Degrees of Unsolvability: A Realizability-Theoretic Perspective

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Two Branches of Computability Theory

- Degree Theory
  - ▷ studies degrees of algorithmic unsolvability of various problems.
  - ▷ initiated by Post (1944), Kleene-Post (1954), ...
  - ▷ many-one degree, truth-table degree, Turing degree, enumeration degree, ...
- Realizability Theory
  - ▷ aims at providing computability-theoretic models of constructive systems.
  - ▷ initiated by Kleene (1945), ...

New Interactions

- Applying Realizability Theory to Degree Theory.
  - Classical theory has some shortcoming: the degree of unsolvability of "*natural problems*" almost entirely determined by counting the "*number of alternations of quantifiers*."
    - i.e., natural problems $\approx$ master codes
  - Using realizability theory, one can reveal the hidden true structure of "*natural problems*."
- Applying Degree Theory to Realizability Theory.
  - Realizability theory discusses the structure of realizability models and their internal logic, and so on.
  - Using degree theory, one can clarify the specific shape of the structure of subtoposes of realizability toposes.
  - Also, degree theory enable us to flexibly construct realizability models of (semi-)constructive systems.

Tutorial 1

Realizability Theory → Degree Theory

Realizability Interpretation

- Key Observation: Formulas involve the notion of witness:
  - ▷ A formula $\exists x \varphi(x)$ may involve existential witnesses
  - ▷ For $\varphi \vee \psi$, information about which is correct.

- Kleene (1945): Realizability Interpretation
  - $\langle a, b \rangle$ realizes $\varphi \wedge \psi \iff a$ realizes φ and b realizes ψ.
  - $\langle i, a \rangle$ realizes $\varphi \vee \psi$
    $\iff$ if $i = 0$ then a realizes φ, otherwise a realizes ψ.
  - e realizes $\varphi \to \psi \iff$ if a realizes φ then $e * a$ realizes ψ.
  - $\langle t, a \rangle$ realizes $\exists x \in \mathbb{N}\ \varphi(x) \iff a$ realizes $\varphi(t)$.
  - e realizes $\forall x \in \mathbb{N}\ \varphi(x) \iff$ for any n, $e * n$ realizes $\varphi(n)$.

    ▷ Here, $e * a$ means the result of feeding input a to program e

  This gives an interpretation of intuitionistic arithmetic.

Definition (Post 1944)

For problems A and B, we say that *A is reducible to B* if there exists a well-behaved function h such that

$$(\forall x) \qquad A(x) \text{ is true} \iff B(h(x)) \text{ is true.}$$

- well-behaved: *computable* or *polytime computable* or *continuous* or *Borel measurable* or ...

(1) For Computability Theorists:
- Problems are subsets of ω; well-behaved means *computable*.
- This reducibility is known as many-one reducibility.

(2) For Descriptive Set Theorists:
- Problems are subsets of ω^ω; well-behaved means *continuous*.
- This reducibility is known as Wadge reducibility.

(3) For Complexity Theorists:
- Problems are subsets of Σ^*; well-behaved means *PTIME*.
- This reducibility is known as Karp reducibility.

As for natural problems, (1) and (2) have a roughly similar structure.

Completeness for Natural Decision Problems

A problem A is *Γ-complete* if $A \in \Gamma$ and any $B \in \Gamma$ is reducible to A.

Empirical Fact (for many-one/Wadge reducibility)

Any natural decision problem is Σ^0_n- or Π^0_n-complete for some $n \in \mathbb{N}$ whenever it is arithmetically definable.

- Σ^0_2-complete problems:
  - Decide if a given countable poset is bounded.
  - Decide if a given countable poset has finite width.
- Π^0_2-complete problems:
  - Decide if a given countable graph is connected.
  - Decide if a given countable linear order is dense.

This merely count the "*number of alternations of quantifiers*."

