On cardinal invariants on universally null sets

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Abstract

We investigate the cardinal invariants on universally null sets. In particular, we prove that the cofinality of the universally null ideal is consistently greater than the size of continuum, using the techniques of Yorioka, who proved that the cofinality of the strong measure zero ideal and the size of continuum are incomparable.

The cardinal invariants of the Lebesgue null ideal $\mathcal{N}$ and the meager ideal $\mathcal{M}$ are well understood through Cichoń's diagram. For the strong measure zero ideal $\mathcal{SN}$, Yorioka showed that $\mathrm{cof}(\mathcal{SN})$ and the cardinality of the continuum $\mathfrak{c}$ are incomparable.

The universally null ideal $\mathcal{UN}$ is a natural counterpart: it captures smallness with respect to all atomless Borel probability measures at once, rather than a single fixed measure. However, its cardinal invariants have not been studied as much.

In this paper, we show that $\mathrm{cof}(\mathcal{UN}) > \mathfrak{c}$ is consistent, adapting Yorioka's techniques to the measure-theoretic setting.

Definition 1. Let $\mathsf{Meas} = \{\mu : \mu$ is a atomless Borel probability measure on $2^\omega\}$. For $\nu, \mu \in \mathsf{Meas}$, let $\nu \ll \mu$ iff $\mu(A) = 0$ implies $\nu(A) = 0$ for every Borel subset A of 2^ω.

Definition 2. $\mathcal{UN} = \{A \subseteq 2^\omega :$ for every $\mu \in \mathsf{Meas}$, we have $\mu(A) = 0\}$. A member in $\mathcal{UN}$ is called a universally null set.

Definition 3. Let $\mathcal{I}$ be a σ-ideal on the Cantor space containing all singletons.

1. Let $\mathrm{add}(\mathcal{I})$ be the least cardinality of a subfamily $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} \notin \mathcal{I}$.
2. Let $\mathrm{cov}(\mathcal{I})$ be the least cardinality of a subfamily $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} = 2^\omega$.
3. Let $\mathrm{non}(\mathcal{I})$ be the least cardinality of a subset $A \subseteq 2^\omega$ such that $A \notin \mathcal{I}$.
4. Let $\mathrm{cof}(\mathcal{I})$ be the least cardinality of a subfamily $\mathcal{A} \subseteq \mathcal{I}$ which is cofinal: for every $X \in \mathcal{I}$, there is $Y \in \mathcal{A}$ such that $X \subseteq Y$.

Let $\mathcal{N}$ and $\mathcal{M}$ denote the Lebesgue null ideal and the meager ideal on 2^ω, respectively. Also for $\mu \in \mathsf{Meas}$, $\mathcal{N}_\mu$ denote the null ideal with repect to the measure μ.

Definition 4. Let $(P, \leqslant)$ be a preordered set. Define $\mathfrak{d}(P, \leqslant)$ be the least cardinality of a subset $A \subseteq P$ which is a dominating family in P: for every $p \in P$, there is $q \in A$ such that $p \leqslant q$. Also, define $\mathfrak{b}(P, \leqslant)$ be the least cardinality of a subset $A \subseteq P$ which is a unbounded family in P: for every $p \in P$, there is $q \in A$ such that $\neg(q \leqslant p)$.

Let $\leqslant^*$ be the almost domination order of ω^ω. We just write $\mathfrak{d}(\omega^\omega, \leqslant^*)$ and $\mathfrak{b}(\omega^\omega, \leqslant^*)$ as $\mathfrak{d}$ and $\mathfrak{b}$, respectively. Also, for a cardinal κ, $\mathfrak{d}_\kappa$ denotes $\mathfrak{d}(\kappa^\kappa, \kappa^\kappa, \leqslant)$, where $\leqslant$ is the pointwise domination order.

The important fact is the following.

