A topological highness notion

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Abstract

We isolate a class of continuous function from Polish spaces to Cantor space, called high functions. We prove characterizations of this class similar to those of high degrees in degree theory and give an application to the theory of Borel realizations of countable Borel equivalence relations.

Soare [Soa72] defined the notion of a **high degree** as a Turing degree $\mathbf{d}$ such that $\mathbf{d}' \geq \mathbf{0}''$. In the context of the Δ^0_2 degrees, highness is a notion of strength—for no Δ^0_2 degree, $\mathbf{d}' > \mathbf{0}''$ and so with respect to the Turing jump, high Δ^0_2 degrees are maximally powerful. Since their first mentioning, characterizations in terms of domination properties [Mar66] and enumerations of recursive sets [Joc72]. Since then, highness has been considered in other computability theoretic contexts such as algorithmic randomness [FSY11] and computable structure theory [CFT23; CFT].

We suggest a topological notion of highness for continuous functions $f : X \to 2^\omega$ for X Polish space by proving characterization reminiscent to those by Martin [Mar66] and Soare [Soa72] in the degree theory case. In order to state our characterization we need a few preliminaries.Let (Φ_e) be a listing of the 0-ary Turing operators which correspond to the effectively continuous functions $2^\omega \to \omega$ where ω is thought of as having the discrete topology. Then, given $x \in 2^\omega$ we let **the Turing jump of** x be the characteristic function of the set

$$x' = \{e : \Phi_e^x \downarrow\}.$$

The map $x \mapsto x'$ is Baire class 1, and we will denote it by $\mathsf{J} : 2^\omega \to 2^\omega$. The jump can be iterated, and we denote $\mathsf{J}^n(x)$ as $x^{(n)}$, the nth jump of x, where it is customary to denote the double jump $x^{(2)}$ as x''. Another notion arising in our characterization is the following domination property. Given a bijection $f : X \to 2^\omega$ we say that a function $g : X \times \omega \to \omega$ **dominates computable functions with respect to** f if for every Turing operator e, and all $x \in X$ and all but finitely many $n \in \omega$, $\Phi_e^{f(x)}(n)\downarrow$ implies $\Phi_{e,g(x,n)}^{f(x)}(n)\downarrow$.

Theorem 1. *Suppose that X is a Polish space and $f : X \to 2^\omega$ is continuous. Then the following are equivalent.*

1. *The function $\mathsf{J}^2 \circ f$ is Σ^0_2-measurable.*
2. *There are continuous functions $(g_i : X \to 2^\omega)$ such that $(\lim g_i) = \mathsf{J}^2 \circ f$.*
3. *For every $P \in \Pi^0_2(2^\omega)$, $f^{-1}(P) \in F_\sigma$.*
4. *There is a function $g : X \times \omega \to \omega$ dominating computable functions with respect to f.*

Definition 1. A continuous function $f : X \to 2^\omega$ is **high** if it satisfies one of Items 2 to 4 in Theorem 1.

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The following example is probably the canonical example of a high function.

Example 2. *The inverse of the Turing Jump, J^{-1} is a high function.*

Proof. Wlog assume that $(x')^{[1]} = x$, (i.e., the Turing functional with the first index halts exactly if $x(n) = 1$), then J^{-1} is clearly continuous. $J^2 \circ J^{-1} = J$ and thus Σ^0_2-measurable. At last, note that

$$range(J) = \{x \in cantor : \forall n \forall m \Phi^n(m) \downarrow \leftrightarrow x^{[n]}(m) = 1\}$$

is G_δ and thus Polish. □

We will prove Theorem 1 in Section 1. Our theorem will hopefully peak the interest of computability theorists as it raises other questions such as finding a topological analogue for the degree theoretic notion of lowness, or considering the possible complexities of the inverses of high functions. However, our main motivation for it was an application to the theory of countable Borel equivalence relations, in particular the theory of Borel realizations of Turing equivalence. Due to space restrictions we refer the reader to [Fri+23] for missing definitions.

