Incompleteness Theorems for Observables in General Relativity

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This is joint work with

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Lorentzian metrics: abstractly

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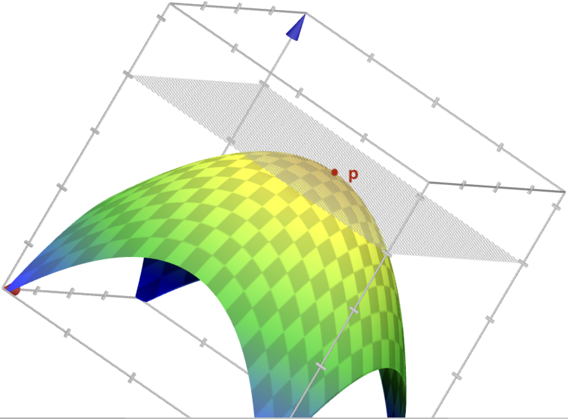

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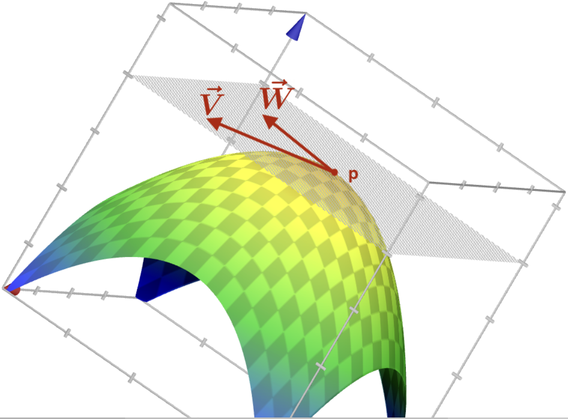

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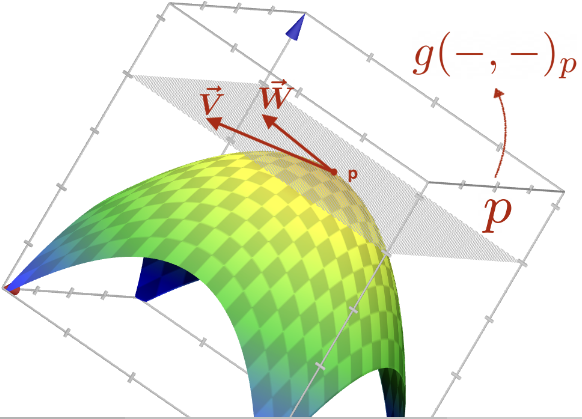

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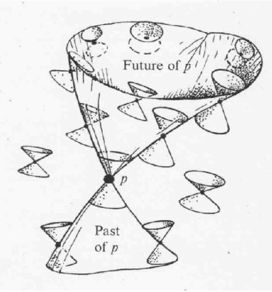

R. Penrose, "The Road to Reality"

Lorentzian metrics: concretely

A **Lorentzian metric** on $\mathbb{R}^4$ is given by a smooth map $g_{\mu\nu} \colon \mathbb{R}^4 \to \mathbb{R}^{4\times 4}$

$$(x^0, x^1, x^2, x^3) \mapsto \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

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$$g = g_{\mu\nu}\, dx^\mu dx^\nu$$

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Example

If $\quad \eta_{\mu\nu} := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad$ then $\quad \eta = -dt^2 + dx^2 + dy^2 + dz^2$

Einstein field equations

A **spacetime** is a Lorentzian metric $g_{\mu\nu}\colon \mathbb{R}^4 \to \mathbb{R}^{4\times 4}$ which satisfies:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}} \tag{1}$$

for some "distribution of matter" given by the **Stress-Energy tensor** $T_{\mu\nu}$.

$$g_{\mu\nu} \rightsquigarrow \Gamma^{\rho}_{\mu\nu} \rightsquigarrow R^{\rho}_{\mu\sigma\nu} \rightsquigarrow R_{\mu\nu} \rightsquigarrow R$$

Compare to Poisson's equation for Newton's law of gravity:

$$\boxed{\nabla^2 \varphi = 4\pi G \rho}$$

Example

$$g_{\mu\nu} := 1/(2\omega^2)\big[-(dt + e^x dy)^2 + dx^2 + 1/2e^{2x}dy^2 + dz^2\big]$$