Fact 5 (folklore)**.** $\mathfrak{b}(\mathsf{Meas}, \ll) = \aleph_1$ and $\mathfrak{d}(\mathsf{Meas}, \ll) = \mathfrak{c}$.

1 Generalizing Yorioka's proof

Theorem 6. If $\mathrm{add}(\mathcal{N}) = \mathfrak{c}$, then $\mathrm{cof}(\mathcal{UN}) \geqslant \mathfrak{d}_{\mathfrak{c}}$.

Proof. This proof is based on Yorioka's proof that CH implies $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\aleph_1}$ in [4].

Let $\kappa := \mathrm{add}(\mathcal{N}) = \mathfrak{c}$. Build a cofinal sequence $\langle \mu_\alpha : \alpha < \kappa \rangle$ in $(\mathsf{Meas}, \ll)$ and a matrix $\langle A_\alpha^\beta : \alpha, \beta < \kappa \rangle$ such that

1. $A_\alpha^\beta \subseteq 2^\omega$ is a G_δ dense subset such that $\mu_\alpha(A_\alpha^\beta) = 0$.
2. For each $\alpha < \kappa$, the sequence $\langle A_\alpha^\beta : \beta < \kappa \rangle$ is cofinal increasing in $\mathcal{N}_{\mu_\alpha}$.
3. For each $\alpha < \kappa$ and each $B \subseteq 2^\omega$ which has μ_α-measure 0, we have $\bigcap_{\gamma<\alpha} A_\gamma^0 \setminus B \neq \varnothing$.

We claim that such a sequence and a matrix exist. Let $\langle \mu_\alpha^* : \alpha < \kappa \rangle$ in $(\mathsf{Meas}, \ll)$ be a cofinal family. Assume we have constructed $\langle \mu_\alpha : \alpha < \alpha' \rangle$ and $\langle A_\alpha^\beta : \alpha < \alpha', \beta < \kappa \rangle$. By $\mathrm{add}(\mathcal{M}) = \kappa$, we have $\bigcap_{\gamma<\alpha'} A_\gamma^0$ is comeager, so it contains a perfect subset. Thus there is $\nu \in \mathsf{Meas}$ such that $\nu(\bigcap_{\gamma<\alpha'} A_\gamma^0) = 1$. Let $\mu_{\alpha'} := (\mu_{\alpha'}^* + \nu)/2$. Also take a cofinal increasing sequence $\langle A_{\alpha'}^\beta : \beta < \kappa \rangle$ using $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \kappa$. This finishes the description of the construction.

Claim 6.1. For every $F\colon \kappa \to \kappa$, there is $G\colon \kappa \to \kappa$ and $\langle x_\alpha : \alpha < \kappa \rangle$ satisfying:

(A) $F \leqslant G$ (pointwise domination).

(B) For every $\alpha < \kappa$, we have $\{x_\gamma : \gamma \leqslant \alpha\} \subseteq A_\alpha^{G(\alpha)}$.

(C) For every $\alpha < \kappa$, $x_\alpha \in \bigcap_{\gamma<\alpha} A_\gamma^{G(\gamma)} \setminus A_\alpha^{F(\alpha)}$.

Proof. Assume we have constructed $\langle G(\gamma), x_\gamma : \gamma < \alpha \rangle$. By (3), there is $x_\alpha \in 2^\omega$ such that $x_\alpha \in \bigcap_{\gamma<\alpha} A_\gamma^0 \setminus A_\alpha^{F(\alpha)}$. Since $\bigcap_{\gamma<\alpha} A_\gamma^0 \subseteq \bigcap_{\gamma<\alpha} A_\gamma^{G(\gamma)}$, we have (B). Take $G(\alpha) < \omega_1$ above $F(\alpha)$ such that $\{x_\gamma : \gamma \leqslant \alpha\} \subseteq A_\alpha^{G(\alpha)}$. Here, we used $\kappa = \mathrm{non}(\mathcal{N})$. //