A Few More Details

- Σ^0_2-complete problems:
  - Decide if a given countable poset is bounded.
    - ▷ $\varphi(P) \equiv \exists t, b \in P \forall p \in P\ (b \leq_P p \leq_P t)$.
  - Decide if a given countable poset has finite width.
    - ▷ $\varphi(P) \equiv \exists n \in \mathbb{N}\ \forall p_0, \dots, p_n \in P\ \exists i, j \leq n\ (i \neq j \text{ and } p_i \leq_P p_j)$.
- Π^0_2-complete problems:
  - Decide if a given countable graph is connected.
    - ▷ $\varphi(G) \equiv \forall u, v \in G\ \exists \gamma\ (\gamma \text{ is a path connecting } u \text{ and } v)$.
  - Decide if a given countable linear order is dense.
    - ▷ $\varphi(L) \equiv \forall a, b \in L\ \exists c \in L\ (a <_L b \rightarrow a <_L c <_L b)$.

This merely count the "*number of alternations of (unbdd) quantifiers*."

The Realizability Interpretation of Many One Reducibility

Definition (Levin 1973)

For problems A and B, we say that *A is reducible to B* ($A \le B$) if there exist well-behaved functions h, r_-, r_+ such that

- r_- is a realizer for [$A(x)$ is true $\Longrightarrow$ $B(h(x))$ is true]; that is,
  - ▷ if a is a witness for $A(x)$ then $r_-(a,x)$ is a witness for $B(h(x))$.
- r_+ is a realizer for [$A(x)$ is true $\Longleftarrow$ $B(h(x))$ is true]; that is,
  - ▷ if b is a witness for $B(h(x))$ then $r_+(b,x)$ is a witness for $A(x)$.

In other words, the following is realizable:

$$(\forall x) \qquad A(x) \text{ is true} \iff B(h(x)) \text{ is true}$$

- This is exactly the realizability interpretation of many-one reducibility.
- Levin introduced this notion for the classification of NP-problems.
  - ▷ In Levin's setting, well-behaved ≈ polytime computable.
  - ▷ A witness ≈ a certificate for a NP-problem.
- No Computability-Theorists seem to have studied this notion.

Existential Witnesses

- A "*problem*" is described by a formula.
  - A Σ^0_2-problem $\exists a \forall b \varphi(a,b,x)$ may have an existential witness.
- Σ^0_2-complete problems:
  - **BddPos**: Decide if a countable poset is bounded.
  - **FinWidth**: Decide if a countable poset has finite width.
  - **DisConn**: Decide if a countable graph is disconnected.
  - **NonDense**: Decide if a countable linear order is non-dense.
- Classical reduction cannot distinguish between these four problems.

Theorem (K. 202x) for realizable many-one/Wadge reducibility

BddPos $<$ **FinWidth** $<$ **DisConn** $<$ **NonDense**

- ▷ This does not mean that this Levin-like degree structure is chaotic.
- ▷ Levin-like reducibility reveals the hidden structure of natural problems.
- ▷ There are clear reasons why the strength of these four problems differs.

New Classes of Formulas

What is the hidden structure of $\mathbf{\Sigma}^0_2$-complete natural problems?

- ($\exists\forall$) Some is of the form $\exists a \forall b\ \varphi(a,b,x)$.
- ($\forall^\infty$) Some is of the form $\exists a \forall b \geq a\ \varphi(b,x)$.
- ($\forall^\infty\forall$) Some is of the form $\exists a \forall b \geq a \forall c\ \varphi(b,c,x)$.

Theorem (K. 202x) for realizable many-one/Wadge reducibility

There are at least three levels of $\mathbf{\Sigma}^0_2$-complete natural problems.

$$\forall^\infty, \quad \forall^\infty\forall \quad \text{and} \quad \exists\forall$$

Indeed:

- **BddPos** is $\forall^\infty$-complete.
- **FinWidth** is $\forall^\infty\forall$-complete.
- **NonDense** is $\exists\forall$-complete.

