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Reflections on the Deduction Theorem

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Abstract

In this paper, we outline some problems that arise for the classical notion of deducibility when combining two basic ideas about presentations of a consequence relation, due to Carnap and Wójcicki, in the presence of the Deduction Theorem. We then attempt to address them by appealing to a relevant notion instead.

§1. The next fragment of Rudolf Carnap's *The Logical Syntax of Language* contains many deep ideas about the presentation of a logic in a axiomatic deductive system:

> For reasons of technical simplicity, it is customary not to formulate the entire system of rules of inference, but only a few of these, and in place of the rest to set up certain sentences which are demonstrable (on the basis of the total system of rules), the so-called primitive sentences. The choice of rules and primitive sentences — even when a definite material interpretation of the calculus is assumed beforehand — is, to a large extent, arbitrary. Often, a system can be changed (without changing the content) by omitting a primitive sentence, and, in its place, laying down a rule of inference — and conversely [2, p.29].[1]

Carnap acknowledges here, first, that logical calculi are presented recursively, by means of an appropriate choice of primitive formulas or rules of inference. He also acknowledges that there may be more than one such choice, depending on different sets of rules and axioms that yield the same logic. By the time Carnap wrote his book, there were several alternative formal deductive bases for Classical Propositional Logic, including those of Frege, Russell and Whitehead, among others (Joseph Dopp collected many of them in [3]), and it is not far-fetched to imagine that he had that fact in mind when writing the passage above.

Moreover, Carnap points out that the choice of axioms and rules is largely arbitrary, and that some rules can be freely replaced by axioms, and vice versa. He further argues that the set of sentences and rules identifying a logic must be closed, since, in his view, even primitive formulas in a system must be derivable by some other sentences and rules within the system. It is evident that, after such replacements, the original axioms should remain derivable as theorems in the resulting system, to preserve the identity of the set of theorems the logic proves. Contrary to the common view, which focuses exclusively on the recursive definition of a logic–according to which axioms derive from an empty set of premises–the sentences Carnap considers primitive should instead derive from the entire set they give rise to by means of the primitive inference rules.

Fifty years later, Ryszard Wójcicki distinguishes in [9], on the one hand, that a *propositional logic of formulas* can be defined as a set of formulas of a propositional language that is closed under substitutions (a *theory of a logic*); while, on the other hand, a *propositional logic of*

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[1] In a similar but partial way, von Neumann, as quoted by Jan von Plato in [8, p. 1176]

inferences can be understood as a set of inferences that are sound with respect to a structural consequence operation.

By combining Carnap and Wójcicki's ideas, it is not difficult to imagine that the types of replacements Carnap described above give rise to a wide variety of deductive bases, which can be situated in the spectrum between these two extremes of a unique set of formulas and a unique set of inference rules. This raises some problems: (i) under what conditions can such replacements take place? Whenever an axiom is replaced by a rule or by other axioms, (ii) how can we guarantee that it remains derivable by the resulting deductive base?

§2. To outline these problems more precisely, let $\mathcal{L}$, our formal language, be the smallest set of formulas recursively generated from a countable set of propositional variables $Var = \{p, q, \dots\}$ and a set of connectives $Con = \{\neg, \rightarrow\}$, in such a way that every member of Var is taken to be a well-formed formula; and, whenever φ and ψ are well-formed formulas, $\neg\varphi$ and $\varphi \rightarrow \psi$ are also taken to be well-formed formulas. Greek letters will be used as meta-variables ranging over arbitrary formulas of $\mathcal{L}$. A consequence relation is here understood as any relation $\vdash_C$: $\wp(\mathcal{L}) \mapsto \mathcal{L}$ that satisfies the Tarskian conditions. As usual, a deductive base capturing a consequence relation $\vdash_C$ is a pair $DB^C = \langle Ax^C, R^C \rangle$ where $Ax^C = \{\alpha_1, ..., \alpha_n\}$ is a set comprising all schemas of formulas taken as primitives (axioms); and, $R^C = \{\rho_k, ..., \rho_m\}$ is a set of primitive inference rules for C. Since we will refer only to a single, although unspecified, consequence relation, we omit the superscript (subscript) C whenever convenient.

