Closed Ramsey numbers $R^{cl}(\alpha, 3)$ below ω^2

Necdet Duman[1], Özge Gönül[2], Burak Kaya[3], Jayatra Saxena[4], and Yiğithan Tamer[5]

[1] Middle East Technical University, Ankara, Türkiye
necdet.duman@metu.edu.tr
[2] Middle East Technical University, Ankara, Türkiye
gonul.ozge@metu.edu.tr
[3] Middle East Technical University, Ankara, Türkiye
burakk@metu.edu.tr
[4] Middle East Technical University, Ankara, Türkiye
jayatra.saxena@metu.edu.tr
[5] Middle East Technical University, Ankara, Türkiye
yigithan.tamer@metu.edu.tr

Abstract

Contributing to the study of topological Ramsey numbers of pairs of small countable ordinals, we prove that $\omega^4 \cdot (n-2) + 1 < R^{cl}(\omega \cdot n + 1, 3) < \omega^5$ for every integer $n \geq 3$, where the notation $R^{cl}(\alpha, \beta) = \gamma$ denotes that γ is the smallest ordinal such that for every edge coloring of the complete graph of γ with colors red and blue, there exist a red clique on an order-homeomorphic copy of α or a blue clique on an order-homeomorphic copy of β. Our results contribute to the systematic study of topological and closed Ramsey numbers initiated by Caicedo and Hilton and significantly strengthen the existing lower and upper bounds for these closed Ramsey numbers.

1 Introduction

Partition relations for ordinals and cardinals have been thoroughly investigated since their inception by Erdős and Rado in [ER53] and [ER56]. Following Baumgartner, who extended this partition calculus to countable topological spaces in [Bau86], Caicedo and Hilton initiated a study of topological Ramsey numbers for pairs of small countable ordinals in [CH17]. Recent contributions to this study include [Mer19, Mer20, KS21]. The results we announce here are further contributions to this systematic study. In order to be able explain where our results fit in the existing literature, let us recall the necessary notation and definitions.

Let α, β be ordinals and let $X \subseteq \alpha$, where α and β are endowed with their order topologies and X is endowed with its subspace topology. We say that X is *order-homeomorphic* to β in the case that there exists an order-isomorphism $f : X \to \beta$ that is also a homeomorphism.

Definition 1. *Let α, β, γ be ordinals. We shall write $\gamma \to_{cl} (\alpha, \beta)^2$ if for every function $\mathfrak{c} : [\gamma]^2 \to \{0, 1\}$ there exists a subspace $X \subseteq \gamma$ such that*

- *X is order-homeomorphic to α and $[X]^2 \subseteq \mathfrak{c}^{-1}(0)$, or*
- *X is order-homeomorphic to β and $[X]^2 \subseteq \mathfrak{c}^{-1}(1)$.*

where $[X]^2 = \{Y \subseteq X : |Y| = 2\}$ denotes the set of two element subsets of X.

In the case that such an ordinal exists, the closed Ramsey number $R^{cl}(\alpha, \beta)$ is defined to be the least ordinal γ such that $\gamma \to_{cl} (\alpha, \beta)^2$.

In [CH17], using a weak topological variant of the Erdős-Milner theorem, Caicedo and Hilton proved that $R^{cl}(\alpha, k)$ is countable for every countable α and finite k. Of particular focus in [CH17, Section 6] was the case $\alpha < \omega^2$, for which the authors have shown that $R^{cl}(\alpha, k) < \omega^\omega$.

Following the work of Caicedo and Hilton, Mermelstein exactly computed the closed Ramsey numbers $R^{cl}(\omega \cdot 2, 3) = \omega^3 \cdot 2$ and $R^{cl}(\omega^2, 3) = \omega^6$ in [Mer19] and [Mer20], introducing the machinery of canonical colorings. Using this machinery, Kaya and Sağlam later showed in [KS21] that $\omega^2 \cdot n < R^{cl}(\omega + n, 3) < \omega^2 \cdot (n + 1)$ for every $n \geq 3$, which precisely determines the leading term of $R^{cl}(\omega + n, 3)$ in its Cantor normal form. The existing state of the art for $R^{cl}(\alpha, 3)$ with $\omega < \alpha \leq \omega^2$ is summarized below.

