Models of an Abstract Elementary Class as a Generalized Polish Space

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Abstract

In first order logic, it is known that you can define a topology so that the countable models of some theory T form a Polish Space. In this paper we use the Baldwin- Boney Relational Presentation Theorem (from [3]) to generalize this result to the models of an Abstract Elementary Class (AEC). More specifically, we define a topology on the models of an AEC of size $\lambda \geq \kappa$, where κ is the Löwenheim-Skolem number and λ has to satisfy a set-theoretic assumption and prove that these models form a Generalized Polish Space.

1 Introduction

1.1 Countable Models of First-Order Theories

First we survey some results about the topological space of countable models of a first-order theory T.

Definition 1.1. *A topological space X is called* **Polish** *if it is completely metrizable and second countable.*

In first order logic, if T is a theory, we can define a topology on the countable models of T and prove that it is a Polish Space.

Definition 1.2. *Let $\mathcal{L}$ be some countable language, $Mod_\omega(\mathcal{L})$ be the space of all $\mathcal{L}$-structures on ω (since all of our models are countable, we can think that their universe is always ω) and $\mathcal{F}$ be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$. We define $T_\mathcal{F}$ to be the* **topology** *on $Mod_\omega(\mathcal{L})$, whose basic open sets are of the following form:*

$$\{M | M \models \phi(n_1, n_2, ..., n_k)\}$$

for $k, n_1, ..., n_k \in \omega$ and $\phi \in \mathcal{F}$.

The following classical result, which can be found in [6], is the result we will generalize to AEC's.

Theorem 1.3. *The space $Mod_\omega(\mathcal{L})$ equipped with the topology defined in Definition 1.2 is a Polish Space.*

Observations : **1**. Let $T \subseteq \mathcal{F}$ a theory, then we define $Mod_\omega(T) \subseteq Mod_\omega(\mathcal{L})$ to be all the countable models of T. From [6], we know that $Mod_\omega(T)$ is a closed subspace (equipped with the subspace topology) of $Mod_\omega(\mathcal{L})$ and therefore a Polish Space.
2. We can get the first order case, if we take as $\mathcal{F}$ the first order logic.
3. Note that the above is not the only way to define a topology on first-order formulas and prove that $Mod_\omega(\mathcal{L})$ is a Polish Space. We can also define a similar topology on quantifier-free first-order formulas. The reader can refer to [6] or [8] for more information.

The above topology on the countable models allows us to see some classical model theoretic results from a different point of view. For example in [6] we find the following two results (which also hold in the first order case). We do not have the answer yet, but we would like to see if those results can be generalized for AEC's (see Open Question 1.7).

Corollary 1.4. *(Omitting Types Theorem) Let* $\mathcal{F} \subseteq \mathcal{L}_{\omega_1,\omega}$ *be a countable fragment. Let* $\Sigma(x_1, x_2, ..., x_n)$ *be a complete non-principal type over* $\mathcal{F}$ *and let* $k_1, k_2, ..., k_n \in \omega$*. Let* $T \subseteq \mathcal{F}$ *be a complete theory. Then the set*

$$\{M \in Mod_\omega(T) | M \text{ omits } \Sigma(\bar{x})\}$$

is comeager in $(Mod_\omega(T), T_{\mathcal{F}})$*.*

Theorem 1.5. *Let* $\mathcal{F}$ *be a countable fragment of* $\mathcal{L}_{\omega_1,\omega}$*,* $T \subseteq \mathcal{F}$ *a complete theory and* $M_0 \in Mod_\omega(T)$*. The set of* $N \in (Mod_\omega(T), T_{\mathcal{F}})$ *which are isomorphic to* M_0*, is comeager if and only if* M_0 *is an atomic model.*

1.2 Abstract Elementary Classes

S. Shelah introduced the notion of Abstract Elementary Classes (AEC) in the 1980's (see [9]), in order to study classes of structures that are not first order axiomatizable. AEC's are a non-syntactic generalization of Elementary Classes. The reader can consult [2] for more details and proofs of the results presented here.

