$\mathcal{M}^2$-computability of Semialgebraic Functions *

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Abstract

This paper continues ongoing research about applications of the subrecursive class $\mathcal{M}^2$ to computable analysis. We prove that all continuous semialgebraic real functions with open domains are conditionally computable with respect to the class **M2O** of $\mathcal{M}^2$-substitutional operators. We also prove that the definite integral with $\mathcal{M}^2$-computable limits of a semialgebraic, bounded and restricted analytic real function is uniformly computable with respect to **M2O**.

1 Introduction

Our goal is to provide evidence that several well-known classes of real numbers and real functions are efficiently computable. This type of problem has been extensively investigated in discrete complexity theory, where efficiency is characterized in terms of resource bounds such as time or space. However, our framework differs from this classical setting: instead of standard machine-based measures, we work with inductively defined subclasses of the second Grzegorczyk class.

To introduce a notion of computational complexity over the real numbers, we adopt the standard approach from computable analysis based on Cauchy representations with proper convergence rates. For real functions, we consider two kinds of computable realizers that operate on such Cauchy names. In the uniform case, we restrict ourselves to simple term-like operators. In the more general conditional case, we additionally allow a controlled form of unbounded search, which becomes necessary when dealing with functions defined on non-compact domains.

In their paper [9], Tent and Ziegler also consider two kinds of relative computability for real functions: low and uniformly low. Their approach is to work directly with the rational approximations without using realisers.

As shown by Skordev in [6], the uniform versions of these two approaches are equivalent.

But this is not the case for the more general notions. In fact, the conditional computability of real functions was introduced in [7] in order to produce a more natural notion, since the non-uniform notion of Tent and Ziegler is not closed under composition and it even contains real functions, which are not computable in the sense of computable analysis.

The aim of this paper is to improve the results of Tent and Ziegler for the complexity of semialgebraic functions, from the class $\mathcal{L}^2$ of lower elementary functions, to the class $\mathcal{M}^2$.

2 The class $\mathcal{M}^2$ and its higher-order counterpart M2O

All functions in this section are total functions of type $\mathbb{N}^k \to \mathbb{N}$.

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The classes we are interested in were implicitly defined by Grzegorczyk in [4], where he asked whether bounded primitive recursion can be replaced by limited minimum operation or by bounded summation when defining the second Grzegorczyk class $\mathcal{E}^2$.

The *initial functions* are: the projections $\vec{x} \mapsto x_i$, the successor $x \mapsto x+1$, the product function $(x,y) \mapsto xy$ and the modified subtraction $(x,y) \mapsto \max(x-y,0)$.

The function g is produced from f using *limited minumum operation* if $g(\vec{x},y) = \mu_{z \leq y}[f(\vec{x},z)=0]$ ($g(\vec{x},y) = y+1$ in case such a z does not exist).

Definition 1. *The class $\mathcal{M}^2$ is the smallest class of total functions in $\mathbb{N}$, which contains the initial functions and is closed under composition and limited minimum operation.*

It is well-known that $\mathcal{M}^2 \subseteq \mathcal{L}^2 \subseteq \mathcal{E}^2$, but whether these inclusions are strict is a long-standing open problem. The class $\mathcal{L}^2$ was introduced by Skolem in [5] (it is obtained using Definition 1 by replacing limited minimum operation with bounded summation).

We consider a fixed coding Π of triples. The decoding functions will be denoted by: $(\Pi(x,y,z))_0 = x, (\Pi(x,y,z))_1 = y, (\Pi(x,y,z))_2 = z$. The coding and the decoding functions can be chosen in $\mathcal{M}^2$. Note that the second level of Grzegorczyk hierarchy presumably does not contain coding of lists of arbitrary length, which is what makes complexity analysis at this level particularly challenging.

Since our aim is to consider computability of functions with real arguments and values, we will need a higher-order counterpart for the functional class $\mathcal{M}^2$.

A *k-operator* will be a total mapping of type $\mathcal{T}_1^k \to \mathcal{T}_1$, where $\mathcal{T}_1$ is the class of all unary total functions in $\mathbb{N}$.

