The Indeterminacy of Isomorphism Invariance

1 Introduction

According to the model-theoretic account of logical consequence, a formula φ is a logical consequence of a set of formulae Γ just in case φ comes out true in every model in which Γ does, no matter how one interprets the non-logical vocabulary contained in Γ and φ (provided only this interpretation is uniform). The invoked distinction between logical and non-logical vocabulary, not characterised further, is drawn in merely informal terms. The most popular attempt at drawing this distinction in a principled manner is by means of invariance criteria ((10), (19), (17; 18), (16),(11)), the most discussed of which is isomorphism invariance.

The most-known criticisms have so far questioned whether isomorphism invariance delivers the right set of logical constants ((4), (6), (5)). This paper develops a new objection, which calls in to question the determinacy itself of isomorphism invariance as a criterion of logicality. Whilst the semantic backdrop in which discussions about isomorphism invariance are conducted usually countenances a single domain of objects, serious philosophical reasons have been adduced to motivate the need for a dual-domain semantics (more on this in §2). Semantics of this kind characterise the best known systems of free logic, where:

> A free logic is a formal system of quantification theory [...] which allows for some singular terms in some circumstances to be thought of as denoting no existing object, and in which quantifiers are invariably thought of as having existential import ((2: 148-149)).

My contention is that there is not much we can hope to learn about the logicality of quantifiers in these settings; at least by isomorphism invariance *alone*. Here, *isomorphism invariance* may be elucidated in two different ways, warranting contradictory claims about the logical status of quantifiers (§3). Possible attempts at establishing one elucidation over the other will be considered, and shown to leave matters unsettled (§4). Defenders of isomorphism invariance have an urgent case to answer concerning the applicability of their preferred criterion of logicality.

2 Dual-Domain Semantics

According to an influential account due to (14), the particular quantifier "$\exists$" has existential import, being associated to a domain of existent objects. In classical predicate logic, this interpretation of "$\exists$" yields rather counterintuitive consequences. Since the semantics stipulates that every term denotes something in the domain of "$\exists$", terms cannot but denote existent objects. Even worse, since the semantics rules out models with an empty domain, it turns out to be a logical truth that something exists.

Free logic was developed in the second half of the 20th century to address such oddities. Whilst free logicians held on to the existential import of "$\exists$", they relaxed the existential import of singular terms by allowing *empty terms*, not denoting existent objects[1]. The simplest account of empty terms is given in so-called *positive free logic* (PFL hereafter), where atomic formulae containing such terms are allowed to be true, and also to be false.

[1] This is unlike other revisionary approaches, such as the one distinctive of the noneist tradition, which *also* dispensed with the existential reading of "$\exists$" ((15), (3), (13)).

The most common types of PFL-models are given in terms of dual-domain structures $\mathscr{M}^P = \langle D_O, D_I, \nu \rangle$ where: D_I (inner domain) is a possibly empty set of existent objects; D_O (outer domain) is a non-empty set of objects such that $D_O \supseteq D_I$; ν is a total valuation function from a standard first-order signature to D_O. Whilst the outer domain comprises all the objects in a given model, the inner domain only comprises the existent ones (if any); and the valuation function assigns to every term an object in the outer domain regardless of its existential status. Then, $\ulcorner \exists x A \urcorner$ is true in an $\mathscr{M}^P$-model just in case at least one object *from the inner domain* satisfies the condition A. Truth-conditions for "$\forall$" ensue in the most obvious way; but under focus hereafter will be "$\exists$".

Specifically, is PFL-"$\exists$" logical, on the isomorphism invariance criterion *alone*? It was anticipated that this question does not have a determinate answer, due to the presence of multiple elucidations of *isomorphism invariance* delivering opposite results. The next section justifies this outcome.

3 The Status of "$\exists$"

Isomorphism invariance is primarily a property of worldly entities, namely extensions of linguistic expressions, and secondarily of the expressions denoting such worldly entities. An obstacle to providing a *general* definition of isomorphism invariance is that, for some familiar expressions, such as the usual Boolean truth-functional operators, it is not at all obvious what their extension would be. Luckily, the argument presented in this section requires only that a definition of isomorphism invariance for the extension of "$\exists$" be in place, and here we can build on pretty solid bases.