Now assume $\mathrm{cof}(\mathcal{UN}) < \mathfrak{d}_\kappa$. Then there is a basis $\mathcal{B} \subseteq \mathcal{UN}$ with $|B| < \mathfrak{d}_\kappa$. For each $B \in \mathcal{B}$, we take $F_B\colon \kappa \to \kappa$ such that $B \subseteq \bigcap_{\gamma<\kappa} A_\gamma^{F_B(\gamma)}$. Since $|\mathcal{B}| < \mathfrak{d}_\kappa$, we can take $F\colon \kappa \to \kappa$ such that $F \not\leqslant F_B$ for every $B \in \mathcal{B}$. By the claim above, we can take $G\colon \kappa \to \kappa$ and $\langle x_\alpha : \alpha < \kappa \rangle$ satisfying (A), (B) and (C). By (B) and (C), we have $\{x_\alpha : \alpha \in \kappa\} \subseteq \bigcap_{\gamma<\kappa} A_\gamma^{G(\gamma)}$. Thus $\{x_\alpha : \alpha \in \kappa\} \in \mathcal{UN}$.

Also we now show $\{x_\alpha : \alpha \in \kappa\} \not\subseteq B$ for every $B \in \mathcal{B}$. Fix $B \in \mathcal{B}$. By $F \not\leqslant F_B$, there is $\gamma^* < \kappa$ such that $F_B(\gamma^*) < F(\gamma^*)$. Then, $B \subseteq \bigcap_{\gamma<\kappa} A_\gamma^{F_B(\gamma)} \subseteq A_{\gamma^*}^{F_B(\gamma^*)} \subseteq A_{\gamma^*}^{F(\gamma^*)}$. On the other hand, by (C), we have $x_{\gamma^*} \notin A_{\gamma^*}^{F(\gamma^*)}$.

These things contradict the fact $\mathcal{B} \subseteq \mathcal{UN}$ is a basis. □

Corollary 7. It is consistent that $\mathrm{cof}(\mathcal{UN}) > \mathfrak{c}$.

Remark 8. In Sacks model, $\mathcal{UN} \subseteq [\mathbb{R}]^{\leqslant\aleph_1}$ holds (Theorem 1.1.4 of [2]). Thus in Sacks model with $2^{\aleph_1} = \aleph_2$, it holds that $|\mathcal{UN}| \leqslant \mathfrak{c}$. Therefore, in this model, $\mathrm{cof}(\mathcal{UN}) \leqslant \mathfrak{c}$ holds. Also $\mathrm{cov}(\mathcal{UN}) \geqslant \aleph_2 = \mathfrak{c}$ holds in this model because $\aleph_1$ many sets of size $\leqslant \aleph_1$ cannot cover the entire space of size $\aleph_2$. So it is consistent that $\mathrm{cov}(\mathcal{UN}) = \mathrm{cof}(\mathcal{UN}) = \mathfrak{c}$.

2 Additivity

Fact 9 (Theorem 8.6 of [1])**.** Let $A \subseteq 2^\omega$. Then $A \in \mathcal{UN}$ if and only if $f(A) \in \mathcal{N}$ for every homeomorphism $f\colon 2^\omega \to 2^\omega$.

Lemma 10. Let $A \in \mathcal{UN}$. Then for every homeomorphism $f\colon 2^\omega \to 2^\omega$, we have $A \in \mathcal{UN}$.

Proof. This is an easy consequence of the above fact. □

In the following proposition, we use a notation: for ideals $\mathcal{I} \subseteq \mathcal{J}$, let $\mathrm{add}(\mathcal{I}, \mathcal{J})$ be the least cardinality of a subfamily $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} \notin \mathcal{J}$.

Proposition 11. $\mathrm{add}(\mathcal{UN}, \mathcal{N}) = \mathrm{add}(\mathcal{UN})$ holds.