And computable/continuous Levin reducibility distinguishes between these.

Higher Levels

Π^0_3-complete problems:

- **Lattice**: Decide if a countable poset is a lattice.
- **Atomic**: Decide if a countable poset is atomic.
- **LocFin**: Decide if a countable graph is locally finite.
- **FinBranch**: Decide if a countable tree is finitely branching.
- **Compl**: Decide if a countable poset is complemented.
- **InfWidth**: Decide if an enumerated poset has infinite width.
- **Cauchy**: Decide if a rational sequence is Cauchy.
- **Normal**: Decide if a real is simply normal in base 2.
- **Perfect**: Decide if a countable binary tree is perfect.

Classical reduction cannot distinguish between these problems.

New Theorem!

The following are $\forall\forall^\infty$-bicomplete:

- **Lattice**: Decide if a countable poset is a lattice.
- **Atomic**: Decide if a countable poset is atomic.

The following are $\forall\forall^\infty\forall$-bicomplete:

- **LocFin**: Decide if a countable graph is locally finite.
- **FinBranch**: Decide if a countable tree is finitely branching.

The following is $\forall\exists\forall$-bicomplete:

- **Compl**: Decide if a countable poset is complemented.

The following is $\exists^\infty\exists\forall$-bicomplete:

- **InfWidth**: Decide if an enumerated poset has infinite width.

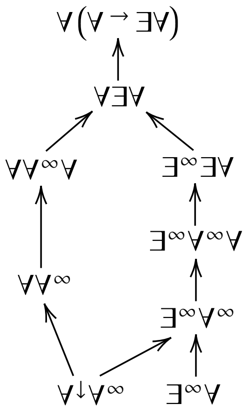
The following are $\forall^\downarrow\forall^\infty$-bicomplete:

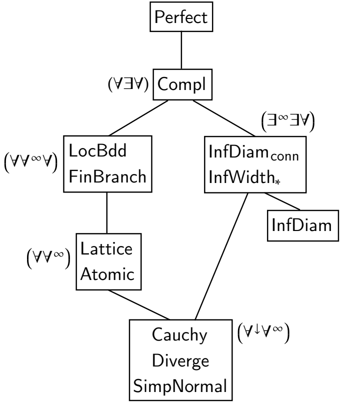
- **Cauchy**: Decide if a rational sequence is Cauchy.
- **Normal**: Decide if a real is simply normal in base 2.

The following is $\forall(\forall \to \exists\forall)$-bicomplete:

- **Perfect**: Decide if a countable binary tree is perfect.

And computable/continuous Levin reducibility distinguishes between these.


$\forall(\forall \to \exists\forall)$
$\forall\exists\forall$
$\forall\forall^\infty\forall$
$\exists^\infty\exists\forall$
$\exists^\infty\forall^\infty\forall$
$\forall\forall^\infty$
$\exists^\infty\forall^\infty$
$\forall^\downarrow\forall^\infty$
$\exists^\infty\forall$


Perfect
$(\forall\exists\forall)$
Compl
$(\exists^\infty\exists\forall)$
$(\forall\forall^\infty\forall)$
LocBdd
FinBranch
InfDiam$_{\text{conn}}$
InfWidth$_*$
InfDiam
$(\forall\forall^\infty)$
Lattice
Atomic
Cauchy
Diverge
SimpNormal
$(\forall^\downarrow\forall^\infty)$

Key Ideas

Historical Background

- The results described so far are new discoveries in classical mathematics.
  - ▷ They are of interest to classical computability theorists.
- However, the origin of this research lies in Veldman's work in intuitionistic mathematics.
  - Of course, a realizability interpretation gives a model of an intuitionistic system.
- Veldman was not simply introducing a *intuitionistic version of many-one/Wadge reducibility*, but was conducting truly new research including new counterexample constructions.
- Veldman's research had been ongoing since the 1980s, but because it was described in a very formal way in the context of intuitionistic mathematics, it seems that classical computability theorists did not realize its importance.