The extension of any (unary) first-order quantifier is standardly identified, in a domain of objects $D^{\mathscr{M}}$ associated to a model $\mathscr{M}$, with the collection of subsets of $D^{\mathscr{M}}$ satisfying a certain condition. This condition, in the case of the familiar "$\exists$" of classical predicate logic, is *being non-empty*; the condition satisfied exactly by those subsets of $D^{\mathscr{M}}$ comprising at least one member. Formally, where "$\exists^{D^{\mathscr{M}}}$" indicates the extension of "$\exists$" in $D^{\mathscr{M}}$,

$$\exists^{D^{\mathscr{M}}} = \{X \subseteq D^{\mathscr{M}} : X \neq \varnothing\};$$

that is, $\exists^{D^{\mathscr{M}}}$ is the set of non-empty subsets of $D^{\mathscr{M}}$.

Moving from the single-domain semantics of classical predicate logic to the dual-domain semantics of PFL, we can no longer identify the extension of "$\exists$" with *being non-empty.* The reason is that there may be non-empty subsets of the outer domain comprising only non-existent objects which, given the existential import of "$\exists$", should not feature in its extension. Simply replacing *being non-empty* with *being non-empty and comprising at least one existent* will obviate. Thus, where "$\exists^{D_O^{\mathscr{M}^P}}$" indicates the extension of "$\exists$" in the outer domain D_O associated to a model $\mathscr{M}^P$:

$$\exists^{D_O^{\mathscr{M}^P}} = \{X \subseteq D_O^{\mathscr{M}^P} : X \cap D_I \neq \varnothing\},$$

that is, $\exists^{D_O^{\mathscr{M}^P}}$ is the set of subsets of the outer domain which have a non-empty intersection with the inner domain. If one wishes, one may follow a more stringent approach, replacing the above condition with one forcing out of the extension of "$\exists$" any set comprising non-existent objects. Truth conditions for "$\exists$" would then have to be reformulated as appropriate. Opting for this approach does not affect the argument presented below.

As a useful warm up before defining isomorphism invariance for "$\exists$" in PFL, let us first define isomorphism invariance for "$\exists$" in classical predicate logic. To begin with, isomorphisms are mappings between pairs of domains (each associated to a model) which preserve their internal structure, subject to the following two constraints: they ensure that any two distinct objects are mapped to distinct values (that is, the map is injective); and they ensure that every value is mapped to by some object (the map is also surjective). Below we will have to apply isomorphisms to *subsets*, S, of a domain; and to collections, T, of such subsets. The images of S and T under an isomorphism π - written, $\pi(S), \pi(T)$ - will be the sets resulting from applying π to the members of S and T respectively. Thus, $\pi(S)$ will be a set of objects, namely, the set which results from applying π to the members of S; and $\pi(T)$ will be a set of sets of objects, comprising exactly the images of every $S \in T$. More rigorous definitions can be given (see e.g. (9)); however, to minimise technicalities, those informal considerations will suffice.

In classical predicate logic, we can unambiguously refer to *the domain of a model*, since the semantics associates exactly one domain to each model. Let thus $\pi : D^{\mathscr{M}_1} \mapsto D^{\mathscr{M}_2}$ be any isomorphism from the domain of $\mathscr{M}_1$ to the domain of $\mathscr{M}_2$. Then, "$\exists$" is isomorphism invariant just in case $\exists^{D^{\mathscr{M}_2}} = \pi(\exists^{D^{\mathscr{M}_1}})$; that is, just in case the extension of "$\exists$" in $D^{\mathscr{M}_2}$ is identical to the image, under the isomorphism, of the extension of "$\exists$" in $D^{\mathscr{M}_1}$. Then, "$\exists$" is isomorphism invariant, since $\exists^{D^{\mathscr{M}_2}}$ and $\pi(\exists^{D^{\mathscr{M}_1}})$ have indeed the same members. On the one hand, every non-empty subset of $D^{\mathscr{M}_2}$ is in the image, under π, of $\exists^{D^{\mathscr{M}_1}}$. This is so because π is surjective, which means that every subset of $D^{\mathscr{M}_2}$ is the image of some subset of $D^{\mathscr{M}_1}$. On the other hand, every subset of $D^{\mathscr{M}_2}$ in the image of $\exists^{D^{\mathscr{M}_1}}$, under π, is non-empty. This is so because every such subset of $D^{\mathscr{M}_2}$ is the image of some non-empty subset of $D^{\mathscr{M}_1}$, and will thus comprise the images of the members of that subset of $D^{\mathscr{M}_1}$. Therefore, $\exists^{D^{\mathscr{M}_2}} = \pi(\exists^{D^{\mathscr{M}_1}})$.