$T_{\mu\nu} =$ "rotating dust" + "negative cosmological constant"

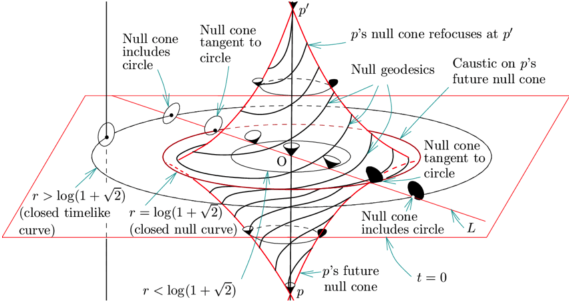

Figure. *Nḿeti, Madarász, Andréka, Andai* (after Hawking, Ellis)

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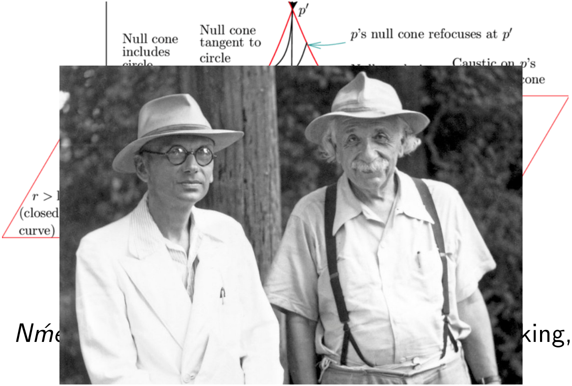

Figure. *Nḿe* ... king, Ellis)

"*Is the universe rotating yet*?" K. Gödel

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Question. Do $g_{\mu\nu}$ and $\widetilde{g}_{\rho\sigma}$ represent different "geometries"?

$$g_{\mu\nu} :=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\widetilde{g}_{\rho\sigma} :=\begin{bmatrix} -1 & -\cos(x_1) & 0 & 0 \\ -\cos(x_1) & 1-\cos^2(x_1) & 2x_2 & 0 \\ 0 & 2x_2 & 4x_2^2+1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

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We say that $g_{\mu\nu}$ and $\widetilde{g}_{\rho\sigma}$ are **diffeomorphic** and write $g_{\mu\nu} \simeq_{\text{diff}} \widetilde{g}_{\rho\sigma}$ if there exists are smooth change of coordinates $\widetilde{x}^{\xi} = \widetilde{x}^{\xi}(x^{\eta})$ so that

$$g_{\mu\nu}(x^{\eta}) = \frac{\partial \widetilde{x}^{\rho}}{\partial x^{\mu}} \frac{\partial \widetilde{x}^{\sigma}}{\partial x^{\nu}} \widetilde{g}_{\rho\sigma}(\widetilde{x}^{\xi}) \text{ for all } x^{\eta}.$$

Same geometry, different coordinate system...

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{vs} \quad \begin{bmatrix} -1 & -\cos(\widetilde{x}_1) & 0 & 0 \\ -\cos(\widetilde{x}_1) & 1-\cos^2(\widetilde{x}_1) & 2\widetilde{x}_2 & 0 \\ 0 & 2\widetilde{x}_2 & 4\widetilde{x}_2^2+1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Consider the change of coordinates $x_\eta = x_\eta(\widetilde{x}^\xi)$ given by:

$$\begin{aligned} x_0 &= \widetilde{x}_0 + \sin(\widetilde{x}_1) \\ x_1 &= \widetilde{x}_1 + \widetilde{x}_2^2 \\ x_2 &= \widetilde{x}_2 - \widetilde{x}_3 \\ x_3 &= \widetilde{x}_3 \end{aligned}$$

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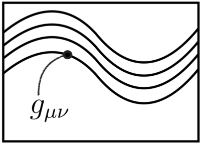
Plug to $\boxed{ds^2 = -(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2}$

Observables

Let $\mathcal{S}$ be a collection of spacetimes and consider the relation $\simeq_{\text{diff}}$ on $\mathcal{S}$:



