \tau_C$.

Therefore, $\tau_C$ is a topology. □

## A.2 Detailed Proof of Example 1

*Detailed proof of Example 1.* Let $\Sigma = \{a, b\}$. We consider one condition $c$, whose valuation is the set of traces containing infinitely many occurrences of $b$:

$$v(c) = \mathrm{Inf}_b = \{t \in \Sigma^\omega \mid t \text{ contains infinitely many occurrences of } b\}.$$

Thus, for a trace $t \in \Sigma^\omega$,

$$t_{\geq n} \in v(c)$$

means that the suffix starting at position $n$ contains infinitely many occurrences of $b$.

We define the condition-based language

$$L_1 = \bigcup_{s \in \Sigma^*} s \cdot a \cdot v(c).$$

Equivalently,

$$L_1 = \{t \in \Sigma^\omega \mid \exists n \in \mathbb{N} \text{ such that } t_n = a \text{ and } t_{\geq n+1} \in v(c)\}.$$

Since $v(c) = \mathrm{Inf}_b$, this means that $L_1$ contains exactly those traces in which some occurrence of $a$ is followed by infinitely many occurrences of $b$. The language $L_1$ is open in $(\Sigma^\omega, \tau_C)$, because each set $s \cdot a \cdot v(c)$ is subbasic open in $\tau_C$, and $L_1$ is a union of such sets.

Now define the language $L_2 = \Sigma^* \cdot b \cdot \Sigma^\omega$. Thus, $L_2$ contains exactly those traces over $\{a, b\}$ in which the letter $b$ eventually appears. This language is open in the Cantor topology $\tau$, since

$$L_2 = \bigcup_{s \in \Sigma^*} s \cdot b \cdot \Sigma^\omega.$$

Define $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ as follows. For an input trace $t = t_0 t_1 t_2 \cdots$, let

$$f(t) = y_0 y_1 y_2 \cdots,$$

where, for every $n \in \mathbb{N}$,

$$y_n = \begin{cases} b, & \text{if } t_n = a \text{ and } t_{\geq n+1} \in v(c), \\ a, & \text{otherwise.} \end{cases}$$

The reduction is defined so that whenever the input has an $a$ followed by a suffix containing infinitely many $b$'s, the output contains $b$ at the same position. In all other cases, the output contains $a$.

– We first show that $L_1 = f^{-1}(L_2)$.

For every $t \in \Sigma^\omega$,

$$\begin{aligned} t \in L_1 &\iff \exists n \in \mathbb{N} \text{ such that } t_n = a \text{ and } t_{\geq n+1} \in v(c) \\ &\iff \exists n \in \mathbb{N} \text{ such that } f(t)_n = b \\ &\iff f(t) \in \Sigma^* \cdot b \cdot \Sigma^\omega \\ &\iff f(t) \in L_2. \end{aligned}$$

Hence, $L_1 = f^{-1}(L_2)$.

– We show that $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ is continuous.

By definition, this means that the preimage under $f$ of every open set in the Cantor topology $\tau$ must be open in $\tau_C$. Since the sets $u \cdot \Sigma^\omega$, $u \in \Sigma^*$, form a basis for the Cantor topology, it is enough to prove that $f^{-1}(u \cdot \Sigma^\omega) \in \tau_C$, for every $u \in \Sigma^*$.

Let

$$u = u_0 u_1 \cdots u_{m-1} \in \Sigma^*.$$

If $u = \varepsilon$, then

$$f^{-1}(\varepsilon \cdot \Sigma^\omega) = f^{-1}(\Sigma^\omega) = \Sigma^\omega,$$

which is open in $\tau_C$. Hence, assume that $m \geq 1$.