$$
\begin{aligned}
& R^{cl}(\omega + 1, 3) = \omega^2 + 1 && \text{[CH17]} \\
& R^{cl}(\omega + 2, 3) = \omega^2 \cdot 2 + \omega + 2 && \text{[CH17]} \\
\omega^2 \cdot n < \ & R^{cl}(\omega + n, 3) < \omega^2(n + 1) \quad \text{for } n \geq 3 && \text{[KS21]} \\
& R^{cl}(\omega \cdot 2, 3) = \omega^3 \cdot 2 && \text{[Mer19]} \\
& \dots && \\
& R^{cl}(\omega^2, 3) = \omega^6 && \text{[Mer20]}
\end{aligned}
$$

In regard to the gap between $R^{cl}(\omega \cdot 2, 3)$ and $R^{cl}(\omega^2, 3)$, the proof of [CH17, Theorem 6.1] provide some implicit upper bounds for the closed Ramsey numbers $R^{cl}(\omega \cdot n + 1, 3)$. However, these implicit upper bounds seem to be already above ω^6. The results we announce here aim to close the aforementioned gap.

2 Main result

Our main result is the following.

Theorem 1 ([DGK+26])**.** *$\omega^4 < R^{cl}(\omega \cdot 2 + 1, 3) < \omega^5$ and $\omega^4 \cdot (n - 2) + 1 < R^{cl}(\omega \cdot n + 1, 3) < \omega^5$ for every integer $n \geq 3$.*

We shall not provide a proof of Theorem 1 here. That said, we will briefly overview the proof strategy for the lower bound and present a coloring witnessing $\omega^4 \nrightarrow_{cl} (\omega \cdot 2 + 1, 3)^2$. In order to be able to do this, we need to recall some basic definitions, most of which can be found in [CH17] and [Mer19].

2.1 From colorings to graphs

Given an ordinal $\gamma = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha_1 \geq \cdots \geq \alpha_n$ are ordinals, the *Cantor-Bendixson rank* of γ is defined to be $CB(\gamma) = \alpha_n$. For an ordinal $\alpha \leq \gamma$, following [Mer19], we also define

$$CNF_\gamma(\alpha) = \min\{1 \leq k \leq n \mid \alpha \leq \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}\}$$

In [CH17, Section 8], the authors introduced the partial order relation $<^*$ on the class of ordinals given by

$$\beta <^* \alpha \text{ if and only if } \alpha = \beta + \omega^\delta \text{ for some } \delta > CB(\beta)$$

By putting an edge between an ordinal and its immediate $<^*$-successor, one can represent an ordinal $\gamma < \omega^\omega$ as a (graph-theoretic) forest, where each term ω^{β_i} in the ungrouped Cantor normal form correspond to a rooted tree of height β_i with root ω^{β_i}. Using the levels of these trees, we can define a *canonical partition* of the ordinal γ as follows

$$V = \left\{V_i^\beta\right\}_{\substack{1\leq i<n \\ 0\leq\beta\leq\alpha_i}} \cup \left\{V_n^\beta\right\}_{0\leq\beta<\alpha_n} \text{ where } V_i^\beta = \{\alpha \in \gamma : CNF_\gamma(\alpha) = i \text{ and } CB(\alpha) = \beta\}$$

The main contribution of [Mer19] is the introduction of *canonical* colorings. Roughly speaking, a canonical coloring $\mathfrak{c} : [\gamma]^2 \to \{0, 1\}$ induces a directed graph on the canonical partition V so that, whenever there is a directed edge from V_i^β to $V_{i'}^{\beta'}$, for every $\alpha \in V_i^\beta$, the set of $\beta \in V_{i'}^{\beta'}$ with $\mathfrak{c}(\{\alpha, \beta\}) = 1$ is "large" in a certain sense. For a precise definition of a canonical coloring and a large subset of a level, see [Mer19, Section 2 and 3]. Among other results, Mermelstein showed that, while proving a positive partition relation $\gamma \to_{cl} (\alpha, \beta)^2$, it is sufficient to consider only canonical colorings [Mer19, Proposition 3.11].

2.2 From graphs to colorings

Having obtained a (directed) graph from a canonical coloring, it is only natural to try to reverse this process. Any (undirected) graph $\mathbf{G} = (V, E)$ on the canonical partition V of the ordinal γ induces a coloring $\mathfrak{c}_\mathbf{G} : [\gamma]^2 \to \{0, 1\}$ given by

$$\mathfrak{c}_\mathbf{G}(\{\alpha, \beta\}) = 1 \text{ if and only if the vertices containing } \alpha \text{ and } \beta \text{ are adjacent in } \mathbf{G}.$$

By creating graphs $\mathbf{G}$ with special properties, one may be able to construct colorings $\mathfrak{c}_\mathbf{G}$ that avoid various prescribed monochromatic homogeneous sets. Indeed, this is precisely the route taken in [KS21, Section 3] and [CH17, Lemma 5.3] to prove negative closed partition relations. The coloring constructed in [Mer19, Section 5] is somewhat in this form as well. While this approach of creating colorings via a graph on the canonical partition has been somewhat successful, its power is limited. This is because we are coloring *all* pairs of ordinals contained in two non-adjacent vertices to the color 0. However, coloring *some* of these pairs to the color 1 could potentially kill off some homogeneous monochromatic sets. For example, in the coloring constructed in [Mer19, Section 5], there are such pairs of color 1.