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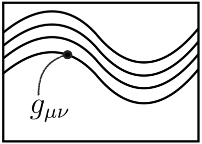

An **observable** is any map $f\colon \mathcal{S} \to R$ that is diffeomorphism invariant:

for all $g_{\mu\nu}, \widetilde{g}_{\rho\sigma} \in \mathcal{S}$ we have $g_{\mu\nu} \simeq_{\text{diff}} \widetilde{g}_{\rho\sigma} \implies f(g_{\mu\nu}) = f(\widetilde{g}_{\rho\sigma})$

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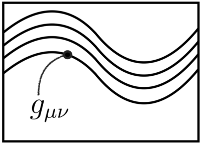

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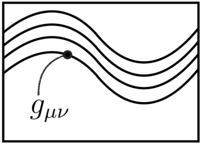

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Canonical Quantization Process

Step 1: Find **complete** set of observables for $\mathcal{S}$.

Step 2: Promote them to an **algebra of operators** on a Hilbert space $\mathcal{H}$.

The problem of observables

"*We define observables as functions (or functionals) of field variables that are invariant with respect to coordinate transformations*."

(1958) P.G. Bergmann, and A.I. Janis

"*A program aiming at the identification and systematic exploitation of the observables has been under way for many years, but its execution is* **hampered by profound technical difficulties, which have not yet been overcome completely**."

(1965) P.G. Bergmann,

"*...presently we can give a formal characterization of observables in general relativity, but we are actually* **not able to explicitly construct many examples** *of quantities that satisfy it*."

(2001) L. Smolin

"*Observables for full general relativity (without special asymptotic symmetries or matter content)* **almost certainly do not exist**."

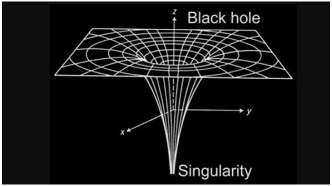
(2015) B. Dittrich, P. A. Höhn, T.A. Koslowski, and M.I. Nelson,

Examples of Observables

- **Komar mass** for static spacetimes

$$g_{\mu\nu} \mapsto \int_M (2T_{\mu\nu} - T g_{\mu\nu}) u^\mu \xi^\nu \, dM$$

It is a complete observable for all Schwarzschild solutions



- **ADM Observables** for asymptotically flat spacetimes

- **Coordinate-like Observables** for spacetimes filled with "generic dust"

Incompleteness of Observables

Theorem (P., Sparling, Christodoulou)

In the same way that "straightedge–and–compass" **cannot** *construct* $\sqrt[3]{2}$*...*

...analysis **cannot** *explicitly define complete observables"*

Incompleteness of Observables

Theorem (P., Sparling, Christodoulou)

Assume that $\mathcal{S} \supseteq \mathcal{S}_\emptyset$ contains the collection of all **vacuum solutions** *$\mathcal{S}_\emptyset$. Then there is no observable $f \colon \mathcal{S} \to R$ that is both* **Borel** *and* **complete**.

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- $f\colon \mathcal{S} \to R$ is **Borel** if it is Borel as a map from $\mathcal{S} \subseteq \mathrm{C}^\infty(\mathbb{R}^4, \mathbb{R}^{4\times 4})$ endowed with the C^∞-compact-open topology to the Polish space R.

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Theorem (P., Sparling, Christodoulou)

"ZF+DC+no **complete** *observables for $\mathcal{S} \supseteq \mathcal{S}_\emptyset$ exist" is consistent.*

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Consider the equivalence relation $\simeq_{\mathbb{Z}}$ on the space $\{0,1\}^{\mathbb{Z}}$ where

$$\alpha \simeq_{\mathbb{Z}} \beta \iff \exists k \in \mathbb{Z} \ \forall n \in \mathbb{Z} \ \alpha(n+k) = \beta(n)$$

i.e., the **orbit equivalence relation** of the *Bernoulli shift* $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$.

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There is **no Borel** *map* $f\colon \{0,1\}^{\mathbb{Z}} \to R$*, taking values in Polish* R*, with*

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Proof Sketch.

