Weak Impredicativity and Frege's *Grundgesetze*

In this talk, I will explore a notion of "weak impredicativity" – ascribed to Gödel (cf. [Gödel(1970)]) in [Van Atten(2015)] –[1], and its potentially fruitful application in revising Frege's *Grundgesetze* [Frege(1893)].

The debate on predicativity/impredicativity shows that these terms can be used to refer to different phenomena [Feferman(2005)].[2] For the sake of simplicity, we focus here on only one coarse-grained distinction between what we will call strong and weak impredicativity.

Impredicativity *simpliciter* (here understood as "strong impredicativity") is usually identified as a violation of a syntactic criterion of predicativity. The latter stems from the seminal Russellian formulation of the Vicious Circle Principle [Russell(1908)] and has subsequently been refined into at least two further notions: "strict predicativity"[3] and "classical predicativity",[4] cf. [Crosilla(2022)], [Contente(2025)]. An interesting debate addresses the relationship between the two and the limitations of the latter. Nevertheless, we will consider them together because they appear to rely on the same ban, with or without the conspicuous exception of natural numbers. In other words, they converge in identifying definitions (including introduction rules) as (strongly) impredicative if the *definiens* does contain a bound variable of the same order as the *definiendum* (including or except for the definition of natural numbers).

On the other hand, there is a competing weaker notion of impredicativity that, already identified in [Gödel(1970)], is today formalised via *finitely stratified polimorphism* [Leivant(1991)], [Van Atten(2016)] or within Martin Löf type theory as a form of constructive predicativity. Finitely stratified polymorphism is distinct from predicative polymorphism in that it is not restricted to working with types of non-polymorphic terms, and from (what we called) strongly impredicative polymorphism in that, while it can in principle work with all types, these must already have been proven. We argue that these forms of polymorphism exhibit a form of impredicativity because it involves a violation of the aforementioned syntactic ban. Nevertheless the violation turns out to be weak because, as an effect of the stratification, these terms cannot take their own type as input.

Two remarks will be useful with respect to this taxonomy. Firstly, we will argue that the notion of weak impredicativity (formalised via stratified polymorphism) strongly resembles a criterion of predicativity as constructivity, also known as "generalised" predicativity [Crosilla(2016)], [Lorenzen(1955)], [Wang(1963)] or "constructive" predicativity [Crosilla(2022)], [Contente Klev(2025)]. In virtue of this criterion many inductive definitions can be classified as predicative – or in our terms "weakly" impredicative. While a Brouwerian inspiration has been identified in the development of finitely stratified polymorphism [Van Atten(2016)], constructive predicativity has so

[1] Gödel 1970: "There are functions of lower type which (within T) can only be defined by using functions of much higher types. This 'impredicativity' is perfectly legitimate, also from the constructivist point of view."

[2] It is correct to say that most of the debate focus on the definition of the phenomenon of predicativity, considering impredicativity just as its negation. Only for the sake of simplicity – with respect to Gödel quotation and the Fregean debate – we will pose the accent on impredicativity.

[3] This notion of predicativity is the most austere, requiring the rejection of large portions of mathematics. In particular, with respect to this criterion, the induction scheme is impredicative; therefore, only fragments of arithmetic can be admitted ([Feferman(1964)]).

[4] This concept arose from Feferman and Kreisel's parallel investigation into the limits of predicativity, i.e. examining the extent to which predicative theories could be developed once natural numbers were accepted as given (despite their strictly impredicative definition). In this approach, natural numbers are assumed to be given, and a ramified hierarchy of second-order arithmetical theories with ramified comprehension is provided. Each stage is represented by a theory indexed by an ordinal available in the previous stage, and the limit of predicativity is identified as the first ordinal that cannot be proved predicatively (which has been proven to be Γ_0).

far been formalised within Martin-Löf type theory [Contente Klev(2025)]. In the talk, we will argue for a convergence of the two approaches. In support of this, while we characterised the notion of (strong) impredicativity as a violation of strict and classical predicativity as originally syntactic, the notions of weak impredicativity and generalised or constructive predicativity appear to stem from an ontological characterisation of the stratified structure of their entities. This confirms an analogy between both notions and the set-theoretic criterion of well-foundedness, cf. [LinneboShapiro(2023)].