Theorem 3. *Suppose that E is a Borel realization of Turing equivalence via a high function $f : X \to 2^\omega$. Then E is an F_σ realization.*

Proof. To see this let

$$A_e = \{x \in 2^\omega : \forall n\, \Phi^x_e(n) \downarrow\} \quad \text{and} \quad R_e = \{(x, y) \in 2^\omega \times 2^\omega : \forall n\, (\Phi^x_e(n) \downarrow = y(n))\}.$$

The former set is in $\Pi^0_2(2^\omega)$, and the latter is in $\Pi^0_1(2^\omega \times 2^\omega)$. Furthermore, for $f^{-1}(\leq_T)$ we have that

$$\begin{aligned}(x, y) \in f^{-1}(\leq_T) &\iff \exists e \left(f(x) \in A_e \;\wedge\; (f(x), f(y)) \in R_e\right)\\ &\iff \bigcup_e \Big(\big(f^{-1}(A_e) \times X\big) \cap (f \times f)^{-1}(R_e)\Big).\end{aligned}$$

As f is high, $f^{-1}(A_e)$ is F_σ and $(f \times f)^{-1}(R_e)$ is closed (as $f \times f$ is continuous). It follows that $f^{-1}(\leq_T)$ is F_σ. Finally, since f is an isomorphism between E and $\equiv_T$, we have

$$x\, E\, y \iff (f(x) \leq_T f(y) \wedge f(y) \leq_T f(x)) \iff (x, y) \in f^{-1}(\leq_T) \wedge (y, x) \in f^{-1}(\leq_T).$$

□

Indeed, we conjecture that if E is an F_σ realization of Turing equivalence via a continuous function $f : X \to 2^\omega$, then f must be high.

At first sight the requirement that f is continuous might appear too strong and not in the spirit of a topological realization as given in [Fri+23]. However, note that the complexity of f^{-1} is crucial, and not the complexity of f. Assuming regularity properties about definable Turing invariant functions, such as Martin's conjecture, we in fact conjecture that any F_σ realization of Turing equivalence must be continuous on a cone. For the special case of compact action realizations we can actually prove this. To do this we need the following easy fact.

Lemma 4. *Assume Martin's conjecture. Suppose that $f : \omega^\omega \to \omega^\omega$ is Borel, injective, and Turing invariant. Then f^{-1} is continuous on a cofinal set.*

Proof. Let $p : 2^\omega \to \omega^\omega$ be the function mapping $x \in 2^\omega$ to the string listing x in order and let $graph : \omega^\omega \to 2^\omega$ be so that $graph(x)$ is the characteristic function of the set $\{\langle n, m\rangle : x(n) = m\}$. Then both p and $graph$ are continuous functions. As f is injective, it is not constant on a cone and neither is $g = graph \circ f \circ p$. So by Martin's conjecture there is a cone c such that $g(x) \geq_T x$, and thus f^{-1} is σ-continuous on c. Playing Martin's usual game, we obtain a pointed perfect tree such that g^{-1} is continuous on $[T]$ (one can also obtain this using e.g. [MSS16, Lemma 3.5] for the indices of the Turing functionals). Then, as $f^{-1} = p \circ g^{-1} \circ graph$, f^{-1} is continuous on $p([T])$. As both p and $graph$ are degree preserving almost everywhere, f^{-1} is continuous on a cofinal set. □

Proposition 5. *Assume Martin's conjecture. If E is an action realization of $\equiv_T$ via $f : \omega^\omega \to 2^\omega$, then f is continuous on a cofinal set.*

Proof. Suppose that E is an action realization of $\equiv_T$ on ω^ω via $f : \omega^\omega \to 2^\omega$ and let (g_i) continuously generate E. Then there is $z \in \omega^\omega$ so that all the (g_i) are uniformly effectively continuous relative to z. For example, we could take z to be the computable join of the sets $S_i = \{(\sigma, \tau) \in \omega^{<\omega} \times \omega^{<\omega} : g_i^{-1}([\tau]) = [\sigma]\}$. Now consider the closed set $C = \{z \oplus x : x \in \omega^\omega\}$ which is homeomorphic to ω^ω via $c : x \mapsto z \oplus x$, let

$$z \oplus x \; E' \; z \oplus y \iff x \; E \; y,$$

and let $g = c \circ f^{-1}$, i.e., $g(x) = f^{-1}(x) \oplus z$. Clearly, g is Turing invariant and thus by Lemma 4 there is a cofinal set so that g^{-1} is continuous. As $g^{-1} = f \circ c^{-1}$, $f = g^{-1} \circ c$. As the latter is a homeomorphism, f is continuous on a cofinal set. □

1 Proof of Theorem 1

The following lemma is merely a standard recursion-theoretic observation.