Shelah was also the first one who proved a Theorem that allows us to replace the entirely semantic description of AEC's by a syntactic one. This Theorem is known as "Shelah's Presentation Theorem" (see [2]).

Theorem 1.6. *(Shelah's Presentation Theorem) Let* τ *be a vocabulary and* $\mathbf{K}$ *be an AEC in* τ *(*$|\tau| \leq LS(\mathbf{K})$*). Then there is a vocabulary* $\tau' \supseteq \tau$ *with cardinality* $|LS(\mathbf{K})|$*, a first order theory* T' *and a set* Γ *of at most* $2^{LS(\mathbf{K})}$ *partial types, such that:*

$$\mathbf{K} = \{M'|_\tau : M' \models T' \text{ and } M' \text{ omits } \Gamma\}.$$

Moreover, the $\prec_{\mathbf{K}}$ *relation satisfies the following:*

- *If* M' *is a* τ'*-substructure of* N'*, where* $M', N' \models T'$ *and omit* Γ*, then* $M'|_\tau \prec_{\mathbf{K}} N'|_\tau$*.*
- *If* $M \prec_{\mathbf{K}} N$ *there is an expansion of* N *to a* τ'*-structure such that* M *is the universe of a* τ'*-substructure of* N'*.*

Since then there have been proved many different "Presentation Theorems" for AEC's. All of them are constructed for a different purpose. In this paper, we use the Baldwin and Boney "Relational Presentation Theorem" (see Section 2.1). Our purpose is to obtain the necessary syntactic tools and define a topology on the models of an AEC, of size λ, for $\lambda \geq LS(\mathbf{K})$.

For our purposes, the Baldwin-Boney's "Relational Presentation Theorem" has certain advantages than Shelah's original theorem. First, every model in the AEC has a unique expansion in the expanded vocabulary and second, there is no set of partial types to omit.

1.3 Generalized Descriptive Set Theory

An important observation is that in this paper we will not necessarily define a topology on a class of countable models, but rather on a class of uncountable models. In the first-order case, we work with subspaces of ω^ω. In this paper, we work with subspaces of κ^κ, where κ is some regular uncountable cardinal. (see Section 3). Since κ^κ can not be a Polish Space, we work a proper generalization of this notion called G-Polish Space (see Section 2.2). The study of G-Polish spaces falls into Generalized Descriptive Set Theory and has been a very active field during the last decade, since it has connections with a lot of other fields in Mathematics.

There are two "natural" ways to generalize the notion of Polish Spaces, **strong κ-Choquet spaces** (see [1]) and **G-Polish spaces** (see [1] or Section 2.2). The definitions are not equivalent in general, but in many cases we can equivalently use either of them. In our paper we use the definition of G-Polish Space, but we could have used strong κ-Choquet spaces too. One can consult [1, 4] and [5] for more results in this area.

In this paper, we connect for the first time AEC's with Generalized Descriptive Set Theory. More specifically, we use the Baldwin-Boney Relational Presentation Theorem in order to obtain the necessary syntactic tools and define a topology on the models of size $LS(\mathbf{K})$ of an AEC $\mathbf{K}$. Then, working similarly to [6], we prove that these models form a Generalized Polish Space. Finally, in the last Section, we generalize this result to models of an AEC of size equal to λ, where $\lambda \geq LS(\mathbf{K}) = \kappa$, under the assumption that for $\mu = (\lambda + \kappa^+)^\kappa$ it holds that $\mu^{<\mu} = \mu$.

One natural question in the context of our paper is how many of the results from first order logic generalize in this new setting. For instance:

Open Question 1.7. *Can we generalize the notion of atomic models to fit the framework of AEC's in such a way that the analogue of Theorem 1.5 for atomic models of size equal to the Löwenheim-Skolem number holds true?*

In Section 2.1 we present the Baldwin- Boney Relational Presentation Theorem and in Section 2.2 we define Generalized Polish Spaces and present some initial results for this notion. In Section 3 we prove the main result of the paper. More specifically, we define a topology on the models of an AEC of size equal to $\lambda \geq \kappa = LS(\mathbf{K})$ (under the assumption that for $\mu = (\lambda + \kappa^+)^\kappa$, $\mu^{<\mu} = \mu$), and prove that they form a Generalized Polish Space.