A trace $t \in \Sigma^\omega$ belongs to $f^{-1}(u \cdot \Sigma^\omega)$ exactly when $f(t)$ starts with $u$. That is,

$$t \in f^{-1}(u \cdot \Sigma^\omega) \iff f(t)_i = u_i, \text{ for every } i < m.$$

For a fixed position $i < m$, define

$$B_i = \{t \in \Sigma^\omega \mid f(t)_i = b\}.$$

By the definition of $f$,

$$f(t)_i = b \iff t_i = a \text{ and } t_{\geq i+1} \in v(c).$$

This means that $t$ has the form

$$t = s \cdot a \cdot r,$$

where $s \in \Sigma^i$ is the arbitrary prefix before position $i$, the letter at position $i$ is $a$, and the suffix $r$ after that letter belongs to $v(c)$. Here, $\Sigma^i$ denotes the set of finite words over $\Sigma$ of length $i$. Hence,

$$B_i = \bigcup_{s \in \Sigma^i} s \cdot a \cdot v(c).$$

Equivalently,

$$B_i = f^{-1}(\Sigma^i \cdot b \cdot \Sigma^\omega).$$

Thus, $B_i$ is the part of the inverse image corresponding to the requirement that the output has $b$ at position $i$. Since each set $s \cdot a \cdot v(c)$ is subbasic open in $\tau_C$, the set $B_i$ is open in $\tau_C$.

Similarly, for a fixed position $i < m$, define

$$A_i = \{t \in \Sigma^\omega \mid f(t)_i = a\}.$$

The output letter at position $i$ is $a$ exactly when the condition for outputting $b$ fails. This can happen in two ways: either the input letter at position $i$ is $b$, or the input letter at position $i$ is $a$ but the suffix starting after that letter does not satisfy $c$. Therefore,

$$A_i = \left( \bigcup_{s \in \Sigma^i} s \cdot b \cdot \Sigma^\omega \right) \cup \left( \bigcup_{s \in \Sigma^i} s \cdot a \cdot (\Sigma^\omega \setminus v(c)) \right).$$

Equivalently,

$$A_i = f^{-1}(\Sigma^i \cdot a \cdot \Sigma^\omega).$$

The first union captures the inputs whose letter at position $i$ is $b$, while the second union captures the inputs whose letter at position $i$ is $a$ but whose following suffix does not satisfy $c$. Both parts are open in $\tau_C$, since they are unions of subbasic open sets. Hence, $A_i \in \tau_C$.

We now argue about the full preimage of $u \cdot \Sigma^\omega$. Since $f(t)$ begins with $u$ exactly when, for each position $i < m$, the $i^{th}$ output letter is $u_i$, we have

$$f^{-1}(u \cdot \Sigma^\omega) = \bigcap_{\substack{i<m \\ u_i = a}} A_i \cap \bigcap_{\substack{i<m \\ u_i = b}} B_i.$$

Each set $A_i$ and each set $B_i$ is open in $\tau_C$, and the above expression is a finite intersection of such open sets. Since topologies are closed under finite intersections, it follows that $f^{-1}(u \cdot \Sigma^\omega) \in \tau_C$. Thus, the preimage under $f$ of every basic Cantor-open set is open in $\tau_C$ and hence $f$ is continuous.

Consequently, $L_1 \leq_W^{\tau_C} L_2$.

□

## A.3 Neighborhoods Determined by Honest Annotated Prefixes

We prove that the neighborhoods associated with finite annotated prefixes form a neighborhood basis in $\tau_C$ along every honest annotated trace.

