An elementary proof of the Lusin-Novikov theorem

Forte Shinko

University of California, Berkeley CA, USA
forteshinko@berkeley.edu

Abstract

We present an elementary proof of the Lusin-Novikov uniformization theorem in descriptive set theory, which in particular works entirely in the classical (boldface) setting and does not use analytic sets.

> At the end of the day, we are all trying to find a Cantor set.
>
> — Անուշ Ծերունյան

1 Introduction

The Lusin-Novikov uniformization theorem, which is a Borel analogue of the axiom of choice for families of countable sets, is one of the most ubiquitous tools in modern descriptive set theory. It is used in virtually every paper concerning countable Borel equivalence relations (see [Kec25]) and the Borel combinatorics of locally countable graphs (see [KM]), which have many connections and applications to areas outside of logic such as ergodic theory. Despite this, the complexity of its proof has resulted in it being inaccessible to non-experts, as well as being omitted from introductory courses in descriptive set theory. More precisely, all proofs of the Lusin-Novikov theorem in the literature use (co)analytic sets (or Π^1_1 sets in the effective setting), and moreover rely on other key results in descriptive set theory: the classical (boldface) proof in [Kec95, 18.10] relies on the coanalyticity of the set of unicity, the classical proof in [Mil12, Theorem 32] relies on the $\mathbf{G}_0$-dichotomy, and the effective proof (see e.g. [Mos09, 4F.6]) relies on the entire effective theory and nontrivial results therein such as the effective perfect set theorem.

Our main contribution is a direct and concise proof of the Lusin-Novikov theorem which is at the same time elementary: it only uses Borel sets and does not rely on any other results from descriptive set theory. We hope therefore that this note will make the theorem more accessible to both students and non-experts alike.

We will prove the Lusin-Novikov uniformization theorem in the following form, where for a function $f : X \to Y$, a **partial f-transversal** is a subset $A \subseteq X$ such that $f \restriction A$ is injective:

Lusin-Novikov theorem. *Let $f : X \to Y$ be a continuous map of Polish spaces. Then exactly one of the following holds:*

(1) There is a countable cover of X by Borel partial f-transversals.

(2) Some fiber of f contains a Cantor set (a subspace homeomorphic to $2^{\mathbb{N}}$).

Our proof can be seen as a generalization of the proof of Cantor's perfect set theorem [Kec95, 6.5], which is the special case when Y is a point.

The Lusin-Novikov theorem is often stated with the additional condition that the image is Borel when every fiber is countable (see e.g. [Kec95, 18.10]). Of course, this version follows from ours via from the Lusin-Suslin theorem; in the spirit of exposition, we sketch a particularly modular proof of the latter due to Ruiyuan (Ronnie) Chen.

Lusin-Suslin theorem. *Every injective Borel map of standard Borel spaces has Borel image.*

Proof sketch [Che]. It suffices to show that every monomorphism with non-empty domain is split, since if $r \circ i$ is the identity, then the image of i is the set of fixed points of $i \circ r$. But this is a direct consequence of the following facts:

- In every category,
  - every monomorphism with injective domain is split;
  - the injective objects are closed under products and retracts.
- In the category of non-empty standard Borel spaces and Borel maps,
  - the object 2 is injective (immediate from Lusin's separation theorem [Kec95, 14.7]);
  - every object is a retract of $2^{\mathbb{N}}$ (immediate from the existence of a Borel embedding into $2^{\mathbb{N}}$, see second remark in [Che]).

□

2 Deficiency

Let $f : X \to Y$ be a Borel map of standard Borel spaces.

A Borel subset of X is ***f*-deficient** if is a countable union of Borel partial f-transversals.

We extend deficiency to families of sets. A finite disjoint family $\mathcal{F}$ of Borel subsets of X is ***f*-deficient** if there is a Borel cover $(V_A)_{A\in\mathcal{F}}$ of Y such that for all $A \in \mathcal{F}$, we have that $A \cap f^{-1}(V_A)$ is f-deficient. Note that we can always pass to subsets to turn the cover into a partition. Note also that $A \subseteq X$ is f-deficient iff $\{A\}$ is f-deficient.

We provide some intuition for deficiency. To prove Lusin-Novikov, we will attempt to construct a Cantor scheme which produces a Cantor set inside a single fiber. Suppose that we have constructed finitely many levels of a Cantor scheme, with $\mathcal{F}$ being the most recent level. One way that this Cantor scheme could fail to produce the desired Cantor set is if every fiber has countable intersection with some $F \in \mathcal{F}$. So f-deficiency of $\mathcal{F}$ says not only that this failure occurs, but that the failure is witnessed in a "uniform Borel manner".

We will need two lemmas for the construction of the Cantor scheme, the first of which is the crux of the argument. We define $\Delta_X := \{(x, x') \in X^2 : x = x'\}$.

Lemma 1. *Let $\mathcal{F}$ be a finite disjoint family of Borel subsets of X. Suppose that there are $A \in \mathcal{F}$ and Borel subsets $(B_n)_{n\in\mathbb{N}}$ and $(C_n)_{n\in\mathbb{N}}$ of X with $A^2 \setminus \Delta_X = \bigcup_{n\in\mathbb{N}} B_n \times C_n$, such that for all $n \in \mathbb{N}$, we have that $(\mathcal{F} \setminus \{A\}) \cup \{B_n, C_n\}$ is f-deficient. Then $\mathcal{F}$ is f-deficient.*

