Diagonalization and Paradox

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Table of Contents

Introduction

Higher-Order Languages

Functional types:

- e is the type of singular terms
- t is the type of formulas
- If σ and τ are types, then $\sigma \to \tau$ is a type the type of function terms which take expressions of type σ and yield expressions of type τ.

If $\overline{\sigma}$ is a sequence of types $\sigma_1, \cdots, \sigma_n$, then we write $\overline{\sigma} \to \tau$ for the type $\sigma_1 \to \cdots \to \sigma_n \to \tau$, for example, $t, t \to t$ is the type $t \to t \to t$ of a binary sentential operator.

Signature:

We have variables and constants of different types, which include:

$$
\begin{aligned}
t &: & \bot, \top \\
t \to t &: & \neg \\
t \to t \to t &: & \wedge, \vee, \to \\
(\sigma \to t) \to t &: & \forall_\sigma
\end{aligned}
$$

Terms:

- A constant or variable of a type is a term of that type.
- If α is a term of type $\sigma \to \tau$ and β is a term of type σ, then $\alpha\beta$ is a term of type τ.
- If $\overline{x}$ are variables of types $\overline{\sigma}$ and α is a term of type τ, then $\lambda\overline{x}.\alpha$ is a term of type $\overline{\sigma} \to \tau$.

Some definitions:

$$\begin{aligned} \exists_\sigma : &\qquad \lambda X^{\sigma\to t}.\neg\forall x^{\sigma}\neg Xx \\ x =_\sigma y : &\qquad \forall X^{\sigma\to t}(Xx \leftrightarrow Xy) \\ X \equiv_{\overline{\sigma}\to t} Y : &\qquad \forall \overline{x}^{\overline{\sigma}}(X\overline{x} \leftrightarrow Y\overline{x}) \end{aligned}$$

Higher-Order Logic

Deductive system:

$$
\begin{aligned}
&\text{Tautologies} \\
&\varphi, \varphi \to \psi / \psi && \text{(MP)} \\
&\forall x \varphi \to \varphi[\epsilon / x] && \text{(UI)} \\
&\varphi \to \psi, \varphi \to \forall x \psi \quad (x \text{ not free in } \varphi) && \text{(UG)} \\
&(\lambda \overline{x}.\varphi)\overline{\epsilon} \leftrightarrow \varphi[\overline{\epsilon}/\overline{x}] && \text{(E}\beta\text{)}
\end{aligned}
$$

Comprehension is a theorem schema:

$$\vdash \exists X \forall \overline{x} (X\overline{x} \leftrightarrow \varphi) \quad (X \text{ not free in } \varphi)$$

A diagonal argument

Some items of type σ are taken as proxy for items of type $\sigma \to t$.

No relation R of type $\sigma \to \sigma \to t$ correlates a proxy x^σ for the concept $X^{\sigma \to t}$ to all and only instances y^σ of X. That is:

$$\neg \exists R^{\sigma \to \sigma \to t} \forall X^{\sigma \to t} \exists x^\sigma \forall y^\sigma (Rxy \leftrightarrow Xy).$$

Consider a *diagonal concept* $\lambda x^\sigma . \neg Rxx$, which applies to an item of type σ if and only if it is not correlated with itself. A proxy the diagonal concept is correlated to itself if and only if it is not.

In symbols:

Theorem

$\neg\exists R^{\sigma\to\sigma\to t}\forall X^{\sigma\to t}\exists x^{\sigma}\forall y^{\sigma}(Rxy \leftrightarrow Xy)$

Proof.

Given a witness d^{σ} for $\lambda x^{\sigma}.\neg Rxx$:

$$\forall y^{\sigma}(Rdy \leftrightarrow (\lambda x^{\sigma}.\neg Rxx)y)$$

$$Rdd \leftrightarrow (\lambda x^{\sigma}.\neg Rxx)d$$

$$Rdd \leftrightarrow \neg Rdd$$

$\square$

A cardinality gloss?

The existential generalization below is unsatisfiable in extensionally full models of the language in which we let $D^{\sigma \to t}$ consist of all functions from D^{σ} into $\{0, 1\}$:

$$\exists R^{\sigma \to \sigma \to t} \forall X^{\sigma \to t} \exists x^{\sigma} \forall y^{\sigma} (Rxy \leftrightarrow Xy)$$

Such a relation would encode a surjection D^{σ} *onto* $D^{\sigma \to t}$. But we know, by Cantor's theorem, that there are strictly more functions from D^{σ} into $\{0, 1\}$ than there are members of D^{σ}.

Unfortunately, that does not by itself explain why the formula remains unsatisfiable in less than extensionally full models of the language.