Secondly, despite the original ontological characterisation of weak impredicativity (inspired by Brouwer and Martin-Löf) and the type theoretical handling of this notion in the relevant literature, I will explore its potential proof theoretical recasting in terms of conceptual explanation, based on [Poggiolesi(2025a)], [Poggiolesi(2025b)]. In this account, both explanatory proofs and explanatory steps in mathematical proofs that are informally classified as definitions are characterised by an increase in conceptual complexity from the premises to the conclusion (or the *definiens* and the *definiendum*).[5] For our purposes, I will argue that the distinction between weak and strong impredicativity can be spelled out by proving that only (the rules for) the former exhibit an explanatory power.

In the second part of the talk, we will focus on the role of impredicative comprehension in Frege's *Grundgesetze*[6] and on its possible predicativist revisions. The predicative subsystems that have been proposed so far – [Heck(1996)], [Wehmeier(1999)], [FerreiraWehmeier(2002)], [Boccuni(2010)], [Ferreira(2018)], [Boccuni(2022)] – are driven by a strict notion of predicativity (cf. note 3)[7]. These solutions can restore the consistency of the system, but only derive a fragment of second-order Peano arithmetic (PA^2), which is equivalent to Robinson arithmetic (Q). As with strict predicativism, these systems require us to renounce induction, even though it is not assumed as an axiom here, but rather derived as a theorem. Furthermore, the derivation of arithmetic does not proceed, not even partially, via Frege's definitions, since most of the original definitions, particularly the crucial notion of "ancestral" (cf. note 8), are precluded. Therefore, even with respect to the Neologicist proposal (cf. [HaleWright(2001)]), a predicativist restriction would undermine the derivation of Frege's theorem, i.e. the derivation of second-order Peano axioms from second-order logic augmented with Hume's principle and Frege's definitions (ancestral, weak ancestral, predecessor and natural number).[8] In other words, banning strong impredicativity via a strict notion of predicativity seems to be an expensive way of achieving consistency, as it excludes a feature that is not only involved in the derivation of Russell's paradox, but is also necessary for the logicist derivation of Frege's theorem, i.e. the Fregean derivation of arithmetic.

[5]More precisely, proofs are considered to be explanatory if they meet the following criteria: the grounds of the explanation are captured by undischarged assumptions and the explanatory steps are formalized by explanatory rules – i.e. introduction rules that work deep inside the formulas [Poggiolesi(2025a)]. Explanatory rules might be of two types: logical explanatory rules [Poggiolesi(2025a)], but also mathematical explanatory rules, that are obtained by transforming non-logical generalizations or definitions, that have the form of biconditional axioms, into deep-rules [Poggiolesi(2025b)].

[6] For the sake of simplicity, we will directly work on Heck's first-order reformulation of the system, based on Archè Logic [Heck(2011)], in which second-order comprehension axiom scheme is substituted by the corresponding schematic introduction and elimination rules for new predicates that exclude explicit second order quantification. Remarkably, Heck proves that this system suffices as an underlying logic for Frege's theorem because it is proof theoretically equivalent to second-order logic with Π^1_1 comprehension.

[7]The classical approach of taking natural numbers for granted would not be a good starting point for the logicist programme of deriving arithmetic from a logical system augmented with some definitions.

[8]Ancestral of R: $\mathrm{R}^*(\mathrm{xy}) \equiv_{def} \forall X((\forall z(Rxz \rightarrow Xz)) \wedge (Her(X,R) \rightarrow Xy))$Weak Ancestral of R: $R^+(x,y) \equiv_{def} R^*(x,y) \vee x = y$. Predecessor: $\mathrm{P(x,y)} \equiv_{def} \exists X \exists z(Xz \wedge y = \sharp X \wedge x = \sharp[Xw \wedge w \neq z])$. Natural Number: $Nx \equiv_{def} P^+(0,x)$.

Conversely, we will emphasise that the impact of a "weakly impredicative" approach has yet to be explored, and we will suggest that it would offer a relevant revision of Frege's *Grundgesetze*.

