Broad Infinity and Generation Principles

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Outline

1. Broad Infinity: a new axiom scheme
2. Ordinals
3. Rubrics

Basics: ZF and ZFA

ZF is a set theory that lacks the axiom of choice (AC).

ZFA is a variant that allows urelements, aka atoms.

Classes are represented by formulas with parameters.

Examples

$\mathcal{T}$ is the class of all things (universal class).

Set is the class of all sets.

They are the same in ZF.

Basics: encodings

We define $\langle x, y \rangle \stackrel{\text{def}}{=} \{\{x\}, \{x, y\}\}$

observing that $\langle x, y \rangle = \langle x', y' \rangle$ implies $x = x'$ and $y = y'$.

We define Nothing $\stackrel{\text{def}}{=} \emptyset$ and Just$(x) \stackrel{\text{def}}{=} \{x\}$

observing that Just$(x) \neq$ Nothing,
and that Just$(x) =$ Just(x') implies $x = x'$.

A K-tuple within C is a function $K \to C$,

think of it as a column with K entries.

Zermelo's axiom of Infinity, 1908

This is included in ZF and ZFA.

There is a set X such that

- Nothing $\in X$
- for any $x \in X$, we have Just$(x) \in X$.

Axiom scheme of Simple Broad Infinity

This isn't provable in ZFC (assuming ZF is consistent).

For any function $F : \mathcal{T} \to \mathsf{Set}$, there is a set X such that

- Nothing $\in X$
- for any $x \in X$ and Fx-tuple y within X, we have Just$\langle x, y \rangle \in X$.

Every time we construct a new element, we gain a new arity.

ZF + Broad Infinity is called Broad ZF.

$\mathbb{N}$ without Infinity

We go through this more slowly.

Remove Infinity, Powerset and Foundation from the base theory.

Let $\mathbb{N}$ be the class of all (Zermelo) natural numbers.

It's the least class X such that

- Nothing $\in X$
- for any $x \in X$, we have Just$(x) \in X$.

For example, 3 is represented as Just(Just(Just(Nothing))).

Infinity says that $\mathbb{N}$ is a set.

Four variants of Infinity

The following are equivalent to Powerset + Infinity:

- Simple Wide Infinity
  Uses one arity.
- Full Wide Infinity
  Uses symbols of various arities.

The following are equivalent:

- Simple Broad Infinity
  Every time we construct a new element, we gain a new arity.
- Full Broad Infinity
  Every time we construct a new element, we gain new symbols of various arities.

Note: the Simple versions seems to be too weak in intuitionistic set theory.

Simple Wide Infinity

Uses one arity

Let K be an arity—a set.

We write $\mathsf{SimpleWide}(K)$ for the class of all simple K-wide numbers.

It's the least class X such that

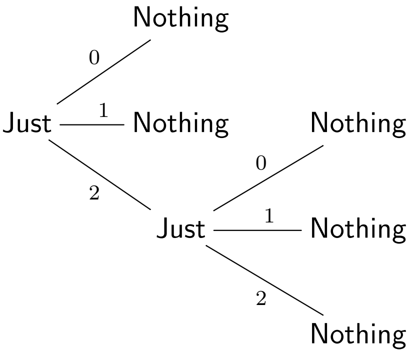
- Nothing $\in X$
- for any K-tuple x within X, we have $\mathsf{Just}(x) \in X$.

Simple Wide Infinity says that $\mathsf{SimpleWide}(K)$ is a set.

Wide number as two-dimensional tree

Arity $= \{0, 1, 2\}$.

$$\text{Just}\begin{bmatrix} \text{Nothing} \\ \text{Nothing} \\ \text{Just}\begin{bmatrix} \text{Nothing} \\ \text{Nothing} \\ \text{Nothing} \end{bmatrix} \end{bmatrix}$$



- Vertical dimension for tupling.
- Horizontal dimension for internal structure.
- Root at the left.