The origin of research into the realizability interpretation of many-one/Wadge reducibility is Veldman's series of studies:

- W. Veldman, Investigations in intuitionistic hierarchy theory, Ph.D. Thesis, Katholieke Universiteit Nijmegen, 1981.
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- W. Veldman, The fine structure of the intuitionistic Borel hierarchy, Rev. Symb. Log. 2 (2009) 30-101.
- W. Veldman, Projective sets, intuitionistically. J. Log. Anal. 14 (2022), Paper No. 5, 85 pp.

The Result that Triggered this Research

Σ^0_2-completeness of Fin is "*trivial*" to those of us familiar with classical theory, but it is not necessarily true in intuitionistic mathematics.

Theorem (Veldman 2008)

In a certain intuitionistic system,
$\mathit{Fin} = \{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m > n.\ x(m) = 0\}$ is not Σ^0_2-complete.

- It is a very interesting theorem...
  but what the essence of this theorem is was unclear.

Our new perspective:

- It is not only Σ^0_2-definable, but also $\forall^\infty$-definable
  - $\forall^\infty \cdots$ "*for all but finitely many ...*"
- Indeed, Fin is a $\forall^\infty$-complete problem.
- However, a $\forall^\infty$-definable problem cannot be Σ^0_2-complete.

Qualitative Differences between Classes of Formulas

- $\forall^\infty \cdots \exists n \forall m \geq n \varphi(m,x)$
- $\forall^\infty \forall \cdots \exists n \forall m \geq n \forall k \varphi(m,k,x)$
- $\exists \forall \cdots \exists n \forall m \varphi(n,m,x)$

- Question: Why is $\forall^\infty$ different from $\exists\forall$?
- Answer: Amalgamability!
  - Given finitely many candidates for realizers, if at least one of them is correct, then it is always possible to construct a correct realizer.
  - (Example) If at least one of $n_0, n_1, \ldots, n_k$ is an existential witness for a $\forall^\infty$-formula $\theta := \exists n \forall m > n \varphi(m,x)$, then $\max\{n_0, n_1, \ldots, n_k\}$ is a correct existential witness for θ.
- Indeed, $\forall^\infty\forall$ has this property.
  - No $\forall^\infty\forall$-definable problem is Σ^0_2-complete.

Qualitative Differences between Classes of Formulas II

- $\forall^\infty \cdots \exists n \forall m \geq n \varphi(m,x)$
- $\forall^\infty \forall \cdots \exists n \forall m \geq n \forall k \varphi(m,k,x)$
- $\exists \forall \cdots \exists n \forall m \varphi(n,m,x)$

- Question: Why is $\forall^\infty$ different from $\forall^\infty\forall$?
- Answer: Unique witness property!
  - Given a realizer, one can always construct a "special" realizer.
  - (Example) If an existential witness n for a $\forall^\infty$-formula $\theta := \exists n \forall m > n \varphi(m,x)$ is given, then one can find the least existential witness for θ.
  - (Proof) Given a witness n for θ, find the least s such that any $m \in [s,n]$ satisfies the decidable formula $\varphi(m,x)$.
- $\forall^\infty\forall$ does not have this property.
  - No $\forall^\infty$-definable problem is $\forall^\infty\forall$-complete.

Natural $\forall^\infty$-Definable Problems

- **Fin**: Decide if an infinite sequence is eventually zero.
- **Period**: Decide if an infinite sequence is eventually periodic.
- **BddPos**: Decide if a countable poset is bounded.
  - A poset is bounded if it has the top and bottom elements.

Fin, **Period** and **BddPos** are $\forall^\infty$-complete.