In general, we suggest to admit only weakly impredicative formulas to specify new predicates. More precisely, we will identify and test a consistent and powerful predicative restriction of the comprehension axiom schema: $\forall\alpha R(D,\alpha) \leftrightarrow \forall y(\phi(y) \rightarrow R(\alpha, y))$ – where R is the characteristic relation for elements of the type of the *definiendum* (D), i.e. the relation used to provide its identity conditions. We will demonstrate that such a scheme is compatible with the Fregean vocabulary and the weakest form of impredicativity discussed so far. Then, we will refine this intuitive restriction by employing a criterion inspired by that of finitely stratified polymorphism. Applying this criterion requires a preliminary work to recast Frege's system in a suitable theory. More precisely, we transpose the abovementioned schematic and first-order version of *Grundgesetze* (cf. note 6) into a system consisting of classical first-order natural deduction, augmented with two non-logical rules for the abstraction operator [Tennant(1978)], [Tennant(2024)]. This also formalises and develops a recent inferentialist trend in the abstractionist debate [Heck(2011)], [Tennant(1978)], [Tennant(2024)], [Wright(2025)].[9]

In this framework, Fregean (strongly) impredicative definitions – such as that involved in Russell's paradox – will be banned. However, those that meet the requirement of weak predicativity (such as the definition of ancestral) will continue to be admitted and handled via suitable introductory and elimination rules. This framework allows us to test the hypothesis that the resources necessary for Frege arithmetic are impredicative with respect to the notion of strict predicativity, but not with respect to that of generalised predicativity; they can therefore be preserved as safely impredicative. Finally, the suggested revision will result in a revised proof of a weakly impredicative version of Frege's theorem.

In the final part of the talk, we will explore the impact of these results on three key questions in the abstractionist debate. Firstly, we will consider the distinction between so-called "idle" and "active" impredicativity (cf. [Wright(2021)]). As Wright (cf. [Wright(2021)]) mentioned, all non-problematic cases of "idle" impredicativity involve unrestricted quantified statements whose truth conditions are the same as those of the corresponding predicative statements. Conversely, definitional flaws, or even paradoxical evaluations, affect cases of "active" impredicativity, in which the truth conditions of unrestricted quantified statements cannot be reduced to those of the corresponding predicative version. This distinction is crucial not only for the Fregean project, but also more generally for understanding the different forms of circularity. We will argue that our distinction between weak and strong impredicativity precisely recasts that between "idle" and "active" impredicativity.

Secondly, we will examine the relationship between weak (or idle) impredicativity and generality (cf. [Linnebo(2022)]). A quick overview of the different treatments of generality will help us to distinguish between alternative taxonomies of the generality expressed by the quantifiers. To the dichotomy between absolute and relative generality ([RayoUzquiano(2006)]), we should add an orthogonal dichotomy between instance-based and non-instance-based generality ([Linnebo(2022)]). We will argue that in both absolute and relative conceptions of generality, discrimination between active and idle impredicativity (cf. [Russell(1908)], [Poincare(1903)]) relies on the improper request of an instance-based verification. This preliminary consideration will enable us to explore the relationship between weak impredicativity and different interpretations of non-instance-based generality, proving that the former implies the latter.

Finally, we will reconsider the choice between static and dynamic approaches (cf. [Linnebo(2009)],

[9] This revision of the abstractionist theories aligns with the proof-theoretic tradition of incorporating non-logical axioms into existing first-order logical proof systems, by transforming them into rules that preserve the logical properties of the theories.

[Linnebo(2018)]) in light of weak impredicativity. As is well-known, the opposing strands of static and dynamic abstractionism were typically characterised by their stance on impredicativity. First-order impredicativity is not only tolerated, but also considered an essential component of the Neologicist epistemology and metaphysics of abstraction principles (cf. [Wright(2021)]: quantifying over the entire domain is innocent with respect to paradoxes and compatible with identifying abstracts, precisely in virtue of the non-instantial meaning of generality mentioned above. Conversely, based on his analysis of Russell's paradox and the traditional Dummettian objection of vicious circularity (cf. [Dummett(1991)]) raised against the abstraction principles, Linnebo proposed a predicative revision of the abstractionist projects, resulting in a dynamic approach — metaphorically speaking, quantifying over "old" and safe objects to identify new abstract ones. Our proposal supports the claim that at least one form of impredicativity (i.e. the weak form) is always legitimate and can underpin the logicist project. This seems to corroborate the Neologicist position. However, a more careful analysis of the criterion of weak impredicativity reveals that it closely resembles a dynamic process in both its required stratification and its constructive flavour. We will indeed emphasise an analogy between weak impredicativity and "determination", i.e. a seminal criterion of second-order dynamic abstraction (cf. [Linnebo(2009)]).

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