Full Wide Infinity (van den Berg)

Uses symbols of various arities.

Let $S - (K_i)_{i \in I}$ be a signature—a family of sets.
Each $i \in I$ is a symbol and K_i is its arity.

We write $\mathsf{Wide}(S)$ for the class of all S-wide numbers.

It's the least class X such that

- for any $i \in I$ and K_i-tuple x within X, we have $\langle i, x \rangle \in X$.

Full Wide Infinity says that $\mathsf{Wide}(S)$ is a set.

Simple Broad Infinity

Every time we construct a new element, we gain a new arity.

Let F be a broad arity—a function $\mathcal{T} \rightarrow$ Set.

We write SimpleBroad(F) for the class of all *simple F-broad numbers*.

It's the least class X such that

- Nothing $\in X$
- for any $x \in X$ and Fx-tuple y within X, we have Just$\langle x, y \rangle \in X$.

Simple Broad Infinity says that SimpleBroad(F) is a set.

Broad number as three-dimensional tree

The broad arity sends Just$\langle$Nothing, $[\,]\rangle$ to $\{0, 1\}$, and everything else to $\emptyset$.

$$\text{Just}\langle\text{Just}\langle\text{Just}\langle\text{Nothing}, [\,]\rangle, \begin{bmatrix} \text{Nothing} \\ \text{Just}\langle\text{Nothing}, [\,]\rangle \end{bmatrix}\rangle, [\,]\rangle$$

- Vertical dimension for tupling.
- Horizontal dimension for Just$\langle -, - \rangle$.
- Depth dimension for internal structure.
- Root at the front.
- Nothing-marked leaves at the rear.

Full Broad Infinity

Every time we construct a new element, we gain new symbols of various arities.

Write $\mathcal{S}$ for the class of all signatures.

Let G be a broad signature—a function $\mathcal{T} \to \mathcal{S}$.

Write Broad(G) for the class of all G-broad numbers. It's the least class such that

- Nothing $\in X$
- for any $x \in X$ such that $Gx = (K_i)_{i \in I}$, and any $i \in I$ and K_i-tuple y within X, we have Just$\langle x, i, y \rangle \in X$.

Full Broad Infinity says that Broad(G) is a set.

Summary

I look at the changing sea and sky
And try to picture Infinity

(Noel Coward)

Updated version

I look at the 3D trees outside
And picture Broad Infinity

Basics of ordinals

Ord is the class of all ordinals.

A limit is an ordinal that is neither 0 nor a successor.

An initial ordinal is not in bijection with a smaller ordinal. Examples are the finite ordinals, ω and ω_1.

When AC is assumed, they are called cardinals.

Two principles from the literature

We shall consider the following principles:

- Blass's axiom: There are unboundedly many regular limits. (Provable from AC.)
- Mahlo's principle: There are stationarily many regular limits.

We first define regularity and stationarity.

Each of these has many equivalent definitions.

Regular limits

A limit κ is *regular* when, for all $\alpha < \kappa$,

the supremum function $\mathsf{Ord}^\alpha \to \mathsf{Ord}$ restricts to a function $\kappa^\alpha \to \kappa$.

$$\begin{aligned} \text{That is: for all } \alpha &< \kappa \\ \text{and } \beta_i &< \kappa \text{ for all } i < \alpha \\ \text{we have } \bigvee_{i \in I} \beta_i &< \kappa \end{aligned}$$

Non-example: $\aleph_\omega$.

Regular implies initial, so ω is the only regular limit that is countable.

Blass's axiom

Blass's axiom says there are arbitrarily large regular limits.

Provable from AC.

But not without.

Gitik's result

Assuming ZFC + "There are arbitrarily large strongly compact cardinals" is consistent,

ZF cannot prove that there is an uncountable regular limit.

Stationary class

For a function $F\colon \mathsf{Ord} \to \mathsf{Ord}$,

a limit λ is F-closed when R restricts to a function $\lambda \to \lambda$.