• Notice that $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ has a dense orbit. This implies the "0–1 law": if $B \subseteq \{0,1\}^{\mathbb{Z}}$ is $\mathbb{Z}$-invariant and Borel, then one of B, B^c is comeager.

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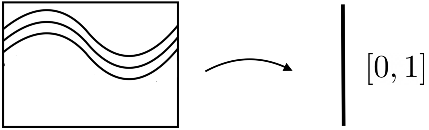
- Assume f exists and get comeager $C \subseteq \{0,1\}^{\mathbb{Z}}$ so that $f(C) = \{x\}$



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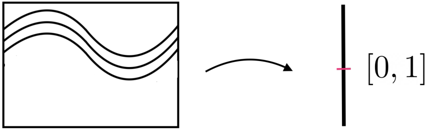

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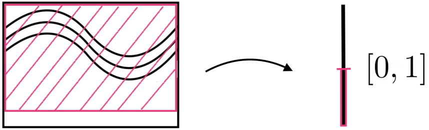

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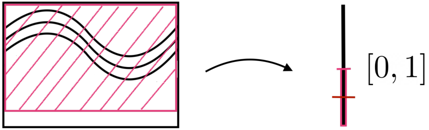

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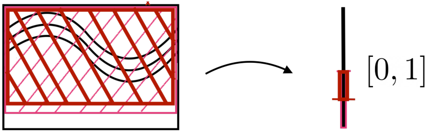

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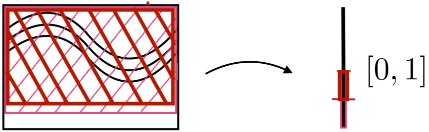

- Since $\mathbb{Z}$ is countable, there exist $\alpha \not\simeq_{\mathbb{Z}} \beta$ in C. But $f(\alpha) = x = f(\beta)$

General Strategy

Let $\mathcal{S}$ be a collection of spacetimes.

In order to prove that:

"there is no observable $f\colon \mathcal{S} \to R$ that is both **Borel** & **complete**"

it suffices to prove that:

there exists a **Borel reduction** from $(\{0,1\}^{\mathbb{Z}}, \simeq_{\mathbb{Z}})$ to $(\mathcal{S}, \simeq_{\mathrm{diff}})$,
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Definition

$\mathcal{S}$ is **rich** if there exists a Borel reduction from $(\{0,1\}^{\mathbb{Z}}, \simeq_{\mathbb{Z}})$ to $(\mathcal{S}, \simeq_{\mathrm{diff}})$

Examples of Rich Families: part I

Theorem (Christodoulou, Sparling, P.)

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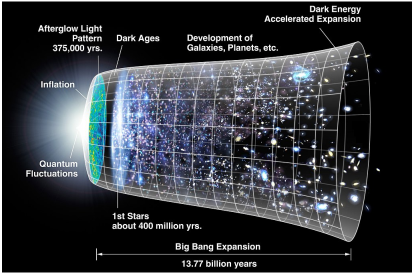
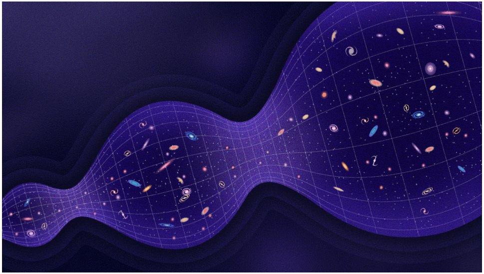


Source: Wikipedia



Source: Samuel Velasco/Quanta Magazine

Examples of Rich Families: part II

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The family $\mathcal{S}_\emptyset$ *of all* **vacuum solutions** *on* $\mathbb{R}^4$ *is rich.*

Examples of Rich Families: part II

"*The problem already lies in the local degrees of freedom of the background theory in 4D.*"

Theorem (Christodoulou, Sparling, P.)

The family $\mathcal{S}_\emptyset$ of all **vacuum solutions** *on $\mathbb{R}^4$ is rich.*

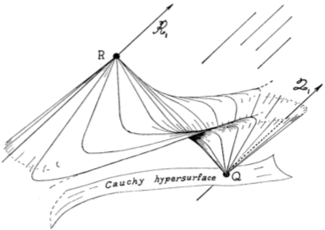
Remark. There is a unique vacuum solution on $\mathbb{R}^3$!