2 Preliminaries

2.1 The relational presentation theorem

In this section we present the Relational Presentation Theorem for AEC's, following J. Baldwin and W. Boney. The reader who is familiar with [3] should skip this section.

Definition 2.1. *Let $\mathbf{K}$ be an AEC and μ a cardinal. Then, $\mathbf{K}_\mu$ is the class of all models in $\mathbf{K}$ of size μ.*

First, we fix some notation. Let $\mathbf{K}$ be an AEC in a vocabulary τ and let κ denote the Löwenheim-Skolem number $LS(\mathbf{K})$. We assume that $\mathbf{K}$ contains no models of size $< LS(\mathbf{K})$. The same arguments given here could also be given for $\kappa > LS(\mathbf{K})$.

We fix some compatible enumerations for models $M \in \mathbf{K}_\kappa$. Compatible enumerations means that each M has an enumeration of its universe, denoted $\mathbf{m}^M = (m_i^M : i < \kappa)$, and, if $M \cong N$, there is some fixed isomorphism $f_{M,M'} : M \cong M'$ such that $f_{M,M'}(m_i^M) = m_i^{M'}$ and if $M \cong M' \cong M''$, then $f_{M,M''} = f_{M',M''} \circ f_{M,M'}$. These enumerations are not contained in τ.

For each isomorphism type $[M]_{\cong}$ and $[M \prec_{\mathbf{K}} N]_{\cong}$ with M, N $\in \mathbf{K}_\kappa$, we add to τ new predicates $R_{[M]}(\mathbf{x})$ and $R_{[M\prec_{\mathbf{K}}N]}(\mathbf{x},\mathbf{y})$ which are κ-ary and $(\kappa \times 2)$-ary respectively and we form $\tau^* \supseteq \tau$.

Next, we describe the presentation theory T^*. The purpose of this theory is to identify strong submodels of size κ and strong submodel relations between these models via the new predicates $R_{[M]}$ and $R_{[M\prec_{\mathbf{K}}N]}$. This is done by expressing properties that connect the canonical enumerations with structures in $\mathbf{K}$ using the next axioms ($\mathbf{x}$ is a sequence of length at most κ).

$$R_{[M]}(\mathbf{x}) \text{ holds iff } x_i \mapsto m_i^M \text{ is an isomorphism}$$

$$R_{[M\prec_{\mathbf{K}}N]}(\mathbf{x},\mathbf{y}) \text{ holds iff } x_i \mapsto m_i^M \text{ and } y_i \mapsto m_i^N \text{ are isomorphisms and } x_i = y_j \text{ iff } m_i^M = m_j^N$$

Note that by the coherence of the isomorphisms, the choice of representative from $[M]_{\cong}$ doesn't matter. Also, we might have $M \cong M', N \cong N', M \prec_{\mathbf{K}} N$ and $M' \prec_{\mathbf{K}} N'$, but not $(M,N) \cong (M',N')$. In this case $R_{[M\prec_{\mathbf{K}}N]}$ and $R_{[M'\prec_{\mathbf{K}}N']}$ are different predicates.

In this paper, we will not use the strict definition of T^*. One fact that we will use is that $|T^*| = I(\mathbf{K},\kappa) + \kappa$. Furthermore, we need to know that T^* is an $\mathcal{L}_{(2^\kappa)^+,\kappa^+}(\tau^*)$-theory. For more details about T^*, the reader should refer to [3].

The following Theorem is the Baldwin-Boney Relational Presentation Theorem. We will not use every result of it, but we include the whole Theorem for the completeness of the paper. In fact, we will use only the first two results of the Theorem.