**Lemma 2.** *Let $t_1 \in \Sigma^\omega$, and let $\widehat{t_1} \in A(v)$. Then, the family $\left(U_{\widehat{t_1}\restriction n}\right)_{n\geq 1}$ is decreasing.*

*Proof.* If $m > n$, then $\widehat{t_1}\restriction m = ((a_0, d_0), \ldots, (a_{m-1}, d_{m-1}))$ contains all the pairs used in $\widehat{t_1}\restriction n = ((a_0, d_0), \ldots, (a_{n-1}, d_{n-1}))$ and possibly more. Therefore, the definition of $U_{\widehat{t_1}\restriction m}$ contains all the constraints appearing in $U_{\widehat{t_1}\restriction n}$, and possibly more. Hence, $U_{\widehat{t_1}\restriction m} \subseteq U_{\widehat{t_1}\restriction n}$. □

*Proof of Lemma 1 (Neighborhood basis).* Let $\widehat{t} \in A(v)$, and let $t = \pi(\widehat{t})$. Let $O \in \tau_C$ with $t \in O$. Since $\mathcal{B}$ is a basis for $\tau_C$, there exists some $B' \in \mathcal{B}$ such that $t \in B' \subseteq O$. It is therefore enough to show that $U_{\widehat{t}\restriction N} \subseteq B'$, for some $N \geq 1$.

Since $B' \in \mathcal{B}$, we may write

$$B' = \bigcap_{j=1}^{m} F_j,\ F_j \in \mathcal{F}.$$

We show that for each $j \in \{1, \ldots, m\}$ there exists $n_j$ such that $U_{\widehat{t}\restriction n} \subseteq F_j$, for every $n \geq n_j$.

We distinguish the following cases.

*Case 1:* $F_j = s \cdot \Sigma^\omega$, for some $s \in \Sigma^*$.

Since $t \in F_j$, we have $s \prec t$. Hence, whenever $n \geq |s|$, every word in $U_{\widehat{t}\restriction n}$ begins with the first $n$ letters of $t$, and therefore begins with $s$. Thus, $U_{\widehat{t}\restriction n} \subseteq s \cdot \Sigma^\omega = F_j$, for every $n \geq |s|$.

*Case 2:* $F_j = s \cdot v(c)$, for some $s \in \Sigma^*$.

Let $|s| = m$. Since $t \in s \cdot v(c)$, we have $s = t\restriction m$ and $t_{\geq m} \in v(c)$. Because $\widehat{t} \in A(v)$, the truth value at position $m$ is $T$. Therefore, for every $n > m$, $(t\restriction m) \cdot v(c) = s \cdot v(c)$ appears in the definition of $U_{\widehat{t}\restriction n}$. Hence, $U_{\widehat{t}\restriction n} \subseteq s \cdot v(c)$, for every $n > m$.

*Case 3:* $F_j = s \cdot (\Sigma^\omega \setminus v(c))$, for some $s \in \Sigma^*$.

Let $|s| = m$. Since $t \in F_j$, we have $s = t\restriction m$ and $t_{\geq m} \notin v(c)$. Because $\widehat{t} \in A(v)$, the truth value at position $m$ is $F$. Therefore, for every $n > m$, $(t\restriction m) \cdot (\Sigma^\omega \setminus v(c)) = s \cdot (\Sigma^\omega \setminus v(c))$ appears in the definition of $U_{\widehat{t}\restriction n}$. Hence, $U_{\widehat{t}\restriction n} \subseteq s \cdot (\Sigma^\omega \setminus v(c))$, for every $n > m$.

Now let $N = \max\{n_1, \ldots, n_m\}$. Then, $U_{\widehat{t}\restriction N} \subseteq F_j$, for every $j = 1, \ldots, m$. Hence,

$$U_{\widehat{t}\restriction N} \subseteq \bigcap_{j=1}^{m} F_j = B' \subseteq O.$$

Therefore, $\left(U_{\widehat{t}\restriction n}\right)_{n\geq 1}$ is a neighborhood basis at $t$. □

## A.4 Proof of the Game-Reduction Equivalence

We now give the full proof of Theorem 1.

*Proof of Theorem 1 (Game-reduction equivalence for $\tau_C$).* $(\Rightarrow)$ Assume $L_1 \leq_W^{\tau_C} L_2$. Then, there exists a continuous function $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$, such that $L_1 = f^{-1}(L_2)$.