Definition 2. *We define inductively the class of $\mathcal{M}^2$-substitutional operators:*

1. *For all n, the n-operator F defined by $F(\vec{f})(x) = x$ is $\mathcal{M}^2$-substitutional.*
2. *For any n and $k \in \{1,\dots,n\}$, if F_0 is an n-operator which is $\mathcal{M}^2$-substitutional, then the n-operator F defined by $F(\vec{f})(x) = f_k(F_0(\vec{f})(x))$ is also $\mathcal{M}^2$-substitutional.*
3. *For any n,k and $a : \mathbb{N}^k \to \mathbb{N}$, $a \in \mathcal{M}^2$, if $F_1,\dots,F_k$ are n-operators which are $\mathcal{M}^2$-substitutional, then so is the operator F defined by $F(\vec{f})(x) = a(F_1(\vec{f})(x),\dots,F_k(\vec{f})(x))$.*

Thus an n-operator F is $\mathcal{M}^2$-substitutional iff $F(f_1,\dots,f_n)(x)$ can be defined by a term over x using functional symbols for the arguments $f_1,\dots,f_n$ and for functions from $\mathcal{M}^2$.

The class of $\mathcal{M}^2$-substitutional operators will be denoted by **M2O**.

3 Relative computability of real numbers and functions

For $n \in \mathbb{N}$, we denote $\nu(n) = \frac{(n)_0-(n)_1}{(n)_3+1}$. We consider n as *a code* for the rational number $\nu(n)$. The tuple $\vec{n}$ will be called a code for $\vec{q} = \nu(\vec{n})$ if n_i is a code for q_i for all i.

The function $f : \mathbb{N} \to \mathbb{N}$ will be called *a name* for the real number ξ, if $|\nu(f(n)) - \xi| < \frac{1}{n+1}$ for all n. The tuple $\vec{f}$ will be called a name for $\vec{\xi}$ if f_i is a name for ξ_i for all i.

A real number ξ is called $\mathcal{M}^2$*-computable* if there exists a name $f \in \mathcal{M}^2$ for ξ.

Definition 3. *A real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$ will be called* uniformly **M2O**-computable *if there exists a k-operator F, belonging to* **M2O***, such that whenever $\vec{\xi} \in D$ and $\vec{f}$ is a name for $\vec{\xi}$, the function $F(\vec{f})$ is a name for $\theta(\vec{\xi})$.*

We will denote by $\hat{e}$ the unary constant with value e.

Definition 4. *A real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^k$ will be called* conditionally **M2O**-computable *if there exist a k-operator E and a $(k+1)$-operator F, both belonging to* **M2O**, *such that whenever $\vec{\xi} \in D$ and $\vec{f}$ is a name for $\vec{\xi}$: (1) there exists e, such that $E(\vec{f})(e) = 0$; (2) for any e, which satisfies $E(\vec{f})(e) = 0$, the function $F(\vec{f}, \hat{e})$ is a name for $\theta(\vec{\xi})$.*

Clearly, a computable realiser can be obtained from this definition using unbounded search for the value e. Results in [8, 7] imply that all elementary functions of calculus are conditionally **M2O**-computable and also uniformly **M2O**-computable, when restricted to compact domains.

A relation $R \subseteq \mathbb{R}^k$ is *semialgebraic*, if it is definable in the ordered field $(\mathbb{R}, 0, 1, +, -, \cdot, <)$ without parameters. A function $\theta : D \to \mathbb{R}, D \subseteq \mathbb{R}^k$ is called *semialgebraic* if its graph $\theta(\vec{\xi}) = \eta$ is semialgebraic. Our main source for semialgebraic sets and functions is [1].