In order to rectify the situation, we shall partition V_i^β's into further pieces that will be recursively partitioned into further pieces and so on. After connecting some of these nested pieces with graph-like connections, we will induce a coloring. Due to the resemblance of these nested pieces to Russian stacking dolls, we shall call such colorings matryoshka colorings.

2.3 Matryoshka colorings

Let us first define the *matryoshka boxes* of ω^ω. Given integers $0 \leq j < k$, we define a *matryoshka box of degree k and rank j* to be a set $M \subseteq \omega^\omega$ of the form

$$M = \{\alpha \in \omega^\omega : \ CB(\alpha) = j \text{ and } \alpha \leq^* \delta\}$$

for some $\delta < \omega^\omega$ with $CB(\delta) = k - 1$. It is readily verified that the set of $\mathcal{M}_k$ of matryoshka boxes of degree k are strictly well-ordered by the relation

$$M \prec N \text{ if and only if}$$
$$\text{rank}(M) < \text{rank}(N), \text{ or, } \text{rank}(M) = \text{rank}(N) \text{ and } \sup(M) < \inf(N)$$

and has order type $\omega^\omega \cdot k$. In what follows, let $M_k(\theta, j)$ denote the box $(\theta, j)^{\text{th}}$ matryoshka box of degree k.

Let $\gamma < \omega^\omega$ and let $\mathfrak{c} : [\gamma]^2 \to \{0, 1\}$ be a coloring. Given integers $0 \leq j_2 < j_1 < k$, the coloring $\mathfrak{c}$ is said to have

- *forward (respectively, backward) connections of degree k from rank j_1 to j_2* if $\mathfrak{c}(\{\alpha, \beta\}) = 1$ for all $\alpha \in M_k(\theta_1, j_1)$ and $\beta \in M_k(\theta_2, j_2)$, whenever $\theta_1 < \theta_2$ (respectively, $\theta_1 > \theta_2$) and $M_k(\theta_1, j_2)$ and $M_k(\theta_2, j_2)$ are matryoshka boxes of degree k and rank j_2 that are contained in the same matryoshka box of degree $k+1$.

- *downward connections of degree k from rank j_1 to j_2* if $\mathfrak{c}(\{\alpha, \beta\}) = 1$ for all $\alpha \in M_k(\theta, j_1)$ and $\beta \in M_k(\theta, j_2)$.

We will write

$$j_1 \overset{k}{\searrow} j_2, \quad j_1 \overset{k}{\swarrow} j_2 \quad \text{and} \quad j_1 \overset{k}{\downarrow} j_2$$

to denote that $\mathfrak{c}$ has forward, backward and downward connections of degree k from rank j_1 to j_2 respectively. The coloring $\mathfrak{c}$ is called *matryoshka* if it can be expressed using finitely many forward, backward or downward connections.

Matyroshka colorings are particularly useful to systematically create colorings of ordinals of the form ω^k, which may then be glued to together to create colorings of ordinals below ω^ω. Indeed, one may build a basic calculus of forward, backward and downward connections to run computer searches for the existence and non-existence of various homogeneous monochromatic sets. For example, the matryoshka coloring in the following theorem is found by a computer search.

Theorem 2 ([DGK+26])**.** *The matryoshka coloring $\mathfrak{c} : [\omega^4]^2 \to \{0, 1\}$ given by*

$$3 \overset{4}{\downarrow} 1 \quad 2 \overset{4}{\searrow} 0 \quad 3 \overset{4}{\swarrow} 0 \quad 3 \overset{4}{\searrow} 2$$

$$2 \overset{3}{\downarrow} 1 \quad 2 \overset{3}{\searrow} 0 \quad 2 \overset{3}{\swarrow} 0$$

$$1 \overset{2}{\downarrow} 0$$

witnesses the negative closed partition relation $\omega^4 \nrightarrow_{cl} (\omega \cdot 2 + 1, 3)^2$.

The proof of Theorem 2 is a long case-by-case exhaustion argument that the reader is welcome to verify.

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