We define a strategy

$$\sigma_{II} : \widehat{\Sigma}^+ \to \Sigma \cup \{\mathsf{skip}\}$$

for Player II on the length of the finite history.

Let $\widehat{s}_1 \in \widehat{\Sigma}^+$ be a non-empty finite play of Player I and

$$U_{\widehat{s}_1} = s_1 \cdot \Sigma^\omega \;\cap\; \bigcap_{\substack{i<n \\ d_i = T}} \big((s_1{\upharpoonright}i) \cdot v(c)\big) \;\cap\; \bigcap_{\substack{i<n \\ d_i = F}} \big((s_1{\upharpoonright}i) \cdot (\Sigma^\omega \setminus v(c))\big).$$

Since $U_{\widehat{s}_1}$ is a finite intersection of subbasis elements, we have $U_{\widehat{s}_1} \in \mathcal{B} \subseteq \tau_C$.

If $U_{\widehat{s}_1} = \varnothing$, we define

$$\sigma_{II}(\widehat{s}_1) = \mathsf{skip}.$$

Assume now that $U_{\widehat{s}_1} \neq \varnothing$. Let $p_{\widehat{s}_1} \in \Sigma^*$, be the longest common prefix of all words in $f(U_{\widehat{s}_1})$, that is,

$$f(U_{\widehat{s}_1}) \subseteq p_{\widehat{s}_1} \cdot \Sigma^\omega.$$

Let $s_2 \in \Sigma^*$ be the finite word consisting of all non-skip symbols produced by Player II so far. Define

$$\sigma_{II}(\widehat{s}_1) = \begin{cases} \text{the } (|s_2|+1)^{\text{st}} \text{ letter of } p_{\widehat{s}_1}, & \text{if } |p_{\widehat{s}_1}| > |s_2|, \\ \mathsf{skip}, & \text{if } |p_{\widehat{s}_1}| \leq |s_2|. \end{cases}$$

Thus, $\sigma_{II}$ depends only on the finite history $\widehat{s}_1$, so it is a strategy.

**Claim 1.** *Along every honest play of Player I, Player II does not play* skip *forever.*

Assume, towards a contradiction, that Player II outputs only finitely many non-skip symbols along the play $\widehat{t}_1 \in A(v)$. Then, there exists $k \in \mathbb{N}$ such that, for every $i$ large enough,

$$|p_{\widehat{s}_i}| \leq k.$$

Hence, the longest common prefixes stop growing beyond length $k$. In particular, there exists a finite word $s_k \in \Sigma^*$ of length $k$ such that, for every $i \in \mathbb{N}$,

$$f\big(U_{\widehat{t}_1{\upharpoonright}i}\big) \subseteq s_k \cdot \Sigma^\omega,$$

and no proper extension of $s_k$ is a common prefix of all words in $f\big(U_{\widehat{t}_1{\upharpoonright}i}\big)$. Therefore, there exist two distinct extensions of $s_k$, namely $s_k \cdot a \cdot \Sigma^\omega$ and $s_k \cdot b \cdot \Sigma^\omega$, where $a, b \in \Sigma$ and $a \neq b$, such that for every $i \in \mathbb{N}$,

$$f\big(U_{\widehat{t}_1{\upharpoonright}i}\big) \cap s_k \cdot a \cdot \Sigma^\omega \neq \varnothing \quad \text{and} \quad f\big(U_{\widehat{t}_1{\upharpoonright}i}\big) \cap s_k \cdot b \cdot \Sigma^\omega \neq \varnothing.$$

Define the disjoint open sets $A = s_k \cdot a \cdot \Sigma^\omega$ and $B = s_k \cdot b \cdot \Sigma^\omega$. Then, for every $i \in \mathbb{N}$,

$$U_{\widehat{t}_1{\upharpoonright}i} \cap f^{-1}(A) \neq \varnothing \quad \text{and} \quad U_{\widehat{t}_1{\upharpoonright}i} \cap f^{-1}(B) \neq \varnothing. \tag{2}$$

Since $f(t_1) \in s_k \cdot \Sigma^\omega$, it must belong to exactly one of $A$ or $B$. Without loss of generality, assume that $f(t_1) \in A$. Since $f$ is continuous and $A$ is open in $\tau$, the set $f^{-1}(A)$ is open in $\tau_C$. Moreover, because $f(t_1) \in A$, we have $t_1 \in f^{-1}(A)$. Hence, $f^{-1}(A)$ is an open neighborhood of $t_1$.