Proof. Let $\mathcal{F}' = \mathcal{F} \setminus \{A\}$. For every $n \in \mathbb{N}$, fix a Borel cover $(V_A)_{A\in\mathcal{F}'} \cup \{V_{B_n}, V_{C_n}\}$ of Y witnessing f-deficiency of $\mathcal{F}' \cup \{B_n, C_n\}$; we can use the same V_A for every n, since if each n had its own V_A^n, we could replace them with $V_A := \bigcup_n V_A^n$. By passing to subsets, we can assume that this cover is a partition. Set $V_A = Y \setminus \bigsqcup_{A\in\mathcal{F}'} V_A$. Note that for every $n \in \mathbb{N}$, we have $V_A = V_{B_n} \sqcup V_{C_n}$. It remains to show that $A \cap f^{-1}(V_A)$ is f-deficient; for this, it suffices to show that

$$A \cap f^{-1}(V_A) \setminus \left(\bigcup_{n\in\mathbb{N}} \left(B_n \cap f^{-1}(V_{B_n})\right) \sqcup \left(C_n \cap f^{-1}(V_{C_n})\right) \right) \tag{*}$$

is a partial f-transversal. Let $x_B, x_C \in A \cap f^{-1}(V_A)$ be distinct. Then there is some $n \in \mathbb{N}$ for which $x_B \in B_n$ and $x_C \in C_n$. If $f(x_B) = f(x_C)$, then since this is in V_A, we have either $f(x_B) \in V_{B_n}$ or $f(x_C) \in V_{C_n}$, so either $x_B \in B_n \cap f^{-1}(V_{B_n})$ or $x_C \in C_n \cap f^{-1}(V_{B_n})$, and thus at most one of x_B and x_C can be in the set (*). So it is a partial f-transversal. □

Lemma 2. *Let $\mathcal{F}$ be a finite disjoint family of Borel sets. Suppose that there is a Borel cover $(W_n)_{n\in\mathbb{N}}$ of Y such that for every $n \in \mathbb{N}$, the family $\{A \cap f^{-1}(W_n)\}_{A\in\mathcal{F}}$ is f-deficient. Then $\mathcal{F}$ is f-deficient.*

Proof. For each $n \in \mathbb{N}$, fix a Borel cover $(V_A^n)_{A\in\mathcal{F}}$ of Y witnessing f-deficiency of $(A \cap f^{-1}(W_n))_{F\in\mathcal{F}}$. For each $A \in \mathcal{F}$, define $V_A := \bigcup_n (W_n \cap V_A^n)$. Then the Borel cover $(V_A)_{A\in\mathcal{F}}$ of Y witnesses f-deficiency of $\mathcal{F}$. □

3 Proof of Lusin-Novikov

Let $f : X \to Y$ be a continuous map of Polish spaces. Fix compatible complete metrics on X and Y. We will make use of the following observations:

(1) For every $\varepsilon > 0$ and every closed $A \subseteq X$, there are closed subsets $(B_n)_n$ and $(C_n)_n$ of X of diameter at most ε such that $A^2 \setminus \Delta_X = \bigcup_n B_n \times C_n$.

(2) For every $\varepsilon > 0$, there is a countable cover of Y by closed sets of diameter at most ε.

Now suppose that there is no countable cover of X by Borel partial f-transversals, i.e that $\{X\}$ is not f-deficient. We will construct non-empty closed subsets $(A_s)_{s\in 2^{<\mathbb{N}}}$ of X such that the following conditions hold:

(i) For all $s, t \in 2^{\mathbb{N}}$ with $s \prec t$, we have $A_s \supseteq A_t$.

(ii) For all $s, t \in 2^{\mathbb{N}}$ with $s \perp t$, we have that A_s and A_t are disjoint.

(iii) For all non-empty $s \in 2^{<\mathbb{N}}$, we have $\operatorname{diam}(A_s) \leq 2^{-|s|}$.

(iv) For all $n > 0$, we have

$$\operatorname{diam}\left(f\left(\bigcup_{|s|=n} A_s\right)\right) \leq \frac{1}{n}.$$

We will proceed by recursion on $|s|$; to ensure that the recursion works, we will additionally require for all $n \in \mathbb{N}$ that the family $\{A_s\}_{|s|=n}$ is not f-deficient.

We start the recursion with $A_\varnothing := X$.

Now suppose we have already constructed $(A_s)_{|s|\leq n}$. By applying Lemma 1 and the first observation, we obtain closed disjoint subsets $B_{0^n\frown 0}$ and $B_{0^n\frown 1}$ of A_{0^n} of diameter at most $2^{-(n+1)}$, such that the family $\{A_s\}_{|s|=n \text{ and } s\neq 0^n} \cup \{B_{0^n\frown 0}, B_{0^n\frown 1}\}$ is not f-deficient. By applying Lemma 1 and the first observation $2^n - 1$ more times (once for each A_s), we obtain a family $\{B_s\}_{|s|=n+1}$ satisfying the first three conditions which is not f-deficient. Then by applying Lemma 2 and the second observation, we obtain a family $\{A_s\}_{|s|=n+1}$ satisfying all four conditions which is not f-deficient. Note that the sets are non-empty since the family is not f-deficient.

This concludes the recursive construction.

Now we are done: the decreasing intersection $\bigcap_{n\in\mathbb{N}}\bigsqcup_{|s|=n} A_s$ is a Cantor set, and it lies in a single fiber, since its image has diameter 0:

$$\begin{aligned}
&\operatorname{diam}\left(f\left(\bigcap_{n\in\mathbb{N}}\bigsqcup_{|s|=n} A_s\right)\right)\\
&= \operatorname{diam}\left(\bigcap_{n\in\mathbb{N}} f\left(\bigsqcup_{|s|=n} A_s\right)\right)\\
&\leq \inf_{n\in\mathbb{N}} \operatorname{diam}\left(f\left(\bigsqcup_{|s|=n} A_s\right)\right)\\
&\leq \inf_{n\in\mathbb{N}} \frac{1}{n}\\
&= 0
\end{aligned}$$

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