Theorem 2.2. *(Relational Presentation Theorem) Let T^* be the $\mathcal{L}_{(2^\kappa)^+,\kappa^+}(\tau^*)$-theory of cardinality $I(\mathbf{K},\kappa) + \kappa$ described above. If $M^* \models T^*$, then we write just M instead of $M^*|_\tau$.*

1. *If $M^* \models T^*$ then $M^*|_\tau \in \mathbf{K}$. Further, for all $M_0 \in \mathbf{K}_\kappa$, we have $M^* \models R_{[M_0]}(\mathbf{m})$ implies that $\mathbf{m}$ enumerates a strong submodel of M.*
2. *Every $M \in \mathbf{K}$ has a unique expansion M^* that models T^*.*
3. *If $M \prec_{\mathbf{K}} N$, then $M^* \subseteq N^*$.*
4. *If $M^* \subseteq N^*$ both model T^*, then $M \prec_{\mathbf{K}} N$.*
5. *If $M \prec_{\mathbf{K}} N$ and $M^* \models T^*$ such that $M^*|_\tau = M$, then there is $N^* \models T^*$ such that $M^* \subseteq N^*$ and $N^*|_\tau = N$.*

2.2 Generalized Polish Spaces

In this section we present the basic facts about Generalized Polish spaces. We follow Claudio Agostini, Luca Motto Ros and Philip Schlicht (cf. [1]) in our presentation.

More specifically, we study the space μ^μ and its subspaces, where μ is an uncountable regular cardinal. This generalizes the results about ω^ω seen in Descriptive Set Theory.

The two main spaces are:

1. The Generalized Baire Space

$$\mu^\mu = \{x | x : \mu \to \mu\}$$

of all sequences with values in μ and length μ, equipped with the bounded topology τ_b, i.e. the topology generated by the sets of the form

$$N_s = \{x \in \mu^\mu | s \subseteq x\}$$

where $s \in \mu^{<\mu}$.

2. The Generalized Cantor Space

$$2^\mu = \{x | x : \mu \to 2\}$$

which is a closed subset o μ^μ, equipped with the relative topology.

Definition 2.3. *A topological space X has* **weight** *μ, if there is a base for its topology of size μ.*

Since the classical Cantor and Baire Spaces are second countable, it is natural to require accordingly that μ^μ and 2^μ have weight μ: this amounts to requiring that $\mu^{<\mu} = \mu$, or equivalently that μ is regular and $2^{<\mu} = \mu$. Thus such assumption is one of the basic conditions in the development of the theory of Generalized Polish Spaces.

In order to generalize Definition 1.1, we need some definitions.

Consider a totally ordered (Abelian) group

$$G = (G, +_G, 0_G, \leq_G)$$

with **degree** Deg(G) = μ, where Deg(G) is the coinitiality of $G^+ = \{\epsilon \in G | 0_G <_G \epsilon\}$ of G. A **G-metric** on a nonempty space X is a function $d : X^2 \to G^+ \cup \{0_G\}$ satisfying the usual rules of a distance function.

Every G-metric space (X, d) is naturally equipped with the d-topology generated by its open balls. If X is already a topological space, we say that the G-metric d is compatible with the topology of X, if the latter coincides with the d-topology. A topological space is **G-metrizable** if it admits a compatible G-metric.

Let (X, d) be a G-metric space. A sequence $(x_i)_{i<\mu}$ of point from X is d-Cauchy if

$$\forall \epsilon \in G^+ \exists \alpha < \mu \forall \beta, \gamma \geq \alpha (d(x_\beta, x_\gamma) <_G \epsilon).$$

A space X is **completely G-metrizable**, if it admits a compatible G-metric and every Cauchy sequence converges.

Definition 2.4. *A space X is* **G-Polish** *if it is completely G-metrizable and has weight $\leq \mu$.*

Lemma 2.5. *(see [1]) The spaces μ^μ and 2^μ (endowed with the bounded topology) are both G-Polish Spaces.*

Definition 2.6. *Let X be a space. A set $A \subseteq X$ is G^μ_δ if it can be written as a μ-sized intersection of open sets of X.*

Theorem 2.7. *(see [1]) Let X be a G-Polish space and $Y \subseteq X$. Then Y is G-Polish if and only if Y is G^μ_δ in X.*

3 Defining a Topology on AEC's

In this section, we assume that $\mathbf{K}$ is an AEC with $\kappa = LS(\mathbf{K})$ in a vocabulary τ, where $|\tau| \leq \kappa$. We define a topology on the models of cardinality $\lambda \geq \kappa$ that belong to $\mathbf{K}$ and prove that they form a G-Polish space.