Following Wójcicki (see [9, p.81]), let a Hilbert-style rule be understood as an instruction of the form '*from* Γ *(where* Γ *is any set of formulas, possibly empty) infer* φ'; typically represented by expressions such as $\Gamma \vdash_{\rho_i} \varphi$, and by figures of the form:

$$\frac{\Gamma}{\varphi}\rho_i \tag{1}$$

Thus, for every $\rho_i \in R^C$, it must hold that $\vdash_{\rho_i} \subset \vdash_C$, and, since axioms may be regarded as the conclusions assigned by a rule of inference to the empty set of formulas, i.e., $\emptyset \vdash_{\rho_i} \alpha_i$, it must also hold that all such rules belong to $\vdash_C$. Since we admit schemas of formulas as axioms, we do not need to add a rule of substitution, allowing the uniform replacement of any occurrence of a propositional variable in a formula with any well-formed formula, as a rule of inference.

Thus, let a *Deductive Base Variation* (a DB-variation) be any pair of deductive bases DB_i^C and DB_j^C for the same consequence relation C, with $Ax_i \neq Ax_j$ or $R_i \neq R_j$. Let a *Carnap Variation* (C-variation) be any DB-variation in which $Ax_i = Ax_j - \Gamma$ and $R_i = R_j \cup \Delta$ or $Ax_i = Ax_j \cup \Gamma$ and $R_i = R_j - \Delta$, for some sets Γ (of formulas) and Δ (of inference rules). Examples of deductive base variations include, for instance, the pair of the Hilbert-Bernays deductive base for Classical Logic and Gentzen's system of Natural Deduction.

§3. Now, to determine whether two bases belong to a DB-variation, it is necessary to establish when two sets of axioms and rules can be considered functionally equipollent, that is, in which cases they yield the same formulas as consequences for the same set of formulas taken as input. Two deductive bases that are proven sound and complete for the same semantics can be readily regarded as belonging to a DB-variation. However, many consequence relations are defined purely syntactically and lack any associated semantics. This, however, should not constitute an obstacle to identifying some variation that simplifies their study, since their deductive bases already contain the necessary information to characterize their main features.

Treating rules as functions allows us to compare them directly. Two functions are *equivalent* if and only if their domains and co-domains coincide, and they assign the same member of the codomain to the same n-tuple of members of their domain. For our purposes, it is important to consider an *arrangement* of axioms or rules as a fixed composition of such functions. Two sets of functions, or of compositions of functions, may be considered *equipollent* if they take the same set of arguments and yield the same set of results.

§4. At first sight, it seems that the Deduction Theorem (DT):

$$\Gamma, \varphi \vdash \psi, \ \textit{if and only if}\ \Gamma \vdash \varphi \to \psi \tag{DT}$$

provides sufficient conditions to significantly simplify the execution of C-variations, since it allows the direct substitutions of rules of inference by axioms, and vice versa. This observation seems to offer a general response to the problem (i) described in §1. It is plausible that Carnap himself had this idea in mind, since he adopts Classical Logic as his logical system by accepting Łukasiewicz's axiomatization in [4] (originally published in Polish in 1929 and also included in [5], published in German in 1930). The symbol $\to$ here denotes a connective in $\mathcal{L}$ that behaves in the object language just as the $\vdash$ symbol in the metalanguage.