A generalization of the diagonal argument

Given a functional expression $\#$ of type $(\sigma \to t) \to \sigma$, the diagonal argument delivers the inconsistency of the schema:

$$\forall X \forall y (\psi(\# X, y) \leftrightarrow Xy)$$

For let D abbreviate: $\lambda x . \neg \psi(x, x)$. Then:

$$\forall y (\psi(\# D, y) \leftrightarrow Dy)$$

$$\psi(\# D, \# D) \leftrightarrow D \# D$$

$$\psi(\# D, \# D) \leftrightarrow \neg \psi(\# D, \# D).$$

Applications

Fregean Abstraction

Frege sought to associate every concept F with an object, ϵF, the *extension* of F. He aimed to associate coextensive concepts with one and the same extension.

$$\forall X \forall Y (\epsilon X = \epsilon Y \leftrightarrow X \equiv Y) \qquad \text{(Axiom V)}$$

X and Y are variables of type $e \rightarrow t$ and ϵ is a functional expression of type $(e \rightarrow t) \rightarrow e$.

Axiom V is inconsistent

Remark

Axiom V $\vdash \exists R^{e\to e\to t}\forall X^{e\to t}\exists x\forall y(Rxy \leftrightarrow Xy)$

Let $E^{e\to e\to t}$ abbreviate:

$$\lambda xy.\exists X^{e\to t}(x = \epsilon X \wedge Xy)$$

That is, y is a member of x. By Axiom V:

$$\forall y(E\epsilon Xy \leftrightarrow Xy).$$

By Existential Generalization:

$$\exists R^{e\to e\to t}\forall X^{e\to t}\exists x\forall y(Rxy \leftrightarrow Xy)$$

Some observations

- The argument is *not* available in predicative fragments of higher-order logic, which do *not* validate:

$$\exists R \forall x \forall y (Rxy \leftrightarrow \exists X (x = \epsilon X \wedge Xy))$$

  The specification of the relation quantifies over concepts which may include concepts constructed from that very quantifier. Such fragments will either restrict UI or Eβ in order to block the argument.

- Axiom V remains consistent in predicative fragments of higher-order, which will nevertheless prove:

$$\neg \exists R \forall X \exists x \forall y (Rxy \leftrightarrow Xy)$$

- Compare with the existential generalization below, which remains satisfiable in models for predicative fragments:

$$\exists R^{e\to(e\to t)\to t}\forall x\forall X(RxX \leftrightarrow x = \epsilon X)$$

- The moral of diagonalization in predicative fragments is *not* that there are *more* first-level concepts than there are objects. In fact, predicativists will deny that.

Interlude: Set Theory

In second-order set theory, the diagonal argument is interpreted to mean that there are *strictly more* classes than sets.

- $\mathrm{ZFC2} \vdash \neg\exists R \forall X \exists x \forall y (R\langle x, y\rangle \leftrightarrow Xy)$.

That is the gloss Paul Bernays originally put on the observation. Yet, there are *no more* definable classes than there are sets.

Indeed, Hamkins (2022) has constructed a formula $\Theta(X, y)$ of the language of second-order set theory, which simulates a function from classes to sets, i.e., $\forall X\ \exists! x \Theta(X, x)$, and is such that

$$\forall X \forall Y \forall x (\Theta(X, x) \wedge \Theta(Y, x) \rightarrow X = Y)$$

The cash value of the diagonal argument in second-order set theory is *not* that there are more classes than there are sets.

Hamkins (2022) and Uzquiano (2019) emphasize the link between the diagonal argument above and familiar limitative results, which seem quite independent independent from cardinality considerations.

The undefinability of satisfaction seems a case in point.

Tarski's Theorem

For each open formula $\varphi(u)$ of the language with a variable u of type e, we may assume there is a closed term $\ulcorner\varphi(u)\urcorner$ of type e.

Theorem (Tarski's Theorem)

There is no expression T of type $e \to e \to t$ such that

$$\forall y(T(\ulcorner\varphi(u)\urcorner, y) \leftrightarrow \varphi(y)).$$

Consider the formula:

$$\neg Tuu$$

Then:

$$\forall y(T(\ulcorner\neg Tuu\urcorner, y) \leftrightarrow \neg Tyy)$$

$$T(\ulcorner\neg Tuu\urcorner, \ulcorner\neg Tuu\urcorner) \leftrightarrow \neg T(\ulcorner\neg Tuu\urcorner, \ulcorner\neg Tuu\urcorner)$$

Structure

On a quasi-syntactic view of propositions, they exemplify a structure that perfectly mirrors the structure of the sentences we use to express them. Two predications, in particular, express the same proposition only if they involve the same constituents.