A class of limits D is stationary when

for all $F\colon \mathsf{Ord} \to \mathsf{Ord}$, there's an F-closed ordinal in D.

This implies

- D is unbounded
- for all $F\colon \mathsf{Ord} \to \mathsf{Ord}$,, there are stationarily many ordinals in D.

Mahlo's principle

Mahlo's principle says there are stationarily many regular limits.

Implies Blass's axiom.

Implies there are arbitrarily large inaccessibles. By taking suitable F.

And α-inaccessibles, hyper-inaccessibles etc.

Limitations of Mahlo's principle

Appealing though Mahlo's principle may be,

I consider it deficient as an axiom scheme, in two respects.

1. It falls short of the ZF standard of simplicity.
2. It's entangled with choice.

Lack of simplicity

Each ZF axiom, other than Extensionality and Foundation,

says that some easily grasped things form a set.

Examples

Infinity	the natural numbers.
Powerset	the subsets of a set.
Separation	the elements of a set that satisfy a property.
Replacement	the images of a set's elements.

Mahlo's principle doesn't do this.

Entanglement with Choice

Gitik: ZF does not imply the existence of an uncountable regular limit.

Arguably, any principle that does imply it is entangled with choice.

In particular, Mahlo's principle.

Counterpoint: a choiceless reflection argument

"For any $F : \mathrm{Ord} \to \mathrm{Ord}$,
the property of being a F-closed regular limit
can be reflected down from Ord to an ordinal."

We avoid such thinking.

Motivating Broad Infinity

I wanted an axiom scheme with these properties:

1. It's equivalent to Mahlo's principle, assuming AC.
2. It asserts that some easily grasped things form a set.
3. It doesn't imply (given only ZF) that an uncountable regular limit exists.

Simple Broad Infinity—successful?

1. Is it equivalent to Mahlo's principle, assuming AC?

   Yes.

2. Does it asserts that some easily grasped things form a set?

   I think so.

3. Does it imply (given only ZF) that an uncountable regular limit exists?

   Not as far as I know.

Plausibility

Simple Broad Infinity is designed to be plausible,

minimizing the mental effort needed to believe it.

Surely desirable for an axiom scheme.

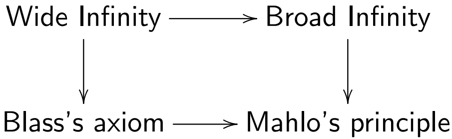
Disentanglement from choice helps to achieve this:

even for a person who find AC intuitively convincing,

it's easier to accept one intuition at a time.

Diagram of subsystems (without AC)

Arrow is inclusion of theories i.e. reverse implication.



Usefulness

My second goal was to find a scheme equivalent to Mahlo's principle that is useful,

minimizing the effort needed to apply it.

The Broad Set Generation scheme:

Every broad rubric on a class generates a subset.

There's also a Wide version, equivalent to Blass's axiom.

Example: wide rubric $\mathcal{R}$ on $\mathbb{N}$

Idea: the rubric tells you when to accept an element of $\mathbb{N}$.

- Rule 0 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_0 + m_1 + p)_{p \geqslant 2m_0}$.
- Rule 1 is nullary and sends $[\,] \mapsto (2p)_{p \geqslant 50}$.

Elements accepted by the rubric

- 100 has derivation $\langle 1, [\,], 50 \rangle$.
- 102 has derivation $\langle 1, [\,], 51 \rangle$.
- 402 has derivations $\left\langle 0, \begin{bmatrix} \langle 1, [\,], 50 \rangle \\ \langle 1, [\,], 50 \rangle \end{bmatrix}, 202 \right\rangle$ and $\left\langle 0, \begin{bmatrix} \langle 1, [\,], 50 \rangle \\ \langle 1, [\,], 51 \rangle \end{bmatrix}, 200 \right\rangle$.
- 7 has no derivation, so it is not accepted.