Proof: Plane Waves

Consider the variables u, v, x, y.

$$g^H_{\mu\nu}: \qquad (u, v, x, y) \mapsto \begin{bmatrix} H(u, x, y) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

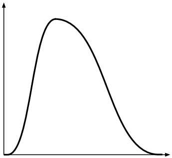
Is a **vacuum solution** whenever $H_{xx} + H_{yy} = 0$.

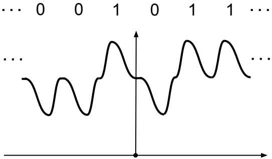


Penrose: "*A Remarkable Property of Plane Waves in General Relativity*"

The reduction

For every $\alpha \in \{0,1\}^{\mathbb{Z}}$ we define a "smooth version" $w^{\alpha}\colon \mathbb{R} \to \mathbb{R}$ of α:





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$$r(\alpha) := g^{\alpha}_{\mu\nu} \quad \text{given by} \quad (u,v,x,y) \mapsto \begin{bmatrix} w^{\alpha}(u)xy & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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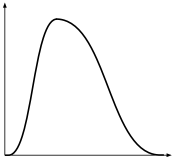

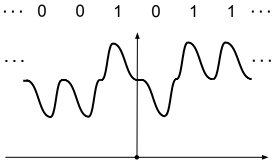

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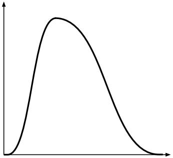

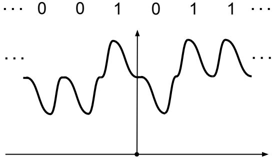

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- Showing that $\alpha E_{\mathbb{Z}} \beta \ \Leftarrow\ r(\alpha) \simeq_{\mathrm{diff}} r(\beta)$ is **hard**.

The difficult direction

Assume that:

$$g := \big(\tilde{w}(\tilde{u})\tilde{x}\tilde{y}\big)d\tilde{u}^2 + 2d\tilde{u}d\tilde{v} + d\tilde{x}^2 + d\tilde{y}^2$$
$$\tilde{g} := \big(w(u)xy\big)du^2 + 2dudv + dx^2 + dy^2$$

are diffeomorphic under the smooth change of coordinates φ specified by

$$\tilde{u} = \tilde{u}(u, v, x, y)$$
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Naive approach: use the definition $\boxed{g_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma}}$

Dead end

The relation $\boxed{g_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\tilde{g}_{\rho\sigma}}$ gives the following equations:

$$
\begin{aligned}
H(u,x,y) &= \tilde{H}(\tilde{u},\tilde{x},\tilde{y})\tilde{u}_u + 2\tilde{u}_u\tilde{v}_u + \tilde{x}_u^2 + \tilde{y}_u^2 \\
0 &= \tilde{H}(\tilde{u},\tilde{x},\tilde{y})\tilde{u}_v + 2\tilde{u}_v\tilde{v}_v + \tilde{x}_v^2 + \tilde{y}_v^2 \\
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\end{aligned}
$$

Good Luck!

Instead: analyze the Killing vector fields!

By analyzing the Lie algebra of Killing fields: every diffeo φ between

$$g = \tilde{H}(\tilde{u}, \tilde{x}, \tilde{y})d\tilde{u}^2 + 2d\tilde{u}d\tilde{v} + d\tilde{x}^2 + d\tilde{y}^2$$

$$\tilde{g} = H(u, x, y)du^2 + 2dudv + dx^2 + dy^2$$

has to be of the following form, for some a, b, c and $f(x), g(u), h(u)$:

$$\begin{aligned}
\tilde{u} &= (u + a)/c \\
\tilde{x} &= x\cos(b) + y\sin(b) + g(u) \\
\tilde{y} &= -x\sin(b) + y\cos(b) + h(u) \\
\tilde{v} &= c[v - x\big(\cos(b)g'(u) - \sin(b)h'(u)\big) \\
&\quad -y\big(\sin(b)g'(u) - \cos(b)h'(u)\big) - f(u)]
\end{aligned}$$

(1960) Jordan, Ehlers, Kundt, based on Robinson

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2 Observables

3 The Proof

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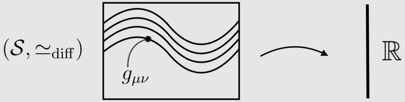
Canonical Quantization

Step 1: Find **complete** set of observables.