For example, *Fa* and *Gb* express the same proposition only if F and G, on the one hand, and a and b, on the other, share the same denotation. More generally:

$$\forall X \forall Y \forall x \forall y (Xx = Yy \rightarrow X = Y \wedge x = y) \qquad (\textsc{Structure})$$

The Russell-Myhill theorem is the observation that STRUCTURE is inconsistent.

Structure is inconsistent

Remark

$\textsc{Structure} \vdash \exists R^{t\to t\to t} \forall X^{t\to t} \exists p \forall q (Rpq \leftrightarrow Xq)$

Let $E^{t\to t\to t}$ abbreviate:

$$\lambda pq. \exists Y^{t\to t}(p = Y\bot \wedge Yq)$$

That is, q falls under the operator which p predicates of $\bot$.

By $\textsc{Structure}$, $X\bot = Y\bot$ only if $X = Y$. Therefore:

$$\forall q(EX\bot q \leftrightarrow Xq).$$

By Existential Generalization:

$$\exists R^{t\to t\to t} \forall X^{t\to t} \exists p \forall q (Rpq \leftrightarrow Xq)$$

Some observations

1. The argument is *not* available in predicative fragments of higher-order logic, which do *not* validate:

   $$\exists R \forall p \forall q (Rpq \leftrightarrow \exists X (p = X\bot \wedge Xq))$$

   Much like before, predicative fragments of higher-order logic will either restrict UI or Eβ in order to block the argument.

2. $\textsc{Structure}$ is consistent in predicative fragments of higher-order logic, e.g., (Hodes 2015) and (Walsh 2016), some of which nevertheless prove:

   $$\neg \exists R^{t \to t \to t} \forall X^{t \to t} \exists p \forall q (Rpq \leftrightarrow Xq)$$

- Compare with the existential generalization below, which remains satisfiable in models for predicative fragments:

$$\exists R^{t\to(t\to t)\to t}\forall p\forall X(RpX \leftrightarrow p = X\bot)$$

- The moral of diagonalization in predicative fragments of higher-order logic is *not* that there are more propositional operators than there are propositions.

Closed Structure

Closed Structure replaces Structure with instances that involve exclusively *closed terms*:

$$\xi\varepsilon = \zeta\eta \to \xi = \zeta \wedge \varepsilon = \eta \qquad \text{(Closed Structure)}$$

For example:

$$\neg\bot =_t \neg\neg\bot \to ((\neg =_{t\to t} \neg) \wedge (\bot =_t \neg\bot))$$

$$\neg\bot =_t \Box\neg\top \to ((\neg =_{t\to t} \Box) \wedge (\bot =_t \neg\top))$$

Fritz, Lederman, et al. (2021) provide models for Closed Structure, which turns out to be inconsistent in a plural extension of the language.

Closed Structure $\nvdash \exists R^{t\to t\to t}\forall X^{t\to t}\exists p \forall q(Rpq \leftrightarrow Xq)$

The argument from Structure relied on the availability of open instances of Structure in order to prove:

$$\forall q(EX\bot q \leftrightarrow Xq).$$

where $E^{t\to t\to t}$ abbreviates again:

$$\lambda pq.\exists Y^{t\to t}(p = Y\bot \wedge Yq).$$

We have articulated two quasi-syntactic approaches to propositional structure, one weakens the deductive system, whereas the other relinquishes open instances of Structure.

The approaches remain constrained by the inconsistency of some instances of the schema:

$$\forall X^{t\to t}(\psi(\varphi(X), q) \leftrightarrow Xq)$$

Think of $\varphi(X)$ as a proxy for X, and let R abbreviate: $\lambda p.\neg\psi(p, p)$.

$$\psi(\varphi(R), \varphi(R)) \leftrightarrow R\varphi(R)$$

$$\psi(\varphi(R), \varphi(R)) \leftrightarrow \neg\psi(\varphi(R), \varphi(R))$$

Depending on the choice of $\psi(\varphi(X), q)$, the inconsistency of corresponding instances of the schema may be derivable by predicative means alone:

$$\forall X^{t \to t}(\psi(\varphi(X), q) \leftrightarrow Xq).$$

On the other hand, proponents of $\textsc{Closed Structure}$ are bound by the inconsistency of *all* instances of the schema.

We now illustrate some of the constraints the diagonal argument may impose on quasi-syntactic approaches to propositional structure. They, we suggest, have nothing to do with cardinality.

Example 1: Substitution

On a quasi-syntactic approach, one would like to distinguish $\xi\varepsilon$ from $\xi\eta$ if $\varepsilon \neq \eta$.