It can be noticed that, in the presence of modus ponens (*mp*), axioms can be replaced by rules that assume the antecedent of the major premise of *mp* and yield its consequent as a result, in a manner analogous to that employed by Gentzen in his preparatory work on Natural Deduction system, as reported by Jan von Plato in [8]. Consider, for instance, the axiom:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{tr}$$

which turns out to yield some similar results than the rule:

$$\frac{\varphi \to \psi \quad \psi \to \chi}{\varphi \to \chi} \tag{tr*}$$

after assuming the two antecedents of *tr*, and two applications of *mp*, like in:[2]

$$\cfrac{\cfrac{\overline{(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))} \quad [\varphi \to \psi]^{I}}{(\psi \to \chi) \to (\varphi \to \chi)}\, mp \qquad [\psi \to \chi]^{II}}{\varphi \to \chi}\, mp \tag{2}$$

A C-variation replacing *tr* and *mp* in a DB_1 with rule (tr*) in a DB_2 does not guarantee, however, that both bases are still equipollent, since (tr*) does not necessarily play the same role as *mp* within DB_1 (both subsets of rules coincide only with respect to the *arrangement* $mp(mp(tr, \varphi \to \psi), \psi \to \chi))$, nor does it yield every substitution of *tr* (which become possible only after admitting DT in DB_2). Consequently, caution is required when taking DT as a general condition for C-variations. The replacement of an axiom with a rule must guarantee that the new DB captures the same consequence relation. Additional provisos may be required to achieve the desired result, with a global approach always in mind. But let us, however, set these complications aside and focus on further problems arising from ascribing such a role to DT for those who accept Carnap's ideas quoted in §1.

It can be routinely showed that DT holds whenever *mp* and the two schemas:

$$\varphi \to (\psi \to \varphi) \tag{3}$$

[2] As is customary by some logicians, assumptions are numbered to identify different applications of DT as an inference rule that takes them as antecedents of new conditional consequences. Derivation (10) is an example. We also draw a line over axioms, to distinguish them from assumptions

and

$$(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \tag{4}$$

belong to a deductive base (see [6, p.30] as an example). In part of that proof, one must show that for any formulas φ and ψ such that $\Gamma, \varphi \vdash \psi$, there exists a formula $\varphi \to \psi$ such that $\Gamma \vdash \varphi \to \psi$. Thanks to schema (3), this result is guaranteed for any axiom, since in that case we can obtain $\psi \to (\varphi \to \psi)$ by substitution, and after one application of modus ponens, that $\varphi \to \psi$, as wished. Axiom (3), in fact, allows us to derive the very well-known rule of weakening (*wk*):

$$\psi \vdash_{wk} \varphi \to \psi \tag{5}$$

This rule captures the idea that axioms are implied by every theorem in the system, when ψ instantiates any axiom and φ any theorem. By DT this yields infinite derived rules from any theorems to any axiom. Thus, it solves our problem (ii) from §1. However, the rule permits any formula as an instance of φ, not only theorems. This leads to losing track of the usual notion of independence of axioms: one axiom α_i in a DB is independent of the others when it cannot be derived from $A^C - \alpha_i$ (see, for instance, [6, p. 40]). However, by means of *wk*, we obtain that for any theorem ψ and any axiom α, $\alpha \vdash \psi$ always holds, even in those cases where some α takes no part in the derivation of ψ within the DB under consideration. If one wishes to avoid such an excessive consequence, we should abandon either (3) as an axiom or prevent the rule *wk* to be derivable.

§5. Unfortunately, Witold Pogorzelski proved in [7] that the minimal conditions for DT, even in the absence of axiom (3), cannot dispense with a generalised version of the *wk*-rule. However, there remains another possible solution: to modify our notion of deducibility in such a way that we keep track of our assumptions, thereby distinguishing cases in which they are entirely unrelated, arbitrary formulas from those in which they are axioms or theorems. To this end, we can adapt Alan Ross Anderson and Nuel Belnap's approach in [1, p. 19, 277], and consider a relation $\Gamma \vdash \varphi$, where the assumptions are *relevant* to φ, to hold whenever there exists a sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$ each of which is either a premise (a member of Γ), an axiom, or the result of applying some rule of inference to preceding formulas, such that the next two conditions hold in a DB:

1. *all assumptions, as well as all conclusions derived from the application of a rule to those assumptions, are adorned by a special symbol (say *)–and only them*; and,