For the smooth development of the theory we isolate the following assumptions which we assume throughout the rest of this paper.

Assumptions: Let $\lambda \geq \kappa$.

1. For $\mu = (\lambda + \kappa^+)^\kappa$, it holds that $\mu^{<\mu} = \mu$. Note that this implies that μ is regular.
2. The universe of all models of size λ, is λ.

We give some definitions first.

Definition 3.1. *Let $\lambda \geq \kappa$ be infinite cardinals.*

We call a set of $\mathcal{L}_{\mu^+,\kappa^+}$-formulas, $\mathcal{F}$, a **fragment** *if there is a set of variables V of cardinality at least κ^+, such that if $\phi \in \mathcal{F}$, then all variables occurring in ϕ are in V and $\mathcal{F}$ satisfies the following properties:*

1. *all atomic formulas using only variables from V and constant symbols are in $\mathcal{F}$*
2. *if $\phi \in \mathcal{F}$ and ψ is a subformula of ϕ, then $\psi \in \mathcal{F}$*
3. *if $\phi \in \mathcal{F}, v$ is free in ϕ and t is a term where every variable is in V, then the formula obtained by substituting t into all free occurrences of v is in $\mathcal{F}$*
4. *$\mathcal{F}$ is closed under $\neg$*
5. *$\mathcal{F}$ is closed under $\exists v$ for $v \in V$ (for finite length, i.e. it may not be closed for $\exists \bar{v}$, where $\bar{v}$ has infinite length)*
6. *If $(\phi_i)_{i<\sigma}$ are formulas in $\mathcal{F}$, then $\bigwedge_{i<\sigma} \phi_i \in \mathcal{F}$, for $\sigma < \mu$*
7. *every formula in $\mathcal{F}$ has at most κ many free variables*

Observation: If T is an $\mathcal{L}_{\mu^+,\kappa^+}$-theory (it could be just a sentence), then there exists a minimum fragment that contains that theory. In order to construct that minimum fragment, one should include all the atomic formulas with variables in V, all the subformulas of T and then include every other formula required by the closure properties (3)-(7).

Lemma 3.2. *Let $\mathcal{F}$ be the minimum fragment of $\mathcal{L}_{\mu^+,\kappa^+}(\tau^*)$ that contains the theory T^* from the Relational Presentation Theorem. Then, $|\mathcal{F}| = \mu$.*

Proof. First, we observe that $|T^*| = I(\mathbf{K}, \kappa) + \kappa \leq 2^\kappa \leq \mu$. Then, we need to bound the size of the fragment $\mathcal{F}$ generated by T^*.

For that purpose observe that since there are κ^+-many variables and at most $|\tau| \leq \kappa$-many constants, the atomic formulas are at most $(\kappa^+)^\kappa = 2^\kappa \leq \mu$-many. Furthermore, the subformulas of sentences in T^* are at most $2^\kappa \leq \mu$ and the formulas produced by substituting variables with terms are at most μ by an easy computation. It is also easy to see that the rest of the closure properties of Definition 3.1 do not produce more than μ-many new formulas. Notice here that the hypothesis $\mu^{<\mu} = \mu$ is needed when dealing with closure property (6). So, we finally have that $|\mathcal{F}| = \mu$. $\square$

We are now ready to define the topology:

Definition 3.3. *1. $Mod_\lambda(\tau)$ is the space of τ-structures whose universe is λ.*

2. For $\mathcal{F} \subseteq \mathcal{L}_{\mu^+,\kappa^+}(\tau)$ a fragment, we let $T_\mathcal{F}$ be the **topology** *on $Mod_\lambda(\tau)$ generated by the basic sets*

$$\{M | M \models \phi(\alpha_1, \alpha_2, ...)\},$$

where $\phi(\bar{x}) \in \mathcal{F}$ and $\alpha_i \in \lambda$, for every i.