> $\xi\eta$ is the substitution of η for ε in $\xi\varepsilon$.

We may tentatively expand the language with a functional expression S of type $t \to \sigma \to \sigma \to t$ of the form

$$S(\varphi(\alpha), \alpha, \beta)$$

designed to express the proposition that results from the substitution of β for α in φ. For example:

$$\begin{aligned} S(\neg\bot, \bot, \top) &= \neg\top \\ S(\neg\bot \to \bot, \bot, \top) &= \neg\top \to \top \\ &\dots \end{aligned}$$

Remark

There is no formula $S(\varphi(\alpha), \alpha, \beta)$ such that

$$\forall \beta (S(\varphi(\alpha), \alpha, \beta) \leftrightarrow \varphi(\beta))$$

Such a formula would validate

$$\forall X (S(X\bot, \bot, q) \leftrightarrow Xq),$$

which is of the problematic form when we let $\varphi(X)$ be $X\bot$ and substitute $S(X\bot, \bot, q)$ for $\psi(\varphi(X), q)$ in

$$\forall X (\psi(\varphi(X), q) \leftrightarrow Xq)$$

Example 2: Witness

We may wonder now whether we could expand the language with a binary sentential operator W of type $(t \rightarrow t \rightarrow t)$ which relates an existential generalization to a proposition if, and only if, the latter is a witness for the truth of the former.

Remark

There is no binary sentential operator W such that

$$\forall X \forall q (W(\exists X, q) \leftrightarrow Xq)$$

We let $\varphi(X)$ be $\exists X$ and substitute $W(\exists X, q)$ for $\psi(\varphi(X), q)$ in

$$\forall X (\psi(\varphi(X), q) \leftrightarrow Xq)$$

Example 3: Instance and Conjunct

Fritz (2021) and Goodman (2022) have shown that we *can* expand the language with a binary sentential operator I which relates a generalization to their instances:

$$I(\exists u\varphi(u), q) \leftrightarrow \exists u\ q = \varphi(u) \tag{\textsc{Instance}}$$

They provide models for the principle, which, Fritz (2021) further observed, is inconsistent with the further principle:

$$p \wedge q = r \wedge t \rightarrow (p = r \wedge q = t) \tag{\textsc{Conjunct}}$$

The argument involves yet another instance of diagonalization:

Recall the inconsistency of the schema:

$$\forall X(\psi(\varphi(X), q) \leftrightarrow Xq)$$

We now let $\varphi(X)$ be $\forall p(Xp \wedge p)$ and substitute the formula $\exists r(r \wedge I(r \wedge q, \forall p(Xp \wedge p)))$ for $\psi(\varphi(X), q)$.

Conjunct and Instance deliver the inconsistent principle:

$$\exists r(r \wedge I(r \wedge q, \forall p(Xp \wedge p))) \leftrightarrow Xq$$

through a series of equivalences:

$$\begin{aligned}
\exists r(r \wedge I(r \wedge q, \forall p(Xp \wedge p))) \\
\exists r(r \wedge \exists t(r \wedge q = (Xt \wedge t))) \\
\exists r(r \wedge \exists t(r = Xt \wedge q = t)) \\
\exists r(r \wedge r = Xq) \\
Xq
\end{aligned}$$

Conclusion

Conclusion

Diagonalization gives us the inconsistency of the schema:

$$\forall X^{\sigma \to t}(\psi(\#X, y^{\sigma}) \leftrightarrow Xy)$$

The diagonal argument makes use of minimal resources and is available in predicative fragments of higher-order logic

The inconsistency of the schema places further constraints on quasi-syntactic approaches to propositional structure many of which have nothing to do with cardinality considerations.

References

- Fritz, P. Lederman, H. et al. (2021) "Closed Structure," *Journal of Philosophical Logic*, Online First.
- Fritz, P. (2021) "Operands and Instances," *The Review of Symbolic Logic*, 16:1, 188-209.
- Goodman, J. (2022) "Grounding Generalizations," *Journal of Philosophical Logic*, Online First.
- Hamkins, J. (2022) "Fregean Abstraction in Zermelo-Fraenkel Set Theory: A Deflationary Approach," arXiv preprint arXiv:2209.07845.
- Hodes, H. (2015) "Why Ramify?" *Notre Dame Journal of Formal Logic*, 56(2), 379–415.

- Uzquiano, G. (2019) "Impredicativity and Paradox," *Thought*, 8:3, 209-221.
- Walsh, S. (2016) "Predicativity, the Russell-Myhill Paradox, and Church's Intensional Logic." *Journal of Philosophical Logic* 45:3, 277–326.