Some developments in the theory of ultrafilters on countable sets

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Abstract

We present some new developments in the theory of ultrafilters by outlining the main results of [5] and [8]. These include new consistency statements on the number of Q-point ultrafilters, as well as an investigation of a new class of ultrafilters, called *Laver ultrafilters*.

1 Introduction

Ultrafilters have played a major role in combinatorial set theory and many other areas of mathematical logic for a long time. Although the usual construction of an ultrafilter on ω via Zorn's lemma suffice for many interesting well-known applications, the construction and existence of special ultrafilters with specific properties have turned out to be a very rich line of research.

One of the most natural examples of such properties is the notion of *Ramsey ultrafilters*, i.e., those ultrafilters that contain a witness for every instance of Ramsey's theorem.

Definition 1.1. An ultrafilter $\mathcal{U}$ on ω is called *Ramsey* if for any coloring $c : [\omega]^2 \to 2$, there is $X \in \mathcal{U}$ such that $c \restriction [X]^2$ is constant.

It is not hard to check that this guarantees the existence of witnesses $X \in \mathcal{U}$ for any coloring $c : [\omega]^n \to r$, where n and r are positive integers. It turns out that the property of being Ramsey is equivalent to the conjunction of two other properties.

Definition 1.2. Let $\mathcal{U}$ be an ultrafilter on ω.

1. $\mathcal{U}$ is called a *P-point* if for all $\{X_n : n \in \omega\} \subseteq \mathcal{U}$, there is $X \in \mathcal{U}$ such that $X \subseteq^* X_n$ for all n, where $A \subseteq^* B$ means $A \setminus B$ is finite.

2. $\mathcal{U}$ is called a *Q-point* if for every sequence of finite and pairwise disjoint intervals $\{I_n\}_{n\in\omega}$ with $\bigcup_{n\in\omega} I_n = \omega$, there is $X \in \mathcal{U}$ such that $|X \cap I_n| \leq 1$ for all $n \in \omega$.

Fact 1.3. $\mathcal{U}$ is Ramsey if and only if it is both a P-point and a Q-point.

Although all of these notions of ultrafilters can be constructed via transfinite recursion under the continuum hypothesis, it turns out that the existence of any of the preceding three notions of ultrafilters is independent of ZFC. Kunen [9] has shown that there are no Ramsey ultrafilters in the model obtained by adding $\aleph_2$ many random reals to a model of ZFC + CH. Later, Miller [11] showed that there are no Q-points in Laver's model for the Borel conjecture [10], and then Shelah [13] constructed a model of set theory in which there are no P-points.

2 There may be exactly n Q-points

Another question that can be asked regarding these notions is whether we can (consistently) control the number of such ultrafilters, upto a suitable notion of isomorphism. We now introduce the well-studied *Rudin-Keisler ordering* on ultrafilters:

Definition 2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on the countable sets I and J, respectively.

1. Let $f : I \to J$ be a map. We define $f^*(\mathcal{U}) := \{Y \subseteq J : f^{-1}(Y) \in \mathcal{U}\}$, an ultrafilter on J.
2. We write $\mathcal{U} \geq_{RK} \mathcal{V}$ if there is a map $f : I \to J$ with $f^*(\mathcal{U}) = \mathcal{V}$.
3. We write $\mathcal{U} \cong \mathcal{V}$ if both $\mathcal{U} \geq_{RK} \mathcal{V}$ and $\mathcal{V} \geq_{RK} \mathcal{U}$ hold.

$\cong$ is the relevant notion of isomorphism for ultrafilters. It is well-known that $\mathcal{U} \cong \mathcal{V}$ holds if and only if there is an injection $f : I \to J$ with $f^*(\mathcal{U}) = \mathcal{V}$.

It is a classical result of Shelah [13] that, consistently, there may be exactly n many Ramsey ultrafilters upto isomorphism, where n is a natural number (in fact, this number can be any cardinal $0 \leq \kappa \leq \omega_2$). It is again a result of Shelah [13] that there may be a unique P-point. Until recently, the problem of controlling the number of Q-points remained open.

Very recently, Halbeisen, Horvath and Shelah [6] constructed a model in which there is a unique Q-point. Their construction started with a ground model $V \vDash \mathrm{ZFC} + \mathrm{CH}$ and with a Ramsey ultrafilter $\mathcal{U} \in V$. They used *Mathias forcing relativized to* $\mathcal{U}$, denoted $\mathbb{M}_{\mathcal{U}}$, to iteratively pseudo-intersect and reconstruct Ramsey ultrafilters extending $\mathcal{U}$, in a countable support iteration of length ω_2.