The proof of the next Theorem follows [6]. We use the Relational Presentation Theorem and consider the topology $T_\mathcal{F}$ on the models of $\mathbf{K}_\lambda$. Finally, we prove that our class is a G^μ_δ subset of the space 2^μ and thus a G-Polish space.

Observation: For us, λ will be the size of the models on which we define a topology. For the next Lemma and the next Theorem, λ is equal to κ so $\mu = (\lambda + \kappa^+)^\kappa = 2^\kappa$.

Lemma 3.4. *If $\mathcal{F}$ is the minimum fragment of $\mathcal{L}_{\mu^+,\kappa^+}(\tau^*)$ that contains the theory T^* from the Relational Presentation Theorem, then $(Mod_\kappa(\tau^*), T_\mathcal{F})$ is a G-Polish Space.*

Proof. Let $C = \{c_i | i < \kappa\}$ be a set of new constants and let $\hat{\tau}^* = \tau^* \cup C$, where $\tau^* \supseteq \tau$ is the extended vocabulary from the Relational Presentation Theorem. Define $\hat{\mathcal{F}}$ to be the fragment generated by $\mathcal{F} \cup C$ in $\mathcal{L}_{\mu^+,\kappa^+}(\hat{\tau}^*)$ and define S to be the set of all sentences in $\hat{\mathcal{F}}$.

First we observe that since we just added κ-many constant symbols, the computations are the same for $\hat{\mathcal{F}}$ as they were for $\mathcal{F}$ in Lemma 3.2. So $|\hat{\mathcal{F}}| = |\mathcal{F}| = \mu$.

We can also see that $|S| = \mu$ (the computations are again the same with the difference that we do not have κ^+-many variables, but we have exactly κ-many constants, so the atomic formulas are $\kappa^\kappa = \mu$-many), and therefore $2^S = \{0,1\}^S$ is a *G-Polish* space.

Let B be the set of all functions $f \in 2^S$ satisfying the following properties:

1. any finite subset of $\{\phi | f(\phi) = 1\}$ is consistent
2. for all ϕ, we have that $f(\phi) = 0$ iff $f(\neg\phi) = 1$
3. $f(\bigvee_{i<\mu} \phi_i) = 1$ iff there is some i with $f(\phi_i) = 1$
4. for all ϕ, we have $f(\exists\bar{x}\phi(\bar{x})) = 1$ iff there is some $\bar{c} \in C$ with $f(\phi(\bar{c})) = 1$ (the length of $\bar{x}$ could be infinite).

Claim: B is a G^μ_δ subset of 2^S.

Proof of claim: It suffices to show that the conditions (1)-(4) correspond to G^μ_δ sets.

(1) Let $[S]^{<\omega}$ be the set of all finite subsets of S and $I \subseteq [S]^{<\omega}$ the set of all inconsistent finite subsets of S. Then (1) corresponds to:

$$\bigcap_{A \in I} \bigcup_{\psi \in A} \{f | f(\psi) = 0\},$$

which is G^μ_δ, since with the topology we defined on the generalized Cantor space the sets $\{f | f(\psi) = 0\}$ are open and the size of the intersection is $|I| \leq \mu$.

Observation: In fact the set $\{f | f(\psi) = 0\}$ is clopen, since its complement can be written as $\{f | f(\psi) = 1\}$, which is also open.

(2) The second condition corresponds to the set:

$$\bigcap_{\phi\in S}\{f|f(\phi)=0\Leftrightarrow f(\neg\phi)=1\}.$$

The sets $\{f|f(\phi)=0\Leftrightarrow f(\neg\phi)=1\}$ are clopen, since for every ϕ the basic open sets $\{f|f(\phi)=0\}$ are clopen. Therefore, the desired set is G^{μ}_{δ}.

(3) The third condition corresponds to the intersection of the following sets (one for each direction):

$$\bigcap_{i<\mu}\{f|f(\phi_i)=1\Rightarrow f(\bigvee_{i<\mu}\phi_i)=1\},$$

which is G^{μ}_{δ} and

$$\{f|f(\bigvee_{i<\mu}\phi_i)=0\}\cup\bigcup_{i<\mu}\{f|f(\phi_i)=1\},$$

which is open.

