The journey of Thanases Pheidas in the realm of analytic functions

PLS14, Thessalonique
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Xavier Vidaux (Universidad de Concepción, Chile)
in memory of Thanases Pheidas

Paris, January 2006

The Structures

Und.: 1978, Denef, $F=\mathbb{R}$

- TP, 1991: $F=\mathbb{F}_p$ (Videla for $p=2$)
- Kim-Roush, 1992: $F=\mathbb{C}(z_1,z_2)$
- Zahidi: $\mathbb{C}(z)$ with ord. $\boxed{\text{alg. ext.}}$
  Shlapentokh
  Eisentraeger

$F(z)$ $\qquad\qquad$ $\mathcal{M}(V)$

Fraction field

complete w/r $|\cdot|$ $\rightsquigarrow$

$F[z]$ Und. by Denef, 1978 $\qquad\qquad$ $\mathcal{A}(V)$

$$\sum_{j=0}^{m} a_j z^j \qquad\qquad \sum_{j\geq 0} a_j z^j \text{ convergent, w/r to } |\cdot|, \text{ on } V \subseteq F$$

3 types of $|\cdot|$:
- archimedean
- non-archimedean
  - char. 0
  - char. p

1st order languages: $\mathcal{L}_R \cup$ something

often: z, ord f ($\Leftrightarrow$ f has a zero at zero)

if $z=(z_1,z_2)$: . eval (f/g): $\left.\frac{f}{g}\right|_{z_1=0}$ analytic in z_2 and $\left.\left.f/g\right|_{z_1=0}\right|_{z_2=0} = 0$.

. $\text{eval}_0(f,g)$: $g\neq 0 \Rightarrow \text{eval}(f/g)$.

$\mathcal{L}_R = \{0,1,+,\times\}$

$\mathcal{L}_z = \mathcal{L}_R \cup \{z\}$

$H10_{\mathcal{L}}(R)$: Is the $\exists^+_-$ theory of R over $\mathcal{L}$ decidable?

Starting point

1951, R. Robinson : full Theory of $\mathcal{A}(V)$,

$\mathbb{C} \supseteq V \supseteq]a,b[\subseteq \mathbb{R}$

Works by L. Rubel over L_T (T for non-constant)

1995 TP : $\cancel{V = \mathbb{C}}$ gap, not fixable

TP, L. Lipshitz : $V = \mathbb{C}_p$

TP' supervisor

Works by TP and X. Kourouniotis for $\mathcal{M}(\mathbb{C}^n)$ with ord.

~65 years

2001 my PhD, supervised by TP : $\mathcal{M}(\mathbb{C}_p)$ with ord

(2006: $Th(\mathbb{C}(z))$: failed attempt)

I join TP and XK on $\mathcal{M}(\mathbb{C}^n)$, ord.

2009 $\cancel{\mathcal{M}(\mathbb{C}^n) \text{ with ord}}$ gap, not fixable

H10 uniformly for $\mathbb{F}_p(z)$, thks to a mistake in a previous paper!

2012 Try eval instead of ord. XK not needed

looks more expressive

~2016 TP, XV : $\mathcal{M}(\mathbb{C}^n)$ for $n \geq 2$, with eval.

$\mathcal{A}(\mathbb{C}^n)$ " , with $eval_0$.

published IRMN in 2022 !

2015 N. Garcia-Fritz, H. Pasten : $\mathcal{A}(K)$, char $k > 2$

2017 HP : $\mathcal{M}(K)$, char $k > 2$ with ord

+ full theory over $\mathcal{L}_R$

open in all other cases

Subrings of $\mathcal{A}(\mathbb{C})$

$$\alpha(f) = \inf \left\{ \alpha \in \mathbb{C} : \log \max_{|z| \leq r} |f(z)| \leq O(r^\alpha) \right\}$$

$\shortparallel$

order of f.

~2020 : D. Chompitaki, N.GF, HP, TP, XV

exp. poly. of finite order $\sum_{j=1}^{m} p_j \cdot \exp(q_j)$

08/2023 : N. GF, HP Lacunary finite order

large gaps in the Taylor extension

06/2024 : HP bounded order α, for any α

11/2023 ; N. GF, HP, XV : $\mathbb{Q}$ with height conditions

dedicated to TP: he had one of the main ideas.

The equations : $f(z) \cdot y^2 = f(x)$

$f(t) = t^2 - 1 \rightsquigarrow$ Pell funcional

$f(t) = t^3 + \delta t^2 + t \rightsquigarrow$ Manin-Demef

Athens, June 2015

California, August 2022

California, August 2022