Definition 2.2. $\mathbb{M}_{\mathcal{U}}$ is the forcing notion consisting of pairs $\langle s, A\rangle$, where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{U}$. We define the ordering as $\langle s, A\rangle \leq \langle t, B\rangle$ if and only if $s \sqsupseteq t$, $s \setminus t \subseteq B$, and $A \subseteq B$. We denote the generic introduced by $\mathbb{M}_{\mathcal{U}}$ by $g_{\mathcal{U}}$, and it satisfies $g_{\mathcal{U}} \subseteq^* X$ for every $X \in \mathcal{U}$.

We generalized this result as follows:

Theorem 2.3 ([5])**.** *Let n be a positive integer. It is consistent that there are exactly n many Q-points. Furthermore, the case of $n = 2$ can also be obtained by using Matet-Mathias forcing $\mathbb{MM}(\mathcal{H})$ restricted to a suitable Matet-adequate family $\mathcal{H} \subseteq \mathrm{FIN}^{[\infty]}$.*

The result for general n is accomplished via *higher dimensional relativized Mathias forcing*, first introduced by Shelah and Spinas [14].

Definition 2.4. Let $\vec{\mathcal{U}} = \langle \mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_{n-1}\rangle$ be a finite sequence of non-isomorphic Ramsey ultrafilters. Let $\mathbb{M}(\vec{\mathcal{U}})$ consist of conditions $\langle s, \vec{X}\rangle$, where $s \in [\omega]^{<\omega}$, and $\vec{X} \in \prod_{k\in n} \mathcal{U}_k \cap [\omega \setminus (\max(s) + 1)]^{\omega}$.

If $\langle m_0, m_1, ..., m_{|s|-1}\rangle$ is the increasing enumeration of s, we denote by $s_k \in [\omega]^{<\omega}$ the set $s_k := \{m_i : i \equiv k \mod n\}$. We let $\langle s, \vec{X}\rangle \leq \langle t, \vec{Y}\rangle$ if

1. $s \supseteq t$,
2. $\forall k \in n : \vec{X}(k) \subseteq \vec{Y}(k)$,
3. $\forall k \in n : s_k \setminus t_k \subseteq \vec{Y}(k)$.

The case of $n = 2$ can also be obtained by iterating a suitable variant of *Matet-Mathias forcing* $\mathbb{MM}$, introduced by García Ávila [4]. This forcing uses the *topological Ramsey space* $\mathrm{FIN}^{[\infty]}$ and as a consequence of Hindman's theorem, it achieves useful properties like pure decision, properness, and Laver property.

Definition 2.5. We define $\mathrm{FIN} := [\omega]^{<\omega} \setminus \{\varnothing\}$. For $s, t \in \mathrm{FIN}$, we write $s <_b t$ if $\max(s) < \min(t)$. A sequence with values in FIN is a *block sequence* if it is $<_b$-increasing. $\mathrm{FIN}^{[\infty]}$ denotes the set of infinite block sequences. For $X \in \mathrm{FIN}^{[\infty]}$, we define $[X] := \{X(n_0) \cup \ldots X(n_k) : n_0 < \ldots < n_k \in \omega, 0 < k \in \omega\}$. For two block sequences X and Y, $X \leq Y$ holds if $X(n) \in [Y]$ for all $n \in \omega$.

The following well-known theorem of Hindman makes this space interesting:

Theorem 2.6 ([7])**.** *For any coloring* $c : \mathrm{FIN} \to 2$*, there is* $X \in \mathrm{FIN}^{[\infty]}$ *such that* $c \restriction [X]$ *is constant.*

The forcing we use in the iteration is the following, where $\mathcal{H} \subseteq \mathrm{FIN}^{[\infty]}$ is a carefully chosen family:

Definition 2.7. $\mathbb{MM}(\mathcal{H})$ consists of pairs $\langle a, X\rangle$, where a is a finite block sequence and $a <_b X \in \mathcal{H}$. $\langle a, X\rangle \leq \langle b, Y\rangle$ holds if $a \sqsupseteq b$, $a \setminus b \leq Y$, and $X \leq Y$.

3 Laver ultrafilters

As the previous section suggests, considering the relativized versions of well-known forcing notions can be very fruitful. Another example of these forcing notions is *Laver forcing* $\mathbb{L}$, first used by Laver [10] to prove the consistency of Borel's conjecture.

$\mathbb{L}$ consists of everywhere infinitely branching trees $T \subseteq \omega^{<\omega}$, ordered by reverse inclusion. One of its most important properties is the *Laver property*.

Definition 3.1. A forcing notion $\mathbb{P}$ has the *Laver property* if the following holds: Assume that $\underset{\sim}{f}$ is a $\mathbb{P}$-name for an element of ω^ω such that there exists $g \in \omega^\omega$ with $\Vdash_{\mathbb{P}} \underset{\sim}{f} < g$. Then, $\mathbb{P}$ forces that there exists some $S \in \prod_{n\in\omega}[g(n)]^{\leq n+1}$ in the ground model with $\underset{\sim}{f}(n) \in S(n)$ for each $n \in \omega$.