(4) The fourth condition also corresponds to the intersection of two sets, one for each direction:

$$\bigcap_{\phi\in S}\bigcap_{\bar{c}\in C}\{f|f(\phi(\bar{c}))=1\Rightarrow f(\exists\bar{x}\phi(\bar{x}))=1\},$$

which is G^{μ}_{δ} since the first intersection has size μ and the second intersection has size κ and

$$\bigcap_{\phi\in S}(\{f|f(\exists\bar{x}\phi(\bar{x}))=0\}\cup\bigcup_{\bar{c}\in C}\{f|f(\phi(\bar{c}))=1\}),$$

which is also G^{μ}_{δ}.

Thus, we have proven that B is a G-Polish space.

The next step is to prove that $Mod_{\kappa}(\tau^*)$ is homeomorphic to B and thus also a G-Polish space. We define the map:

$$\begin{gathered} e: Mod_{\kappa}(\tau^*)\to B\\ M\mapsto T_M, \end{gathered}$$

as follows

$$T_M(\phi(c_{i_1},c_{i_2},...))=1\Leftrightarrow M\models\phi(i_1,i_2,...)$$

Claim: e is homeomorphism.

1-1: If $M_1\neq M_2$, there will be some atomic formula $\psi(x_1,x_2,...)$ and $i_1,i_2,...\in\kappa$ on which they disagree, whence T_{M_1} and T_{M_2} disagree on $\psi(c_{i_1},c_{i_2},...)$.

Onto: Fix some $T\in B$. We define M as follows:

For any relation $R\in\tau^*$

$$M\models R(i_1,i_2,...)\Leftrightarrow T(R(c_{i_1},c_{i_2},...))=1.$$

For any function $F\in\tau^*$

$$F^M(i_1,i_2,...)=l\Leftrightarrow T(F(c_{i_1},c_{i_2},...)=c_l)=1$$

(This is well defined. By (1) and (2) we have $T(\exists x(F(c_{i_1}, c_{i_2}, ...) = x)) = 1$ and then by (4) we have some witness c_l.)

For any constant $c \in \tau^*$

$$c^M = i \Leftrightarrow T(c = c_i) = 1$$

(which is also well defined)

Now we can easily prove

$$M \models \phi(i_1, i_2, ...) \Leftrightarrow T(\phi(c_{i_1}, c_{i_2}, ...)) = 1$$

by induction on $\phi \in \mathcal{F}$.

e is continuous: If G is is a basic subset of B, i.e. there is a $\lambda < \mu$ and $(\phi_\alpha)_{\alpha<\lambda} \in \mathcal{F}$, such that the value $T(\phi_\alpha)$ is the same for every $T \in G$, we will prove that $e^{-1}(G)$ is an open set in $Mod(\tau^*)$.

$e^{-1}(G) = \{M | M \models \bigwedge_{\alpha<\lambda}(\phi^*_\alpha)\}$, where ϕ^*_α is either ϕ_α or $\neg\phi_\alpha$ depending on whether the value of $T(\phi_\alpha)$ is 1 or 0 respectively. It follows that $e^{-1}(G)$ is an open set.

Inverse function is continuous: If $\{M | M \models \phi(i_1, i_2, ...)\}$ is a basic set in $Mod_\kappa(\tau^*)$, for some $\phi \in \mathcal{F}$ and $i_1, i_2, ... \in \kappa$, then its image through e is $\{T \in B | T(\phi(c_{i_1}, c_{i_2}, ...) = 1\}$, which is an open set in B.