Proposition 1. *([1], Proposition 2.6.1.) Let $a \in \mathbb{R}$ and $\theta : (a, +\infty) \to \mathbb{R}$ be a semialgebraic function. There exist $r > a$ and $m \in \mathbb{N}^+$, such that $\forall \xi (\xi \geq r \Rightarrow |\theta(\xi)| < \xi^m)$.*

Proposition 2. *Let $R \subseteq \mathbb{R}^k$ be a semialgebraic relation. There exists $R_\mathbb{Q} : \mathbb{N}^k \to \{0, 1\}$ with $R_\mathbb{Q} \in \mathcal{M}^2$, such that $R(\nu(\vec{n})) \Leftrightarrow R_\mathbb{Q}(\vec{n}) = 0$ for all $\vec{n} \in \mathbb{N}^k$.*

Proof. Let R be defined by the formula $\varphi(x_1, x_2, \ldots, x_k)$. By quantifier elimination φ is equivalent to a quantifier-free formula. Further, φ is equivalent to a Boolean combination of formulas of the form $p_1 < p_2$ or $p_1 = p_2$, where both p_1 and p_2 are polynomials with natural coefficients in variables $x_1, x_2, \ldots, x_k$. We replace any occurrence of x_i in p_1, p_2 by $\nu(n_i) = \frac{(n_i)_0 - (n_i)_1}{(n_i)_3 + 1}$ for $i = 1, 2, \ldots, k$. Clearly, we can rewrite $p_1 < p_2$ $(p_1 = p_2)$ as $p_1' < p_2'$ $(p_1' = p_2')$, where p_1' and p_2' are polynomials with natural coefficients of $(n_1)_0, (n_1)_1, (n_1)_2, \ldots, (n_k)_0, (n_k)_1, (n_k)_2$. Since all such polynomials and the decoding functions belong to $\mathcal{M}^2$, the result follows. □

4 Main Results

We will denote the maximum norm of $\vec{\xi} \in \mathbb{R}^n$ by $|\vec{\xi}|$. Let $O \subseteq \mathbb{R}^n$ be an open set.

Following [9], for any real $e \geq 1$ we define

$$O_e \;=\; \{\vec{\xi} \in O \mid |\vec{\xi}| \leq e \;\&\; \mathtt{dist}(\vec{\xi}, \mathbb{R}^n \setminus O) \geq e^{-1}\}.$$

Clearly, for $e \in \mathbb{N}^+$, $O_1 \subseteq O_2 \subseteq \ldots \subseteq O_e \subseteq \ldots$ and O is the union of this sequence.

Note that if O is semialgebraic, the relation $\vec{\xi} \in O_e$ is semialgebraic in $\vec{\xi}, e$.

Lemma 1. $\vec{x} \in O_e \;\&\; |\vec{x} - \vec{y}| < (2e)^{-1} \;\Rightarrow\; \vec{y} \in O_{2e}$.

Proof. Since $|\vec{x} - \vec{y}| < (2e)^{-1} < e^{-1}$ and $\mathtt{dist}(\vec{x}, \mathbb{R}^n \setminus O) \geq e^{-1}$, we have $\vec{y} \in O$. Furthermore, $|\vec{y}| \leq |\vec{y} - \vec{x}| + |\vec{x}| \leq (2e)^{-1} + e < 1 + e \leq 2e$. Finally, let $\vec{z} \in \mathbb{R}^n \setminus O$. Then $|\vec{x} - \vec{z}| \geq e^{-1}$ and we have $|\vec{y} - \vec{z}| \geq |\vec{x} - \vec{z}| - |\vec{x} - \vec{y}| \geq e^{-1} - (2e)^{-1} = (2e)^{-1}$, which implies $\mathtt{dist}(\vec{y}, \mathbb{R}^n \setminus O) \geq (2e)^{-1}$. □

Lemma 2. *Let $O \subseteq \mathbb{R}^k$ and $\theta : O \to \mathbb{R}$ be a continuous semialgebraic function. There exists $m_1 \in \mathbb{N}^+$, such that for all sufficiently large e: $\forall \vec{x}, \vec{y} \in O_e (|\vec{x} - \vec{y}| < e^{-m_1} \Rightarrow |\theta(\vec{x}) - \theta(\vec{y})| < e^{-1})$.*

