Proof-theoretic aspects of stable logic

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The atomic formulas of a logic are decidable if they satisfy the law of excluded middle, namely $P \vee \neg P$ is a theorem. The atomic formulas of a logic are stable if the satisfy the law of double-negation elimination, namely $\neg\neg P \supset P$ is a theorem. While in classical logic decidability and stability coincide, in intuitionistic logic we may have stable atomic formulas that are not decidable. In [2] decidability is formulated a sequent-calculus rule corresponding to the law of excluded middle for atomic formulas:

$$\frac{P \Rightarrow C \quad \neg P \Rightarrow C}{\Gamma \Rightarrow C}\ dc$$

It is shown that when such a rule is added on top of the rules of the single-succedent calculus for propositional intutionistic logic Int, then the resulting calculus $\mathsf{Int} + dc$ is complete for classical propositional logic. Similarly, stability is formulated as a rule corresponding to the law of double-negation elimination for atomic formulas:

$$\frac{\neg P, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow P}\ st$$

However, $\mathsf{Int}+st$ is *not* complete for classical propositional logic, since the sequent $\Rightarrow P \vee \neg P$ is not derivable in $\mathsf{Int} + st$. And since the rule st is admissible in $\mathsf{Int} + dc$, it follows that $Th(\mathsf{Int} + st) \subset Th(\mathsf{Int} + dc)$. At the same time, we also have $Th(\mathsf{Int}) \subset Th(\mathsf{Int} + st)$, since the sequent $\Rightarrow \neg\neg P \supset P$ is derivable in $\mathsf{Int} + st$ but clearly not in Int. This is why $\mathsf{Int} + st$ can be legitimately thought of as an intermediate logic, called stable logic in [2].

In this work I focus on the proof theory for stable logic and its extensions. In particular, I show how to extend the standard cut-elimination procedure from stable logic to a class of intuitionistic stable theories. Building on previous works by Negri and von Plato, I aptly modify the underlying calculus for first-order intuitionistic logic so as to preserve the admissibility of all the structural rules, including cut, in the presence of a restricted version of the rule of classical *reductio ad absurdum* and of a special case of universal rules. Regarding the former, I shall consider a sound and complete intuitionistic calculus obtained from the standard one by replacing the $L_\bot$

$$\frac{}{\bot, \Gamma \Rightarrow C}\ L_\bot$$

with the following rule of *ex falso quodlibet*:

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow C}\ \text{efq}$$

Regarding the universal rules, I shall quasi-universal formulas, namely formulas of the form $\forall \bar{x}(P_1 \wedge \cdots \wedge P_n \supset Q_m)$. A quasi-universal formula corresponds to a quasi-universal rule of the form:

$$\frac{\Gamma \Rightarrow P_1 \quad \cdots \quad \Gamma \Rightarrow P_n}{\Gamma \Rightarrow Q_m}\ \text{R}$$

The main result will be to prove that the intuitionistic calculus so designed extended with finitely many quasi-universal rules is cut-free. Additionally, along the line of my previous work [1], I prove that the basic intuitionistic calculus, namely the calculus without quasi-universal rules, enjoys Craig's interpolation theorem.

References

[1] Guido Gherardi, Paolo Maffezioli, and Eugenio Orlandelli. Interpolation in extensions of first-order logic. *Studia Logica*, 108:619-648, 2020.

[2] Sara Negri and Jan von Plato. Structural Proof Theory. *Cambridge University Press*, 2001.