Lemma 6. *There is a recursive function $c : \omega \to \omega$ so that for every $x \in 2^\omega$, $n \in x''$ if and only if $\Phi^x_{c(n)}(k) = 1$ for at most finitely many k.*

Proof. Recall that x'' is $\Sigma^0_2(x)$ as a subset of ω, uniformly in x. Thus, there is a recursive predicate R such that

$$n \in x'' \iff \exists m \forall k \exists s R(x \restriction s, n, m, k) \quad (1)$$
$$n \notin x'' \iff \forall m \exists k \exists s \neg R(x \restriction s, n, m, k) \quad (2)$$

Fix n and x. We may view $R(x, n, -)$ as defining a table so that if $n \in x''$, then $R(x, n, -)$ contains a first infinite column. We define $\Phi^x_{c(n)}(0) = 0$ if $\exists s R(x \restriction s, n, 0, 0)$ and $\Phi^x_{c(n)}(0) = 1$ if $\exists s \neg R(x \restriction s, n, 0, 0)$. Assuming that we have defined $\Phi^x_{c(n)}(i)$ we let $j_1 = |\{k \leq i : \Phi^x_{c(n)}(k) = 1\}|$ and $j_0 = \max\{l : \Phi^x_{c(n)}(i - l) \dots \Phi^x_{c(n)}(i) = 0^l\}$. Then

$$\Phi^x_{c(n)}(i+1) = \begin{cases} 0 & \exists s R(x \restriction s, n, j_1, j_0) \\ 1 & \exists s \neg R(x \restriction s, n, j_1, j_0) \end{cases}.$$

If $n \in x''$ there is a least witness m_0 for the outer existential quantifier in Eq. (1). Hence, there is some i_0 so that for all $i > i_0$ $j_1[i] = m_0$ and $j_0[i] > j_0[i-1]$. Thus, by construction, $\Phi^x_{c(n)}$ is only finitely different from the constant 0 string.

On the other hand, if $n \notin x''$, then by definition $j_0[i]$ does not grow monotonically in i, and hence $\lim_i j_1[i] = \infty$, so $\Phi^x_{c(n)}$ has infinitely many 1's. □

Lemma 7. *The following are equivalent.*

1. $\mathsf{J}^2 \circ f$ *is* Σ^0_2*-measurable.*
2. $\mathsf{J}^2 \circ f$ *is a limit of continuous functions.*
3. *For every* $P \in \Pi^0_2(2^\omega)$, $f^{-1}(P) \in F_\sigma$.

Proof. Item 3 $\implies$ Item 1. Using the function c given by Lemma 6, consider the sets

$$J^1_n = \{x : n \in x''\} = \{x : \Phi^x_{c(n)} \in \mathrm{Fin}\} \text{ and } J^0_n = \{x : n \notin x''\} = \{x : \Phi^x_{c(n)} \in \mathrm{Inf}\}$$

where Fin is the standard Σ^0_2 set of elements finitely different from the constant 0 string and Inf its complement. Clearly $J^1_n \in \Sigma^0_2$ and $J^0_n \in \Pi^0_2$. Now, suppose that for all Π^0_2 subsets P of 2^ω, $f^{-1}(P) \in F_\sigma$. Then for every n, and the basic open sets $\{x : x(n) = 1\}$ and $\{x : x(n) = 0\}$ we have

$$(\mathsf{J}^2 \circ f)^{-1}(\{x : x(n) = i\}) = f^{-1}(\overbrace{(\mathsf{J}^2)^{-1}(\{x : x(n) = i\})}^{=J^i_n}) \in F_\sigma.$$

As the sets $\{x : x(n) = i\}$ for $i \in \{0, 1\}$ form a subbase for 2^ω, $\mathsf{J}^2 \circ f : X \to 2^\omega$ is Σ^0_2-measurable, as required.

Item 1 $\implies$ Item 3. Suppose that $P \subseteq 2^\omega$ is Π^0_2, then there is a recursive predicate R such that

$$x \in P \iff \forall n(\exists m > n)R(x \restriction m, n),$$

which implies that there is an index e_0 for an x-c.e. subset of ω with $n \in W^x_{e_0}$ if and only if $(\exists m > n)R(x \restriction m, n)$ and $x \in P$ if and only if $W^x_{e_0} = \omega$. Now, the set $\mathrm{Tot}^x = \{e : W^x_e = \omega\}$ is a $\Pi^0_2(x)$ index set and by the $\Sigma^0_2(x)$ completeness of x'' as a set of natural numbers there is a computable function $c : \omega \to \omega$ such that

$$e \in \mathrm{Tot}^x \iff x''(c(e)) = 0.$$

Assuming that $\mathsf{J}^2 \circ f$ is Σ^0_2-measurable, $(\mathsf{J}^2 \circ f)^{-1}(\{x : x(c(e_0)) = 0\}) \in F_\sigma$ and chasing the equalities above $x \in f^{-1}(P) \iff \mathsf{J}^2(f(x))(c(e_0)) = 0$, implying that $f^{-1}(P) \in F_\sigma$.