This proves the claim. □

Theorem 3.5. *Let* $\mathbf{K}$ *be an AEC in a vocabulary* τ. *If* $\mathcal{F}$ *is the minimum fragment of* $\mathcal{L}_{\mu^+,\kappa^+}(\tau^*)$ *that contains the theory* T^* *from the Relational Presentation Theorem, then* $(\mathbf{K}_\kappa, T_\mathcal{F})$ *is a G-Polish space.*[1]

Proof. We have that:

$$\mathbf{K}_\kappa = \bigcap_{\sigma \in T^*} \{M|_\tau : M \in Mod_\kappa(\tau^*) \text{ and } M \models \sigma\}.$$

From Lemma 3.4, the fact that

$$\bigcap_{\sigma \in T^*} \{M : M \in Mod_\kappa(\tau^*) \text{ and } M \models \sigma\}$$

is a G^μ_δ subset of $Mod_\kappa(\tau^*)$ and that there is a bijection between the models of $\mathbf{K}$ and their expansions that model T^*, we have the desired result. □

Lemma 3.4 and Theorem 3.5 can be generalized for $\lambda > \kappa$ with some minor transformations in their proofs.

Lemma 3.6. *Let* $\lambda > \kappa$. *If* $\mathcal{F}$ *is the minimum fragment of* $\mathcal{L}_{\mu^+,\kappa^+}(\tau^*)$ *that contains the theory* T^* *from the Relational Presentation Theorem, then* $(Mod_\lambda(\tau^*), T_\mathcal{F})$ *is a G-Polish Space and if additionally* $\mathbf{K}$ *is an AEC, then* $(\mathbf{K}_\lambda, T_\mathcal{F})$ *is also a G-Polish Space.*

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[1] To be more accurate, $T_\mathcal{F}$ cannot be defined on $\mathbf{K}_\kappa$, since the models in $\mathbf{K}$ are τ-structures and $T_\mathcal{F}$ is defined on τ^*-structures. But we will overlook this fact since there is a bijection between the models in $\mathbf{K}$ and their expansions in the Relational Presentation theorem.

Proof. We notice only a couple of differences between the two proofs.

Here we add to τ^* the set $C = \{c_i | i < \lambda\}$ of λ-many new constants and define $\hat{\tau}^* = \tau^* \cup C$. We define $\mathcal{F}$, $\hat{\mathcal{F}}$ and S, as we did in Lemma 3.4. With similar computations as in the initial Lemma, we can see that $|S| = \mu$ and that means that $2^S = \{0,1\}^S$ is a G-Polish space.

The rest of the proof, if we replace 2^κ with $\lambda^\kappa = \mu$, is the same and we do not need to add anything new. □

4 Closing Remarks

Since the models of a first order theory, equipped with the elementary substructure relation, form an AEC with countable Löwenheim-Skolem number, it comes natural to ask whether the results in this paper generalize the already known results presented in the introduction, namely that the countable models of a first order theory form a Polish Space.

The answer to this question is negative. In the first-order case, we do not need to extend the vocabulary and appeal to a Presentation Theorem. The space of the countable models is homeomorphic to a subspace of 2^ω, which is a Polish Space. If we study the countable models of a first order theory in the terms of this paper, we first extend the language and then we use the Relational Presentation Theorem. This results in regarding countable models as models of an $\mathcal{L}_{(2^{\aleph_0})^+,\aleph_1}$-theory and the space of countable models in the extended language is homeomorphic to a subspace of 2^{2^ω}, which is a G-Polish Space.

An open question is if we can derive similar results to this paper by using a Presentation Theorem different than the Baldwin-Boney Presentation Theorem. Since there is a plethora of other Presentation Theorems, it is interesting to ask if we can define a different topology on the models of an AEC which also gives rise to a G-Polish space.

From the variety of Presentation Theorems, noteworthy is the theorem in [7], where Samson Leung improves the results of the "Relational Presentation Theorem". One interesting part of Leung's theorem is that it keeps the original language. Another interesting Presentation Theorem is the one that S. Shelah and A. Villaveces prove in [10]. In that theorem, we also do not need to expand the original language. One can ask if these theorems can be used in the present paper instead of the relational presentation theorem, but this is something we have not examined.

Finally, this paper is a first attempt to generalize certain notions from first-order logic to AEC's. One natural question is if we can generalize notions like the atomic models, or define a topology on the models of an AEC so that the model-theoretic properties are closely connected with the topological properties, as it happens in the first-order case. We do not know the answer to this, but we see this paper as part of a larger project.

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