By Lemma 1, the family $\big(U_{\widehat{t}_1{\upharpoonright}i}\big)_{i \geq 1}$ is a neighborhood basis at $t_1$. Applying the lemma with $O = f^{-1}(A)$, there exists $n \geq 1$ such that

$$U_{\widehat{t}_1{\upharpoonright}n} \subseteq f^{-1}(A).$$

It follows that $f\big(U_{\widehat{t}_1{\upharpoonright}n}\big) \subseteq A$. Since $A \cap B = \varnothing$, we conclude that $U_{\widehat{t}_1{\upharpoonright}n} \cap f^{-1}(B) = \varnothing$. But this contradicts (2), since we assumed it holds for every $i \in \mathbb{N}$.

Therefore, Player II cannot output only finitely many letters. Hence, she cannot play skip forever. $\dashv$

**Claim 2.** *$\sigma_{II}$ is a winning strategy.*

Let $\widehat{t}_1 \in \widehat{\Sigma}^\omega$ be any play of Player I, and let $t_2$ be the resulting play of Player II according to $\sigma_{II}$.

If $\widehat{t}_1 \notin A(v)$, then Player II wins immediately by the first case of the winning condition.

Assume $\widehat{t}_1 \in A(v)$, and let $t_1 = \pi(\widehat{t}_1)$. Since $L_1 = f^{-1}(L_2)$, it is enough to show that

$$t_2 = f(t_1).$$

For every $i \geq 1$, define $\widehat{s}_i = \widehat{t}_1 \upharpoonright i$. Let $s_2^{(i)} \in \Sigma^*$ be the finite word obtained by taking all Player II outputs during the first $i$ rounds. Since for every $i \geq 1$,

$$U_{\widehat{s}_{i+1}} \subseteq U_{\widehat{s}_i},$$

we have

$$f(U_{\widehat{s}_{i+1}}) \subseteq f(U_{\widehat{s}_i}),$$

which implies

$$p_{\widehat{s}_{i+1}} \cdot \Sigma^\omega \subseteq p_{\widehat{s}_i} \cdot \Sigma^\omega.$$

Therefore,

$$p_{\widehat{s}_i} \preceq p_{\widehat{s}_{i+1}}, \text{ for every } i \geq 1.$$

Moreover, by construction of $\sigma_{II}$, we have $s_2^{(i)} \preceq p_{\widehat{s}_i}$, for every $i \geq 1$. By the **Claim 1**, Player II does not play skip forever along the play $\widehat{t}_1 \in A(v)$. Hence, the sequence $\big(s_2^{(i)}\big)_{i\geq 1}$ is increasing and unbounded, and therefore determines an infinite word

$$t_2 = \lim_{i\to\infty} s_2^{(i)} \in \Sigma^\omega.$$

We prove that $t_2 = f(t_1)$. To do so, we show that for every $m \geq 1$, $t_2 \upharpoonright m = f(t_1) \upharpoonright m$. Fix $m \geq 1$. Then, $f(t_1) \upharpoonright m \cdot \Sigma^\omega$ is an open neighborhood of $f(t_1)$ in $(\Sigma^\omega, \tau)$. Since $f$ is continuous, the set $f^{-1}\big(f(t_1) \upharpoonright m \cdot \Sigma^\omega\big)$ is an open neighborhood of $t_1$ in $(\Sigma^\omega, \tau_C)$. By Lemma 1, there exists $n \geq 1$ such that

$$U_{\widehat{t}_1\upharpoonright n} \subseteq f^{-1}\big(f(t_1) \upharpoonright m \cdot \Sigma^\omega\big).$$