Proof. For all $e \geq 1$, we have that O_e is compact, therefore θ is uniformly continuous on O_e. For any $e \geq 1$, there exists $d > 0$, such that $\forall \vec{x}, \vec{y} \in O_e (|\vec{x} - \vec{y}| < d^{-1} \Rightarrow |\theta(\vec{x}) - \theta(\vec{y})| < e^{-1})$. Let us denote this last property by $\mathcal{P}(e, d)$. Note that $d < d'$ & $\mathcal{P}(e, d)$ implies $\mathcal{P}(e, d')$. Since $\mathcal{P}$ is semialgebraic in d, e, the function $g : [1, +\infty) \to \mathbb{R}$, defined by $g(e) = \inf\{d > 0 | \mathcal{P}(d, e)\}$, is also semialgebraic. We apply Proposition 1 to obtain $m_1 \in \mathbb{N}^+$, such that $g(e) < e^{m_1}$ for all sufficiently large e. This clearly implies $\mathcal{P}(e, e^{m_1})$ for all such e and the proof is finished. □

Lemma 3. *Let* $O \subseteq \mathbb{R}^k$ *and* $\theta : O \to \mathbb{R}$ *be a continuous semialgebraic function. There exists* $m_2 \in \mathbb{N}^+$*, such that for all sufficiently large* e*:* $\forall \vec{x} \in O_e(|\theta(\vec{x})| < e^{m_2})$*.*

Proof. For any $e \geq 1$, since O_e is compact and θ is continuous, there exists $d > 0$, such that $\forall \vec{x} \in O_e|\theta(\vec{x})| \leq d$. As in Lemma 2, the infimum of all such d is a semialgebraic function of e and we apply Proposition 1 to obtain the upper bound e^{m_2} for sufficiently large e. □

Theorem 1. *Let* $O \subseteq \mathbb{R}^k$ *and* $\theta : O \to \mathbb{R}$ *be continuous and semialgebraic. There exists a* $(k+1)$*-operator* G*, belonging to* ***M2O****, such that for all* $e \in \mathbb{N}^+$*,* $\vec{\xi} \in O_e$ *and any name* $\vec{f}$ *of* $\vec{\xi}$*,* $G(\vec{f}, \hat{e})$ *is a name of* $\theta(\vec{\xi})$*.*

Proof. Let us fix $m_1, m_2 \in \mathbb{N}^+$ according to Lemma 2 and Lemma 3 and also $e_0 \in \mathbb{N}^+$, such that the conclusions of both lemmas hold for $e \geq e_0$. Now if $\vec{q} \in \mathbb{Q}^k \cap O_e$ and $e \geq e_0$, then we have $|\theta(\vec{q})| < e^{m_2}$. We split the interval $[-e^{m_2}, e^{m_2})$ into disjoint subintervals of length e^{-1}. Exactly one of them contains $\theta(\vec{q})$. Accordingly, we define the function $s : \mathbb{N}^{k+1} \to \mathbb{N}$ by

$$s(\vec{n}, e) \; = \; \mu_{i \leq 2e^{m_2+1}}[-e^{m_2} + i \cdot e^{-1} \; \leq \; \theta(\vec{q}) \; < \; -e^{m_2} + (i+1) \cdot e^{-1}],$$

where $\vec{n}$ is a code of $\vec{q}$. Now the relation in the brackets is semialgebraic in $\vec{q}, e, i$ and by using Proposition 2 we can choose a function $R_{\mathbb{Q}} \in \mathcal{M}^2$, such that

$$s(\vec{n}, e) \; = \; \mu_{i \leq 2e^{m_2+1}}[R_{\mathbb{Q}}(\vec{n}, \Pi(e, 0, 0), \Pi(i, 0, 0)) = 0]$$

and we conclude that $s \in \mathcal{M}^2$. We have $|\theta(\vec{q}) - \left(-e^{m_2} + s(\vec{n}, e) \cdot e^{-1}\right)| < e^{-1}$ for all $e \geq e_0$, such that $\vec{q} \in \mathbb{Q}^k \cap O_e$. We are ready to define the operator G in the following way:

$$G(\vec{f}, c)(t) \; = \; \Pi(s(\vec{f}(2u+1), 2u+2), (2u+2)^{m_2+1}, 2u+1),$$

where $u = (2\max(c(0), e_0, t) + 2)^{m_1}$. Clearly, G belongs to **M2O**.