That Item 2 implies Item 1 is trivial. To see that Item 1 implies Item 2 note that by the equivalence of Item 3 and Item 1 both $f^{-1}(J^1_n)$ and $f^{-1}(J^0_n)$ are F_σ, and since they are complements of each other, they are also G_δ. Let $f^{-1}(J^0_n) = \bigcap \bigcup U^n_{i,j}$ and $f^{-1}(J^1_n) = \bigcap \bigcup V^n_{i,j}$ for $U^n_{i,j}, V^n_{i,j}$ basic open sets. Define

$$f_s(x)(n) = \begin{cases} 1 & \exists j_f(\forall i < s)x \in \bigcup_{k<j_f} V^n_{i,k} \wedge (\exists j_i(\forall i < s)x \in \bigcup_{k<j_i} U^n_{i,k} \to j_f < j_i) \\ 0 & \exists j_i(\forall i < s)x \in \bigcup_{k<j_i} U^n_{i,k} \end{cases}.$$

Intuitively, $f_s(x)(n)$ approximates J^1_n by checking the first s open sets of the intersection, and, at the same time the first s open sets in the intersection defining J^0_n. Since $J^1_n =$

$2^\omega \setminus J^0_n$, x must be in one of these approximations. If it is in both, then from a fast Cauchy name of x, $f_s(x)(n)$ will output 1 if it meets the approximation to J^1_n before it meets the approximation of J^0_n, otherwise it outputs 0. It is easy to check that $\lim_s f_s(x) = \mathsf{J}^2 \circ f(x)$. ☐

Lemma 8. *$\mathsf{J}^2 \circ f : X \to 2^\omega$ is Σ^0_2-measurable if and only if there is a continuous function $g : X \times \omega \to \omega$ dominating computable functions.*

Proof. ($\Rightarrow$). By Lemma 7 we may suppose that $\mathsf{J}^2 \circ f = \lim_s j_s$. As for given $x \in X$, $\mathrm{Tot}^{f(x)} = \{e : \forall n \Phi^{f(x)}_e(n)\downarrow\} \leq_T \mathsf{J}^2(f((x))$ uniformly in $f(x)$ via say Φ_{e_t},

$$e \in \mathrm{Tot}^{f(x)} \iff \lim_s \Phi^{j_s(x)}_{e_t}(e) = 1 \text{ and } e \notin \mathrm{Tot}^{f(x)} \iff \lim_s \Phi^{j_s(x)}_{e_t}(e) = 0$$

Our proof is now a simple adaptation of the proof of Martin's high domination theorem. To define $g(x,s)$ for all $e \leq s$ define

$$t(x,e) = \min\{r > s : \Phi^{j_r(x)}_{e_t}(e) = 0 \vee (\forall n \leq s)\Phi^x_{e,r}(n)\downarrow\}, \quad g(x,s) = \max\{t(x,e) : e \leq s\}.$$

Note that $t(x,e)$ and $g(x,s)$ are always defined and continuous. Also, if Φ^x_e is total, i.e., $\lim_s \Phi^{j_s(x)}_{e_t}(e) = 1$, then $g(x,n) > \Phi^{f(x)}_e(n)$ for almost every n, as required.

($\Leftarrow$). We will produce a family of continuous functions $h_s : X \to 2^\omega$ such that $\lim_s h_s(x) = \mathrm{Tot}^{f(x)}$. Then, as $\mathrm{Tot}^{f(x)} \equiv_T \mathsf{J}^2(f(x))$ uniformly in x via say Φ_e, we have that $\Phi_e \circ h_s : X \to 2^\omega$ is continuous for all s and $\lim_s(\Phi_e \circ h_s) = \mathsf{J}^2 \circ f$. To define $h(x,s)$ we let

$$h(x,s)(n) = \begin{cases} 1 & \text{if } (\forall z < s)\Phi^{f(x)}_{n,g(x,s)}(z)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

The function is clearly continuous as f is continuous. If $n \in \mathrm{Tot}^{f(x)}$, then also $\psi_n(m) = \mu s(\forall z < m)\Phi^{f(x)}_{n,s}(z)\downarrow$ is total. Thus, $\lim_s h(x,s)(n) = 1$. If $\Phi^{f(x)}_n$ is not total then, there is m such that $\Phi^{f(x)}_n(m)\uparrow$ and thus $\lim_s h(x,s)(n) = 0$. It follows that $\lim_s h(x,s) = \mathrm{Tot}^{f(x)}$ as required. ☐

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