Proof (using Unique witness property):

- For **Fin, Period**, given a witness, one can find the least witness.
  - For completeness, add a new nonzero term if the current witness is refuted; otherwise keep adding zeros.
- For **BddPos**, the top and bottom elements are unique if they exist.
  - For completeness, add new $\top$ and $\bot$ if the current witness is refuted; otherwise keep the current $\top$ and $\bot$.

Natural $\forall^\infty\forall$-Definable Problems

- **Bdd**: Decide if an infinite sequence has an upper bound.
- **FinWidth**: Decide if a countable poset has finite width.
  - ▷ The width of a poset is the size of a maximal antichain.
- **FinHeight**: Decide if a countable poset has finite height.
  - ▷ The height of a poset is the size of a maximal chain.

Bdd, **FinWidth** and **FinHeight** are $\forall^\infty\forall$-complete.

Proof (using Increasing witness property):

- If n is a witness for $\exists n \forall k \geq n \forall \ell \dots$, so is any $m \geq n$.
- For **Bdd**, if n is an upper bound, so is any $m \geq n$.
  - ▷ For completeness, the value of a new term is the smallest unrefuted witness.

Abstract framework

Categorical Formulation

Our results are implemented as an interpretation of reducibility in a certain category.

Thee main "algebras" $(\mathbb{A}, \mathbb{A}_{eff}, *)$:

- Kleene's first algebra K_1
  - The algebra of computability on natural numbers.
  - $\mathbb{A} = \mathbb{A}_{eff} = \mathbb{N}$ and $e * x = \varphi_e(x)$
  - where φ_e is the eth partial computable function on $\mathbb{N}$.
- Kleene's second algebra K_2
  - The algebra of continuity on infinite strings.
  - $\mathbb{A} = \mathbb{A}_{eff} = \mathbb{N}^{\mathbb{N}}$, and $e * x = \psi_e(x)$
  - where ψ_e is the partial continuous function on $\mathbb{N}^{\mathbb{N}}$ coded by e.
- Kleene-Vesley algebra KV
  - The algebra of computability on infinite strings.
  - $\mathbb{A} = \mathbb{N}^{\mathbb{N}}$, $\mathbb{A}_{eff}$ = computable strings, and $e * x = \psi_e(x)$

Represented Spaces

Let $(\mathbb{A}, \mathbb{A}_{eff}, *)$ be a relative pca, i.e, K_1, K_2, KV or so.

- An represented space is a pair of a set X and a partial surjection $\delta :\subseteq \mathbb{A} \to X$.
  - ▷ That $\delta(p) = x$ means that p is a code of $x \in X$.
- A function $f : X \to Y$ is realizable if there exists $a \in A_{eff}$ such that if p is a code of $x \in X$ then $a * p$ is a code of $f(x) \in Y$

A represented space is also known as a modest set.

- Fact: The category of represented spaces and realizable functions is a locally cartesian closed category with NNO, whose internal logic corresponds to the realizability interpretation.

Kleene (1945): Realizability Interpretation

- $\langle a, b\rangle$ realizes $\varphi \wedge \psi \iff a$ realizes φ and b realizes ψ.
- $\langle i, a\rangle$ realizes $\varphi \vee \psi$
  $\iff$ if $i = 0$ then a realizes φ, otherwise a realizes ψ.
- e realizes $\varphi \to \psi \iff$ if a realizes φ then $e * a$ realizes ψ.
- $\langle p, a\rangle$ realizes $\exists x\, \varphi(x) \iff p$ codes x and a realizes $\varphi(t)$.
- e realizes $\forall x\, \varphi(x) \iff$ if a codes x then $e * a$ realizes $\varphi(x)$.

LCCC structure of the category of represented spaces.