Let us turn now to isomorphism invariance for "$\exists$" in PFL. Since the semantics here countenances an inner/outer domain pair instead of a single domain, isomorphisms adequate for this semantics (more succinctly, *dual-domain isomorphisms*) must be characterised as mappings between inner/outer domain *pairs*, preserving the structure of each member in the pair. Informally, the dual-domain semantics groups objects into inner/outer domain pairs, and *dual-domain isomorphisms* are just switchings of objects preserving the structure of each domain in a pair. However, some dual-domain isomorphisms will switch existent object with existent object and non-existent object with non-existent object, thereby preserving existential status. Others will not preserve this feature, switching objects with different existential status. Thus, in defining isomorphism invariance for "$\exists$", should we consider *all* switchings which preserve the structures of inner/outer domain pairs, or be restricted to those switchings which also preserve existential status?

Each option can be justified by relevant considerations for the purpose of ascertaining the logical status of PFL-"$\exists$". Considering only those switchings of dual-domain isomorphisms which preserve existential status responds to the need of preserving a feature of paramount importance for PFL and its account of "$\exists$". Lifting this restriction, and considering all switchings performed by dual-domain isomorphisms, responds to the need of ensuring that our judgement about the logicality of "$\exists$" upholds the highest standards of generality available. There are thus two types of switchings which may drive our verdict about the logicality of "$\exists$". Consequently, there are two possible elucidations of *isomorphism invariance*. Here is how to make them precise[2].

First, a dual-domain isomorphism $\pi : \langle D_I^{\mathscr{M}^P}, D_O^{\mathscr{M}^P} \rangle \mapsto \langle D_I^{\mathscr{M}^{P^*}}, D_O^{\mathscr{M}^{P^*}} \rangle$ is a structure-preserving mapping from the inner/outer domain pair of $\mathscr{M}^P$ to that of $\mathscr{M}^{P^*}$. The do-

[2]Recently, (7) have applied the isomorphism invariance criterion of logicality to those versions of ontological pluralism which account for concrete and abstract objects as enjoying different kinds of existence, expressed by quantifiers the $\exists^C, \exists^A$ with disjoint domains; for an overview, see (20). Also the pluralist's (dual-domain) semantics, they found, allows for general and restricted elucidations of *isomorphism invariance*. However, they contend that neither elucidation is viable for the pluralist. Going into detail is unnecessary, as the argument revolves around the commitments of ontological pluralism, orthogonal to present concerns.

main of π is $D_I^{\mathscr{M}^P} \cup D_O^{\mathscr{M}^P} = D_O^{\mathscr{M}^P}$, its codomain $D_I^{\mathscr{M}^{P^*}} \cup D_O^{\mathscr{M}^{P^*}} = D_O^{\mathscr{M}^{P^*}}$, and its range $\text{Ran}(\pi) = \{\pi(x) \in D_O^{\mathscr{M}^{P^*}} : x \in D_O^{\mathscr{M}^P}\}$ - the collection of members of the codomain assigned by π to the members of the domain. Since π is surjective, its range is by definition identical with its codomain; however, the importance of the range of a dual-domain isomorphism will become clear very shortly. The more general elucidation of *isomorphism invariance* was characterised as insensitivity to all switchings of objects which preserve the structures of inner/outer domain pairs. Given that any such switching is performed by a dual-domain isomorphism π defined as above, "$\exists$" is isomorphism invariant in this sense just in case $\exists^{D_O^{\mathscr{M}^{P^*}}} = \pi(\exists^{D_O^{\mathscr{M}^P}})$, for every π.