Let $e \in \mathbb{N}^+$, $\vec{\xi} \in O_e$ and $\vec{f}$ be a name of $\vec{\xi}$. We take an arbitrary $t \in \mathbb{N}$ and denote $v = \max(e, e_0, t)$ and $u = (2v+2)^{m_1}$. Note that $u > v \geq e_0$. For all $i = 1, 2, \ldots, k$ we have: $|\nu(f_i(u)) - \xi_i| < (u+1)^{-1}$. Therefore $|\vec{q} - \vec{\xi}| < (2u+2)^{-1} < (2v+2)^{-1}$, where $q_i = \nu(f_i(2u+1))$. By Lemma 1 and $\vec{\xi} \in O_e \subseteq O_{v+1}$, we obtain $\vec{q} \in O_{2v+2} \subseteq O_{2u+2}$. Moreover, by Lemma 3, we have $|\theta(\vec{q})| < (2u+2)^{m_2}$. Now the rational number

$$q \; = \; -(2u+2)^{m_2} + s(\vec{f}(2u+1), 2u+2) \cdot (2u+2)^{-1}$$

satisfies $|\theta(\vec{q}) - q| < (2u+2)^{-1}$ and by definition, $G(\vec{f}, \hat{e})(t)$ is a code of q. Further $\vec{q}, \vec{\xi} \in O_{2v+2}$ and $|\vec{q} - \vec{\xi}| < (2u+2)^{-1} < (2v+2)^{-m_1}$, so by Lemma 2 we have $|\theta(\vec{q}) - \theta(\vec{\xi})| < (2v+2)^{-1}$. Finally,

$$|\nu(G(\vec{f}, \hat{e})(t)) - \theta(\vec{\xi})| \; = \; |q - \theta(\vec{\xi})| \; \leq \; |q - \theta(\vec{q})| + |\theta(\vec{q}) - \theta(\vec{\xi})| \; < \; (2u+2)^{-1} + (2v+2)^{-1} \; < \; (t+1)^{-1},$$

since $u > v \geq t$. So indeed $G(\vec{f}, \hat{e})$ is a name of $\theta(\vec{\xi})$ and the proof is finished. □

Theorem 2. *All continuous semialgebraic functions* $\theta : O \to \mathbb{R}, O \subseteq \mathbb{R}^k$ *with open domains are conditionally* ***M2O****-computable.*

Proof. By Proposition 2 we can choose $S_{\mathbb{Q}} \in \mathcal{M}^2$, such that $\vec{q} \in O_e \;\Leftrightarrow\; S_{\mathbb{Q}}(\vec{n}, \Pi(e, 0, 0)) = 0$, where $\vec{n}$ is a code of $\vec{q}$. Now let us take the operator G from Theorem 1. We define E and F in the following way:

$$E(\vec{f})(e) \; = \; S_{\mathbb{Q}}(\vec{f}(4e+3), \Pi(2e+2, 0, 0)), \quad F(\vec{f}, c) \; = \; G(\vec{f}, 4\widehat{c(0)+4}).$$

