Naturality as a Structural Criterion of the Intension–Extension Relation

Introduction

The distinction between intension and extension belongs to the classical core concepts of logic, philosophy of language, and metaphysics. Since Frege, intension has been associated with meaning, sense, or rule-governed determination, whereas extension is understood as the set of objects to which an expression applies. Despite the established terminology, a fundamental question remains open: under what conditions is the relation between intension and extension objectively determined and not merely the result of an accidental or arbitrary assignment?

This paper advances the thesis that the categorical notion of naturality provides precisely the structural criterion required to explicate this objectivity. Naturality functions as an invariance principle under context change and allows one to distinguish essential from accidental relations between intension and extension.

1. Intension and Extension in the Classical Framework

A unary predicate over a carrier set X can be formally represented as a map

$$f : X \to 2$$

where $2 = \{0, 1\}$ denotes the object of truth values. The intension of the predicate is the rule f itself. The extension arises as the inverse image of the distinguished truth value

$$Ext_X(f) = f^{-1}(\{1\}) \subseteq X$$

2. Context Change and Retraction of Properties

A change of context between carrier sets is described by a map

$$u : X \to Y$$

A predicate on Y, given by $g : Y \to 2$, can be retracted to X by precomposition

$$g \quad g \quad u$$

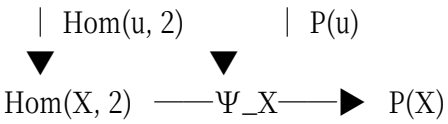
At the same time, the extension of a predicate, understood as a subset of Y, can be retracted to X by inverse image formation

$$S \quad u \quad {}^{-1}(S)$$

3. The Naturality Square

The relation between intension and extension under context change can be expressed by a commutative square:

$$\begin{array}{ccc} \mathrm{Hom}(Y, 2) & \xrightarrow{\Psi_Y} & P(Y) \\ \big| & & \big| \end{array}$$



Formally, the commutativity of the square corresponds to the equation

$$P(u) \quad \Psi_Y = \Psi_X \quad \mathrm{Hom}(u, 2)$$

or, evaluated for a concrete $g : Y \to 2$,

$$u \quad {}^{1}(g \quad {}^{1}(\{1\})) = (g \quad u) \quad {}^{1}(\{1\})$$

4. Essence and Accident

Naturality allows for a precise structural distinction: a relation between intension and extension is essential if and only if it is part of a natural transformation, that is, if it remains invariant under all context changes. A merely pointwise correspondence without naturality, by contrast, is accidental.

5. Conclusion

The requirement of naturality closes a conceptual gap between classical semantics and modern structural theory. It allows the relation between intension and extension to be characterized as objective, non-arbitrary, and structurally stable. In this sense, naturality may be understood as a modern structural replacement for the classical notion of essence.