Proof. $\mathrm{add}(\mathcal{UN}) \leqslant \mathrm{add}(\mathcal{UN}, \mathcal{N})$ is clear. To prove the other direction, let $\kappa < \mathrm{add}(\mathcal{UN}, \mathcal{N})$ and let $\{A_i : i < \kappa\} \subseteq \mathcal{UN}$. Let $A = \bigcup_{i<\kappa} A_i$. We need to show that $A \in \mathcal{UN}$. Let $f\colon 2^\omega \to 2^\omega$ be a homeomorphism. Then, by Lemma 10, we have $f(A_i) \in \mathcal{UN}$ for every $i < \kappa$. Therefore, we have $f(A) \in \mathcal{N}$ using $\kappa < \mathrm{add}(\mathcal{UN}, \mathcal{N})$. Thus, $A \in \mathcal{UN}$ holds. □

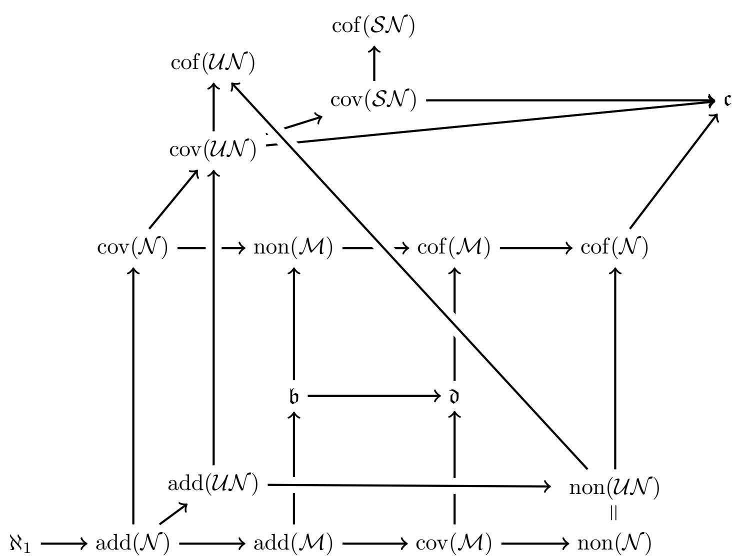
3 Uniformity

Proposition 12. $\mathrm{non}(\mathcal{UN}) = \mathrm{non}(\mathcal{N})$.

Proof. It is trivial using the measure isomorphism theorem. □

4 Discussions

Summarizing the results so far yields the following diagram.



The following questions remain.

Question 13. Is it consistent that $\mathrm{cof}(\mathcal{UN}) < \mathfrak{c}$, or does ZFC prove $\mathfrak{c} \leqslant \mathrm{cof}(\mathcal{UN})$?

Question 14. Is it consistent that $\mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{UN})$, or does ZFC prove $\mathrm{add}(\mathcal{N}) = \mathrm{add}(\mathcal{UN})$?

Let $\mathcal{UM}$ be the ideal of universally meager sets.

Question 15. Does ZFC prove that $\mathrm{add}(\mathcal{UN}) \leqslant \mathrm{add}(\mathcal{UM})$?

The way that naturally comes to mind for increasing $\mathrm{add}(\mathcal{UN})$ is to force $\mathcal{UN} = [2^\omega]^{\leqslant \aleph_1}$. But this statement is incosistent.

Proposition 16. $\mathcal{UN} \neq [2^\omega]^{\leqslant \aleph_1}$.

Proof. Suppose $\mathcal{UN} = [2^\omega]^{\leqslant \aleph_1}$. Then, $\mathrm{non}(\mathcal{UN}) = \mathrm{non}(\mathcal{N}) = \aleph_2$ holds. But, by Grzegorek's theorem [3], there is a universally null set of size $\mathrm{non}(\mathcal{N})$. It contradicts $\mathcal{UN} = [2^\omega]^{\leqslant \aleph_1}$. □

References

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