Step 2: promote them to an **algebra of operators** on a Hilbert space $\mathcal{H}$.

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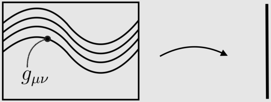
~~Step 1: Find **complete** set of observables.~~

Cannot be done in a constructive fashion if $\mathcal{S} \supseteq \{\text{vacuum solutions}\}$

$(\mathcal{S}, \simeq_{\text{diff}})$  $\mathbb{R}$

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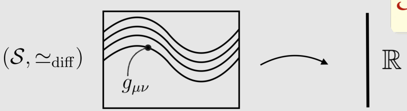

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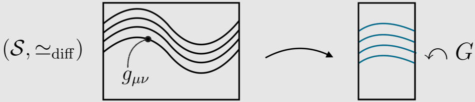
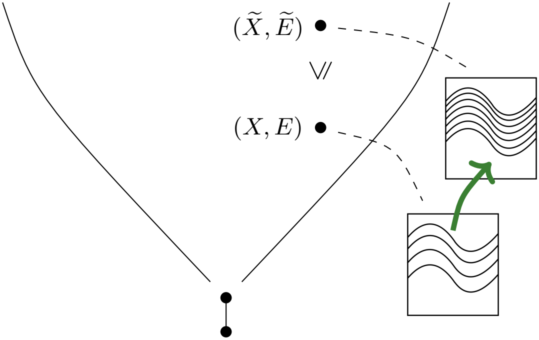
Equivariant forms of Quantization



G–**observables** where G has **nice representation theoretic properties**.

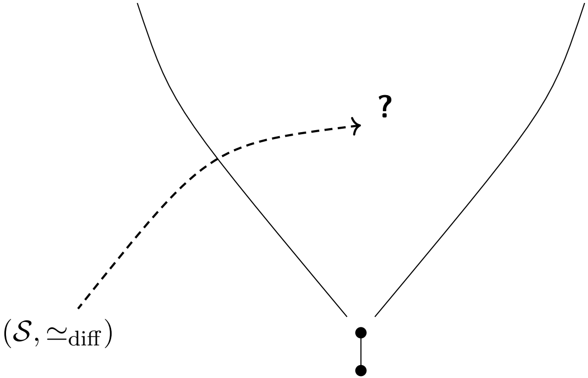
The Borel reduction hierarchy



A **classification problem** (X, E) is an equivalence relation E on Polish X

$(X, E) \leqslant (\widetilde{X}, \widetilde{E})$ iff (X, E) **Borel reduces** to (Y, F)
iff there exists Borel $r \colon X \to Y$ so that $xEx' \iff r(x)Fr(x')$

Problem. Place $(\mathcal{S}, \simeq_{\mathrm{diff}})$ in the **Borel reduction hierarchy**



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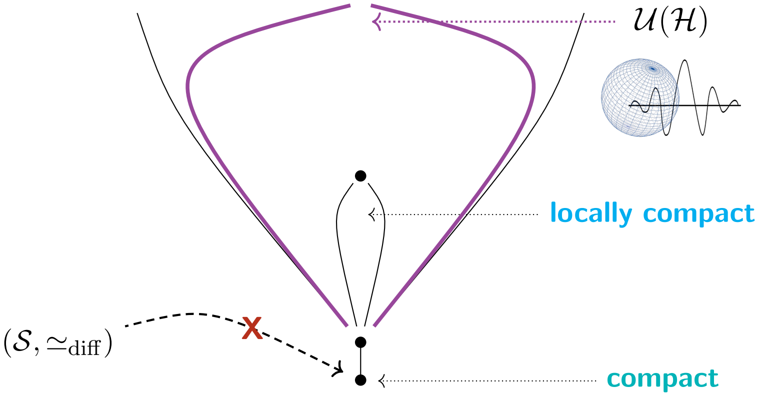

Thαnk you!