2. *the final formula in the sequence, namely φ, is also adorned by *.*

According to Anderson and Belnap, DT is provable when the following axiom and rules:

$$\varphi \to \varphi \tag{6}$$

$$\varphi \to \psi, \psi \to \chi \vdash_{tr*} \varphi \to \chi \tag{7}$$

$$\varphi \to (\psi \to \chi), \psi \vdash_{p} \varphi \to \chi \tag{8}$$

$$\varphi \to (\psi \to \chi), \varphi \to \psi \vdash_{sd} \varphi \to \chi \tag{9}$$

are included among the axioms and rules of inference to any DB. The set of axioms and rules from (6) to (9) corresponds to the implicational fragment of Anderson and Belnap's system *R*, which also includes *mp*. This approach would solve problem (i), provided that one accepts that DT offers sufficient conditions for the free replacement of rules of inference with axioms. Since the rules that may be taken in exchange for axioms can be obtained by the same procedure as in the case of axiom (tr*) above regarding (tr), the question arises: how could axioms be obtainable from inference rules? By axiom (6) and DT, every theorem (including those that were originally axioms) should follow from itself. They would also derive from the axioms in DB. However, this does not by itself clarify whether the axioms replaced by inference rules remain provable in the new DB. The answer is nonetheless affirmative, since any former axiom $\varphi \to \psi$ replaced by a rule of the form $\varphi \vdash_{\rho_i} \psi$, can be derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\varphi \to (\varphi \to (\psi \to \psi))} \quad \dfrac{}{\varphi \to \varphi}}{\varphi \to (\psi \to \psi)}sd \quad \dfrac{[\varphi]^I}{\psi}\rho_i}{\varphi \to \psi}p}{\varphi \to (\varphi \to \psi)}dt, I \quad \dfrac{}{(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)}}{\varphi \to \psi}mp \tag{10}$$

provided that $\varphi \to (\varphi \to (\psi \to \psi))$ and $(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$ are taken as axioms of the same implicational fragment R described above, and 'dt, n' denotes an application of the Deduction Theorem that takes the numbered assumption as the antecedent of a new conditional formula whose consequent is a previously derived conclusion.

Does this mean that axioms can be derived from the entire set of formulas in the theory of such a logic, as intended? The relevance constraint imposed over the notion of deducibility appears to restrict the formulas that can validly imply them. Does the union of these formulas, however, coincide with the entire theory of the logic under consideration? At present, we are not in a position to answer this question.

§6. Carnap's intuition regarding the possibility of exchanging rules of inference for axioms, and vice versa, under certain conditions (apparently, the validity of DT in the logic at issue), together with the view that axioms should be derivable from the very system they generate, though compatible, seems to require further clarification. In the case of Classical Logic, this perspective gives rise to some undesirable, even paradoxical, consequences. Nevertheless, it appears to remain compatible with a relevant notion of deducibility.

References

[1] Alan Anderson and Nuel Belnap. *Entailment: The Logic of Relevance and Necessity, I.* Princeton University Press, United States of America, 1 edition, 1975.

[2] Rudolf Carnap. *The Logical Syntax of Language.* Kegan Paul, Trench, Trubner and Co., London, 1 edition, 1937.

[3] Joseph Dopp. *Notions de Logique Formelle.* Publications Universitaires de Louvain, Louvain, 2 edition, 1967.

[4] Jan Łukasiewicz. *Elements of Mathematical Logic.* Pergamon Press, Poland, 1 edition, 1963.

[5] Jan Łukasiewicz. *Investigations into the Sentential Calculus.* North-Holland Publishing Company, Amsterdam, 1 edition, 1970.

[6] Paolo Mancosu, Sergio Galvan, and Richard Zach. *An Introduction to Proof Theory.* Oxford University Press, United Kingdom, 1 edition, 2021.

[7] Witold Pogorzelski. On the scope of the classical deduction theorem. *The Journal of Symbolic Logic*, 33:77–81, 1968.

[8] Jan von Plato. From axiomatic logic to natural deduction. *Studia Logica*, 102(6):1167–1184, 2014.

[9] Ryszard Wójcicki. *Theory of Logical Calculi.* Kluwer Academic Publishers, Dordrecht, 1 edition, 1988.