Laver property provides strong control on the reals added by the forcing. Furthermore, it prevents the addition of Cohen and random reals, and additionally it is preserved under countable support iterations of proper forcing notions.

For an ultrafilter $\mathcal{U}$ on ω, the relativized variant of $\mathbb{L}$ is defined as follows:

Definition 3.2. $\mathbb{L}_{\mathcal{U}}$ is the forcing notion consisting of trees $T \subseteq \omega^{<\omega}$ such that for each $s \in T$ with $s \supseteq \mathrm{stem}(T)$,

$$\mathrm{succ}_T(s) := \{n \in \omega : s^\frown n \in T\} \in \mathcal{U},$$

ordered by reverse inclusion (here, $\mathrm{stem}(T)$ is the longest node of T which is compatible with every other node of T).

It is a natural question to ask for which ultrafilters $\mathcal{U}$ on the natural numbers $\mathbb{L}_{\mathcal{U}}$ continues to enjoy the Laver property. A partial answer to this question to this question was given by Błaszczyk and Shelah [2], who showed that $\mathbb{L}_{\mathcal{U}}$ does not add Cohen reals if and only if $\mathcal{U}$ is *nowhere dense*. Other work on this topic the very recent paper of Nieto-de la Rosa, Guzmán and Ramos-García [12], who independently explored a closely related research direction.

In our paper [8], we provide simple combinatorial conditions for ultrafilters $\mathcal{U}$ that are equivalent to $\mathbb{L}_{\mathcal{U}}$ having the Laver property.

Theorem 3.3 ([8])**.** *Let $\mathcal{U}$ be an ultrafilter over ω. Then $\mathbb{L}_{\mathcal{U}}$ has the Laver property if and only if for every sequence $\langle \mathcal{P}_n : n \in \omega \rangle$, where each $\mathcal{P}_n$ is a partition of ω into finitely many sets, there exists $X \in \mathcal{U}$ such that for all $n \in \omega$, X has non-empty intersection with at most $n+1$ elements of $\mathcal{P}_n$.*

We call ultrafilters that satisfy this *Laver ultrafilters*. In addition to this condition, we also found a family of ideals $\mathcal{I}_f$ on $2^{<\omega}$ that characterize Laverness in Baumgartner's $\mathcal{I}$-ultrafilter framework:

Definition 3.4 ([1])**.** Let $\mathcal{I}$ be an ideal on a set A. $\mathcal{U}$ is called an *$\mathcal{I}$-ultrafilter* if for all $f : \omega \to A$, there is $X \in \mathcal{U}$ such that $f''X \in \mathcal{I}$.

As a result of this, we were able to compare Laver ultrafilters to other $\mathcal{I}$-ultrafilters for various ideals $\mathcal{I}$, and proved separation results under weak fragments of Martin's axiom.

We used these findings to establish bounds on the *generic existence number* of Laver ultrafilters. Denoted by $\mathfrak{ge}$(Laver), this cardinal invariant of the continuum was defined by Brendle and Flašková [3] to characterize the *generic existence* of certain families of ultrafilters defined via ideals. This number has the following property:

Proposition 3.5. $\mathfrak{ge}$(Laver) $= \mathfrak{c}$ holds if and only if every filter base $\mathcal{F}$ of size $< \mathfrak{c}$ can be extended to a Laver ultrafilter.

We established the following bounds on $\mathfrak{ge}$(Laver):

Theorem 3.6 ([8])**.**

$$cov(\mathcal{M}), non(\mathcal{NA}) \leq \mathfrak{ge}(Laver) \leq non(\mathcal{SN}), \max\{non(\mathcal{E}), \mathfrak{d}\}.$$

Here, $\mathcal{M}$ is the ideal of meager sets, $\mathcal{NA}$ is the ideal of null-additive sets, $\mathcal{SN}$ is the ideal of strong measure zero sets, and $\mathcal{E}$ is the σ-ideal generated by closed nowhere dense sets.

Although Laverness is implied by being a *rapid* P-point (an ultrafilter is rapid if the collection of the enumeration functions of the sets in $\mathcal{U}$ is a dominating family), the converse fails on a high scale. One way of showing this is to construct models of ZFC that contain one type of the ultrafilters, exclude the other. We proved the following results:

Theorem 3.7 ([8])**.** *There are no Laver ultrafilters in the model obtained by iterating the* Silver forcing *in an ω_2-length iteration over a model of* CH [1].

Theorem 3.8 ([8])**.** *There is a model of* ZFC *in which there are no P-points, but Laver ultrafilters exist generically (i.e.,* $\mathfrak{ge}$(Laver) $= \mathfrak{c}$*).*

Acknowledgments

The results summarized in this paper are available publicly on arXiv ([5], [8]). The former is joint work with Lorenz Halbeisen and Silvan Horvath, and will appear in Archive for Mathematical Logic. The latter is joint work with Silvan Horvath, and is currently under review.

[1] It is well-known that rapid ultrafilters exist in this model

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