- $\langle a, b\rangle$ codes $(x, y) \in X \times Y \iff a$ codes $x \in X$ and b codes $y \in Y$.
- $\langle i, a\rangle$ codes $(i, x) \in X + Y$
  $\iff$ if $i = 0$ then a codes $x \in X$, otherwise a realizes $x \in Y$.
- e codes $f \in Y^X \iff$ if a codes $x \in X$ then $e * a$ codes $f(x) \in Y$.
- $\langle p, a\rangle$ codes $(t, x) \in \sum_{u \in I} X_u \iff p$ codes $t \in I$ and a codes $x \in X_t$.
- e codes $f \in \prod_{u \in I} X_u \iff$ if a codes $t \in I$, $e * a$ codes $f(t) \in X_t$.

In the category of represented spaces:

- A formula is interpreted as something like a "witness-search problem (or a realizer-search problem)"

Example: The type $\mathbb{N}^{\mathbb{N}}$ formula "$\varphi(x) \equiv \exists n \forall m \geq n.\ x(m) = 0$" is interpreted as a subobject $\mathit{FIN} \rightarrowtail \mathbb{N}^{\mathbb{N}}$ such that

- the underlying set is $\{x \in \mathbb{N}^{\mathbb{N}} : \exists n \forall m \geq n.\ x(m) = 0\}$
- a name of $x \in \mathit{FIN}$ is a pair of $\langle x, n \rangle$, where n is an existential witness.

Fact: Every subobject of X has a representative of the following form:

- an underlying set A is a subset of X
- a name of $x \in A$ is the pair of a name p of $x \in X$ and some $q \in \mathbb{A}$. This q is considered as a "witness".

Roughly speaking:

- A subobject is a subset with witnesses.
- A regular subobject is a subset without witnesses.

Recall: A problem A is reducible to B (written $A \leq B$) iff

$$\exists \text{ well-behaved } \varphi\ \forall x\ (x \in A \iff \varphi(x) \in B)$$

That is, $A = \varphi^{-1}[B]$.

Its categorical version would be something like:

Def: Let X, Y be objects in a category C having pullbacks.
A mono $A \overset{\alpha}{\rightarrowtail} X$ is reducible to $B \overset{\beta}{\rightarrowtail} Y$ if $A \overset{\alpha}{\rightarrowtail} X$ is a pullback of $B \overset{\beta}{\rightarrowtail} Y$ along some morphism $\varphi\colon X \to Y$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, tail] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

When this notion is interpreted in the category of represented spaces, we obtain (computable/continuous) Levin reducibility.

New Theorem!

The following are $\forall\forall^\infty$-bicomplete:

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- **Atomic**: Decide if a countable poset is atomic.

The following are $\forall\forall^\infty\forall$-bicomplete:

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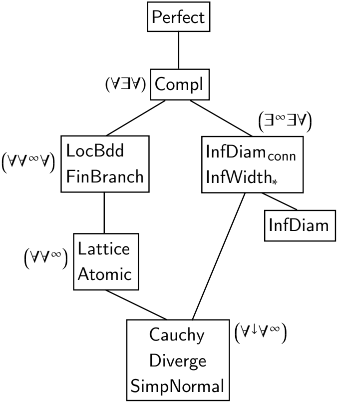
And computable/continuous Levin reducibility distinguishes between these.

$$
\begin{tikzcd}
 & \forall(\forall \to \exists\forall) & \\
 & \forall\exists\forall \arrow[u] & \\
\forall\forall^{\infty}\forall \arrow[ur] & & \exists^{\infty}\exists\forall \arrow[ul] \\
 & & \exists^{\infty}\forall^{\infty}\forall \arrow[u] \\
\forall\forall^{\infty} \arrow[uu] & & \exists^{\infty}\forall^{\infty} \arrow[u] \\
 & \forall^{\downarrow}\forall^{\infty} \arrow[ul] \arrow[ur] & \exists^{\infty}\forall \arrow[u]
\end{tikzcd}
$$



Summary:

- Constructive mathematics gives us ideas for good definitions.
- Classical mathematics gives us ideas for powerful proof techniques.
- The combination of the two, when well harmonized, yields beautiful results.

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