On the other hand, a dual-domain isomorphism $\pi : \langle D_I^{\mathscr{M}^P}, D_O^{\mathscr{M}^P}\rangle \mapsto \langle D_I^{\mathscr{M}^{P^*}}, D_O^{\mathscr{M}^{P^*}}\rangle$ preserves existential status just in case $D_I^{\mathscr{M}^{P^*}}$ is the range of the restriction of π to $D_I^{\mathscr{M}^P}$, namely $(\pi \restriction D_I^{\mathscr{M}^P}) = \{\pi(x) \in D_O^{\mathscr{M}^{P^*}} : x \in D_I^{\mathscr{M}^P}\} = D_I^{\mathscr{M}^{P^*}}$; *and* $D_O^{\mathscr{M}^{P^*}} \backslash D_I^{\mathscr{M}^{P^*}}$ is the range of the restriction of π to $D_O^{\mathscr{M}^P} \backslash D_I^{\mathscr{M}^P}$, namely $(\pi \restriction D_O^{\mathscr{M}^P} \backslash D_I^{\mathscr{M}^P}) = \{\pi(x) \in D_O^{\mathscr{M}^{P^*}} : x \in D_O^{\mathscr{M}^P} \backslash D_I^{\mathscr{M}^P}\} = D_O^{\mathscr{M}^{P^*}} \backslash D_I^{\mathscr{M}^{P^*}}$. Of course, Ran $(\pi) = (\pi \restriction D_I^{\mathscr{M}^P}) \cup (\pi \restriction D_O^{\mathscr{M}^P} \backslash D_I^{\mathscr{M}^P}) = D_O^{\mathscr{M}^{P^*}}$. One then obtains the restricted elucidation of *isomorphism invariance* simply by narrowing down the above definition of *isomorphism invariance* for "$\exists$" to those dual-domain isomorphisms which preserve existential status.

It is now shown below that the two elucidations of *isomorphism invariance* just specified warrant opposite conclusions about the logical status of PFL-"$\exists$". Suppose that *isomorphism invariance* is elucidated in the most general way, that is, as insensitivity to *all* switchings of objects which preserve the structures of inner/outer domain pairs. Let $\mathscr{M}^P$ be a model where $D_I = \{a\}$, $D_O = \{a, b\}$ and, accordingly, $\exists^{D_O^{\mathscr{M}^P}} = \{\{a, b\}, \{a\}\}$. Consider now the dual-domain isomorphism $\pi : \langle D_O^{\mathscr{M}^P}, D_I^{\mathscr{M}^P}\rangle \mapsto \langle D_O^{\mathscr{M}^P}, D_I^{\mathscr{M}^P}\rangle$ from the inner/outer domain pair of $\mathscr{M}^P$ to itself, mapping a to b and b to a. Applied to $\exists^{D_O^{\mathscr{M}^P}}$, π returns the set $\pi(\exists^{D_O^{\mathscr{M}^P}}) = \{\{b, a\}, \{b\}\}$. Therefore, $\exists^{D_O^{\mathscr{M}^P}} \neq \pi(\exists^{D_O^{\mathscr{M}^P}})$. If to be logical is to exhibit the present form of isomorphism invariance, then PFL-"$\exists$" is not logical.

Suppose now that isomorphism invariance is elucidated more restrictedly, by disallowing mappings not preserving existential status. On *this* way of interpreting *isomorphism invariance*, PFL-"$\exists$" comes out logical. Consider a dual-domain isomorphism $\pi : \langle D_O^{\mathscr{M}^P}, D_I^{\mathscr{M}^P}\rangle \mapsto \langle D_O^{\mathscr{M}^{P^*}}, D_I^{\mathscr{M}^{P^*}}\rangle$ from the domains of $\mathscr{M}^P$ to the domains of $\mathscr{M}^{P^*}$, preserving existential status. There are two cases to check as $D_I^{\mathscr{M}^P}$ may or may not be empty. If $D_I^{\mathscr{M}^P}$ is not empty, then the argument is exactly analogous to the one, encountered earlier, with which one establishes isomorphism invariance for "$\exists$" in classical predicate logic.