Evidently, E and F belong to **M2O** and we will show that they satisfy the two requirements in Definition 4 for θ. Let $\vec{\xi} \in O$ and $\vec{f}$ be a name of $\vec{\xi}$. Firstly, we can choose e, such that $\vec{\xi} \in O_{e+1}$. Moreover, $\vec{f}(4e+3)$ is a code of $\vec{q} \in \mathbb{Q}^k$, where $|\vec{q} - \vec{\xi}| < (4e+4)^{-1} < (2e+2)^{-1}$. Lemma 1 entails that $\vec{q} \in O_{2e+2}$, therefore $S_{\mathbb{Q}}(\vec{f}(4e+3), \Pi(2e+2, 0, 0)) = 0$. We conclude that $E(\vec{f})(e) = 0$, which verifies (1) in Definition 4. Secondly, let $e \in \mathbb{N}$ satisfy $E(\vec{f})(e) = 0$. We have that $\vec{f}(4e+3)$ is a code of a $\vec{q} \in \mathbb{Q}^k$, such that $\vec{q} \in O_{2e+2}$. Since we also have $|\vec{q} - \vec{\xi}| < (4e+4)^{-1}$, by Lemma 1, we obtain $\vec{\xi} \in O_{4e+4}$. Now the property of the operator G allows us to conclude that $F(\vec{f}, \hat{e}) = G(\vec{f}, \widehat{4e+4})$ is a name of $\theta(\vec{\xi})$, which verifies (2) in Definition 4. □

Theorem 3. *Let $\alpha < \beta$ be $\mathcal{M}^2$-computable real numbers, $O \subseteq \mathbb{R}^k$ be an open set and suppose that the function $\theta : O \times (\alpha, \beta) \to \mathbb{R}$ is bounded, semialgebraic and restricted analytic. Then the real function $I : O \to \mathbb{R}$ defined by $I(\vec{x}) = \int\limits_{\alpha}^{\beta} \theta(\vec{x}, y)dy$ is uniformly* ***M2O****-computable.*

Proof sketch. We take an upper bound $M \in \mathbb{N}$ of $|\theta|$, such that $M > \max(|\alpha|, |\beta|, (\beta - \alpha)^{-1})$ and we define $e(t) = M(3t+3)$. Of course, $e \in \mathcal{M}^2$. It suffices to show that the real function $I_1(\vec{x}, t) = \int\limits_{\alpha+e(t)^{-1}}^{\beta-e(t)^{-1}} \theta(\vec{x}, y)dy$ is uniformly **M2O**-computable. We apply Theorem 6.1 from [3], after the change of variables $y = (\beta - \alpha - 2e(t)^{-1}) \cdot u + (\alpha + e(t)^{-1})$. Note that by Theorem 1 we have the operator G, corresponding to θ and we can derive the uniform **M2O**-computability of the integrand by using $G(\vec{f}, \widehat{P(e(t))})$, where the polynomial P is obtained by standard semialgebraic techniques. The requirements on the analytic continuation of the integrand follow from the fact that θ is bounded and restricted analytic. □

By using the above methods and also the existence of semialgebraic decompositions with restricted analytic functions, the second author was able to show in [2] that all periods are $\mathcal{M}^2$-computable real numbers.

References

[1] Jacek Bochnak, Michel Coste, and Marie-Francoise Roy. *Real Algebraic Geometry.* Springer-Verlag, Berlin, Heidelberg, 1998.

[2] Dimitar Chaltakov. Periods are $\mathcal{M}^2$-computable real numbers. Master's thesis, Sofia University, 2025.

[3] Ivan Georgiev. On subrecursive complexity of integration. *Annals of Pure and Applied Logic*, 171(4):102777, 2020.

[4] Andrej Grzegorczyk. Some classes of recursive functions. *Rozprawy Matematyczne*, 4:1–45, 1953.

[5] Thoralf Skolem. Proof of some theorems on recursively enumerable sets. *Notre Dame Journal of Formal Logic*, 3(2):65–74, 1962.

[6] Dimiter Skordev. On some computability notions for real functions. *Computability*, 2(1):67–73, 2013.

[7] Dimiter Skordev and Ivan Georgiev. On a relative computability notion for real functions. In B. Löwe, D. Normann, I. Soskov, and A. Soskova, editors, *Models of Computation in Context, CiE 2011, Lecture Notes in Computer Science*, volume 6735, pages 270–279. Springer-Verlag.

[8] Dimiter Skordev, Andreas Weiermann, and Ivan Georgiev. $\mathcal{M}^2$-computable real numbers. *Journal of Logic and Computation*, 22(4):899–925, 2012.

[9] Katrin Tent and Martin Ziegler. Computable functions of reals. *Münster Journal of Mathematics*, 3:43–66, 2010.