Wide rubric on a class C

A wide rubric is a family of wide rules.

A wide rule $\langle K, R \rangle$ on C consists of

- a set K—the arity
- a function R sending each K-tuple $[a_k]_{k \in K}$ within C to a family $(y_p)_{p \in P}$.

Example: broad rubric $\mathcal{S}$ on $\mathbb{N}$

Idea Each tuple yields a wide rubric.

Broad rule A is nullary, and sends $[\,]$ to the wide rubric $\mathcal{R}$.

Broad rule B is unary, and sends [7] to the following wide rubric:

- Rule 0 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_0 + m_1 + 500p)_{p \geqslant 9}$.

and sends [100] to the following wide rubric:

- Rule 0 is ternary and sends $\begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix} \mapsto (m_0 + m_1 m_2 + p)_{p \geqslant 17}$.
- Rule 1 is nullary and sends $[] \mapsto (p+3)_{p \geqslant 1000}$.
- Rule 2 is binary and sends $\begin{bmatrix} m_0 \\ m_1 \end{bmatrix} \mapsto (m_1 + p)_{p \geqslant 4}$.

and sends $[n]$ for $n \neq 7, 100$ to the empty wide rubric.

Elements accepted by $\mathcal{S}$

- 100 has derivation $\langle \mathsf{A}, [\,], 1, [\,], 50 \rangle$.
- 102 has derivation $\langle \mathsf{A}, [\,], 1, [\,], 51 \rangle$
- 107 has derivation $\langle \mathsf{B}, [\langle \mathsf{A}, [\,], 1, [\,], 50 \rangle], 2, \begin{bmatrix} \langle \mathsf{A}, [\,], 1, [\,], 50 \rangle \\ \langle \mathsf{A}, [\,], 1, [\,], 51 \rangle \end{bmatrix}, 5 \rangle$
- 7 has no derivation, so it is not accepted.

Broad rubric on a class C

A broad rubric is a family of wide rules.

A broad rule $\langle L, S \rangle$ on C consists of

- a set L—the arity
- a function S sending each L-tuple $[b_l]_{l \in L}$ within C to a wide rubric.

Application: Grothendieck universes

A Grothendieck universe is a transitive set $\mathfrak{U}$ such that

- $\mathbb{N} \in \mathfrak{U}$.
- For every set of sets $\mathcal{A} \in \mathfrak{U}$, we have $\bigcup \mathcal{A} \in \mathfrak{U}$.
- For every set $A \in \mathfrak{U}$, we have $\mathcal{P}A \in \mathfrak{U}$.
- For every set $K \in \mathfrak{U}$ and K-tuple $[a_k]_{k \in K}$ within $\mathfrak{U}$, we have $\{a_k \mid k \in K\} \in \mathfrak{U}$.

The axiom of Universes says that every set X is included in a Grothendieck universe.

Broad Set Generation directly gives this—no need for a detour through ordinals or cardinals.

Derivation Set principles

To get from Broad Infinity to Broad Set Generation (or equivalently to Mahlo's principle), we use AC.

A weak form of AC known as WISC is sufficient.

Those who don't accept AC can make do with Broad Derivation Set:

For any broad rubric, the class of derivations is a set.

This gives Tarski-style universes, used by type theorists.

The sets in such a universe are indexed by "codes".

Future work

- Intuitionistic set theory
- Constructive set theory
- Restricted vs unrestricted quantification

Summary

- Broad Infinity is a ZF-style principle:
  some easily grasped things (the F-broad numbers) form a set.
- Every time we construct an element, we gain an arity.
- Given AC, it's equivalent to Mahlo's principle.
- Without AC, it seems to be weaker.
- Broad Set Generation is equivalent to Mahlo's principle and directly yields Grothendieck universes.
- Broad Derivation Set is equivalent to Broad Infinity and directly yields Tarski-style universes.
- Each Broad principle has a Wide counterpart that's ZFC-provable.