Hence,

$$f\big(U_{\widehat{t}_1\upharpoonright n}\big) \subseteq f(t_1) \upharpoonright m \cdot \Sigma^\omega,$$

so every word in $f\big(U_{\widehat{t}_1\upharpoonright n}\big)$ has prefix $f(t_1) \upharpoonright m$. Therefore, $f(t_1) \upharpoonright m \preceq p_{\widehat{t}_1\upharpoonright n}$. Since the sequence $\big(p_{\widehat{s}_i}\big)_{i\geq 1}$ is increasing, it follows that

$$f(t_1) \upharpoonright m \preceq p_{\widehat{s}_i}, \text{ for every } i \geq n.$$

On the other hand, the sequence $\big(s_2^{(i)}\big)_{i\geq 1}$ has unbounded length, because Player II does not skip forever. Hence, there exists $i \geq n$ such that $|s_2^{(i)}| \geq m$. For this $i$, we have $s_2^{(i)} \preceq p_{\widehat{s}_i}$ and $f(t_1) \upharpoonright m \preceq p_{\widehat{s}_i}$, with

$$|f(t_1) \upharpoonright m| = m \leq |s_2^{(i)}|.$$

Hence, the first $m$ letters of $s_2^{(i)}$ must agree with the prefix $f(t_1) \upharpoonright m$. Thus, $s_2^{(i)} \upharpoonright m = f(t_1) \upharpoonright m$. Therefore,

$$t_2 \upharpoonright m = s_2^{(i)} \upharpoonright m = f(t_1) \upharpoonright m.$$

Since $m \geq 1$ was arbitrary, we conclude that $t_2 = f(t_1)$. Finally, since $L_1 = f^{-1}(L_2)$, we obtain

$$t_1 \in L_1 \iff f(t_1) \in L_2 \iff t_2 \in L_2.$$

Thus,

$$\pi(\widehat{t_1}) = t_1 \in L_1 \iff t_2 \in L_2.$$

Therefore, Player II satisfies the second case of the winning condition whenever $\widehat{t_1} \in A(v)$. Combining this with the case $\widehat{t_1} \notin A(v)$, we conclude that $\sigma_{II}$ is a winning strategy for Player II in $WG_{\tau_C}(L_1, L_2)$. ⊣

This concludes the proof of the forward direction.

(⇐) Assume Player II has a winning strategy

$$\sigma_{II} : (\Sigma \times \{T, F\})^+ \to \Sigma \cup \{\mathsf{skip}\}$$

in $WG_{\tau_C}(L_1, L_2)$.

For each $t_1 \in \Sigma^\omega$, let $\widehat{t_1} \in A(v)$ and $\pi(\widehat{t_1}) = t_1$. Since $\sigma_{II}$ is a winning strategy and $\widehat{t_1} \in A(v)$, the first case of the winning condition does not apply. Hence, Player II must win by the second case, and therefore produces an infinite output word $t_2 \in \Sigma^\omega$. We define $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ such that $f(t_1) = t_2$. We show that $f$ is continuous and that $L_1 = f^{-1}(L_2)$.

*(i) We first show that $L_1 = f^{-1}(L_2)$.*

Fix $t_1 \in \Sigma^\omega$, and let $\widehat{t_1} \in A(v)$. Since $\sigma_{II}$ is a winning strategy and $\widehat{t_1} \in A(v)$, the winning condition of the game gives

$$\pi(\widehat{t_1}) \in L_1 \iff t_2 \in L_2,$$

where $t_2$ is the word produced by Player II. By definition of $\widehat{t_1}$ and $f$, we have

$$\pi(\widehat{t_1}) = t_1 \qquad \text{and} \qquad t_2 = f(t_1).$$

Therefore,

$$t_1 \in L_1 \iff f(t_1) \in L_2.$$

Since the equivalence holds for any $t_1 \in \Sigma^\omega$, we conclude that $L_1 = f^{-1}(L_2)$.