Suppose, then, that $D_I^{\mathscr{M}^P}$ is empty. If $D_I^{\mathscr{M}^P}$ is empty, then so is $\exists^{D_O^{\mathscr{M}^P}}$, since no subset of the outer domain has a non-empty intersection with the empty set. Moreover, since $D_I^{\mathscr{M}^P}$ and $D_I^{\mathscr{M}^{P^*}}$ have the same size, it follows that $\exists^{D_O^{\mathscr{M}^{P^*}}}$ is also empty. Suppose, then, that $\exists^{D_O^{\mathscr{M}^{P^*}}} \neq \pi(\exists^{D_O^{\mathscr{M}^P}})$. Then, either π will map two distinct members of $\exists^{D_O^{\mathscr{M}^P}}$ to the same member of $\exists^{D_O^{\mathscr{M}^{P^*}}}$, or else π will have some member of $\exists^{D_O^{\mathscr{M}^{P^*}}}$ not mapped to by any member of $\exists^{D_O^{\mathscr{M}^P}}$. But since $\exists^{D_O^{\mathscr{M}^P}}$ and $\exists^{D_O^{\mathscr{M}^{P^*}}}$ are empty, both options are excluded. Hence, if $D_I^{\mathscr{M}^P}$ is empty, $\exists^{D_O^{\mathscr{M}^{P^*}}} = \pi(\exists^{D_O^{\mathscr{M}^P}})$. The upshot is that $\exists$" Is isomorphism invariant in the sense of III. If to be logical is to exhibit *this* sort of invariance, then PFL-"$\exists$" is logical.

Friends of isomorphism invariance appear to be faced with the difficult task of defending one elucidation of *isomorphism invariance* over the other. The next section anticipates some of the possible moves which may be made in the dialectic arising out of here, and shows that they leave matters unsettled.

4 Generality and Standard Practice

Logical expressions, it is often said, are distinctively general. Based on this thought, one may try to conclude that the correct standard of logicality is the one given in terms of the most general form of isomorphism invariance available; namely, invariance under all dual-domain isomorphisms. However, in this dialectical context, an appeal to generality may easily backfire. The reason is that, going by generality considerations alone, isomorphism invariance appears to lack traction in the first place. Indeed, isomorphisms are not the most general type of maps that can be defined between structures. More general forms of invariance are thus definable, and some of them have actually been put forward as candidate criteria of logicality in lieu of isomorphism invariance ((6), (4)). Generality considerations *alone* are a slippery slope.

A recurring thought is that isomorphism invariance allows one to retain the logicality of those expressions standardly recognised as logical: namely, the Boolean connectives, the familiar quantifiers and identity ((11: Ch. 8)). Perhaps, then, the right form of isomorphism invariance is simply the one delivering a verdict aligned with what is standardly assumed about PFL-quantifiers. In the remainder of this section, I will explain why this line of argument is equally unlikely to bring about progress.

One reason is that, unfortunately, there simply are no *standard* assumptions about the logical status of free quantification; commentators have, for better or worse, overlooked this issue. As such, one cannot appeal to the consensus around what is standardly assumed about PFL-quantifiers, simply because such *consensus* appears to be non-existent. However, one thesis which is standardly assumed about PFL is that it *generalises* classical predicate logic, by expanding the class of standard cases with a class of non-standard ones - where some objects are found outwith the domain of quantification ((21), (8: 131 ff.), (1: 5), (12: 479-480)). Could this routine assumption help us shed light on the logical status of PFL-quantifiers?

Hardly so. To simplify matters, let us keep focusing on "$\exists$", as we have done throughout this paper. The familiar existential quantifier is widely regarded as logical. But given that also PFL-"$\exists$" takes existential import, one may argue that the move to PFL does not change of meaning of "$\exists$". Thus, PFL-"$\exists$" may be deemed logical, simply in virtue of its synonymy with a logical constant.

However, even granting the argument, the synonymy of PFL-"$\exists$" with the familiar existential quantifier cannot determine whether PFL-"$\exists$" gets to keep its usual properties in the non-standard cases countenanced by PFL. This is so, for the very simple reason that classical predicate logic does not countenance such cases; so appealing to the synonymy of PFL-"$\exists$" with the familiar existential quantifier has no explanatory force. Merely *assuming* that PFL-"$\exists$" gets to preserve its logicality in such non-standard cases is unwarranted, since notions in general may well lose some of their usual properties in non-standard cases. For example, ordinal addition is commutative in the finite case, but in general it is not; the material conditional detaches insofar as it is confined to consistent cases, but in general it does not; and so on. The synonymy of PFL-"$\exists$" with the familiar existential quantifier may justify its logicality relative to standard cases; but we should not *expect* its logicality to automatically carry over to non-standard cases.

We have seen that generality considerations do not enable us to select either form of isomorphism invariance. Considerations pertaining to what is standardly assumed about PFL do not seem to do any better. Friends of isomorphism invariance owe us an urgent explanation about how their preferred criterion is to be applied, or find a plausible justification for its indeterminacy.

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