*(ii) We now show that $f : (\Sigma^\omega, \tau_C) \to (\Sigma^\omega, \tau)$ is continuous.*

We show that the preimage of every open set in $\tau$ is $\tau_C$-open. Let $u \cdot \Sigma^\omega$, with $u \in \Sigma^*$ be an open set in $(\Sigma^\omega, \tau)$, and let

$$t_1 \in f^{-1}(u \cdot \Sigma^\omega).$$

Then, $f(t_1) \in u \cdot \Sigma^\omega$, i.e.

$$u \prec f(t_1).$$

Let $\widehat{t_1} \in A(v)$ and consider the play in which Player I plays $\widehat{t_1}$ and Player II follows $\sigma_{II}$. Since $f(t_1)$ begins with $u$, there exists some $n \geq 1$ such that after the first $n$ steps, the sequence of letters produced by Player II has $u$ as a prefix.

Let

$$\widehat{t_1}\restriction n = ((a_0, d_0), \ldots, (a_{n-1}, d_{n-1})),$$

and

$$s = a_0 a_1 \cdots a_{n-1}, \quad s\restriction i = a_0 a_1 \cdots a_{i-1}, \quad \text{for } i < n.$$

Define

$$U_{\widehat{t_1}\restriction n} = s \cdot \Sigma^\omega \;\cap\; \bigcap_{\substack{i<n \\ d_i = T}} \big((s\restriction i) \cdot v(c)\big) \;\cap\; \bigcap_{\substack{i<n \\ d_i = F}} \big((s\restriction i) \cdot (\Sigma^\omega \setminus v(c))\big).$$

By construction, $U_{\widehat{t_1}\restriction n} \in \mathcal{B} \subseteq \tau_C$, and because $\widehat{t_1} \in A(v)$, we have $t_1 \in U_{\widehat{t_1}\restriction n}$. We will show that $U_{\widehat{t_1}\restriction n} \subseteq f^{-1}(u \cdot \Sigma^\omega)$.

Let $x \in U_{\widehat{t_1}\restriction n}$, and let $\widehat{x} \in A(v)$ be the honest annotation of $x$. Then, $\widehat{x}$ has the same first $n$ letters as $\widehat{t_1}$, that is

$$\widehat{x}\restriction n = \widehat{t_1}\restriction n.$$

Since $\sigma_{II}$ is a strategy, the moves it suggests depend only on the finite history of Player I. Therefore, against the play $\widehat{x}$, Player II makes exactly the same moves during the first $n$ steps as she did against $\widehat{t_1}$.

But against $\widehat{t_1}$, by choice of $n$, those first $n$ steps already produce an output with prefix $u$. Therefore, against $\widehat{x}$, the first $n$ steps produce the same output prefix $u$. Hence,

$$f(x) \in u \cdot \Sigma^\omega.$$

This gives that $x \in f^{-1}(u \cdot \Sigma^\omega)$, and therefore

$$U_{\widehat{t_1}\restriction n} \subseteq f^{-1}(u \cdot \Sigma^\omega).$$

So we have shown that for every $t_1 \in f^{-1}(u \cdot \Sigma^\omega)$ ($t_1$ was arbitrary), there exists a basic $\tau_C$-open set $U_{\widehat{t_1}\restriction n}$ such that

$$t_1 \in U_{\widehat{t_1}\restriction n} \subseteq f^{-1}(u \cdot \Sigma^\omega).$$

Hence, $f^{-1}(u \cdot \Sigma^\omega)$ is $\tau_C$-open.

Therefore, $f$ is continuous and this completes the backward direction.

□