Upper Logicism

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§1. Introduction

First aim:

To characterize a novel philosophy of arithmetic: UPPER LOGICISM (UPPER) about arithmetic.

Second aim:

To present a couple of results on the relationship between modal type theory and arithmetic which lend support to UPPER.

Third aim:

To offer some reasons in favor of FINITARY PLENITUDE (a distinctive commitment of UPPER).

Motto:

‘Arithmetic is nothing but a part of higher-order modal logic’.

§2. Context

§2.1. The double life of number words

Natural number expressions seemingly occur both as proper nouns and as determiners.

E.g.:

– *Proper noun*: ‘Two is a prime number’

– *Determiner*: ‘Two students are playing football’

Proper nouns are usually taken to have individuals as their meanings.

E.g.:

– In ‘Cristiano Ronaldo is a striker’, ‘Cristiano Ronaldo’ has as its meaning an individual.

(Specifically, its meaning is Ronaldo)

Determiners are usually taken to have as their meanings properties of properties.

E.g.:

– In ‘everything is physical’, ‘everything’ has as its meaning a property of properties.

(Specifically, its meaning is that property that a property has just in case it is had by every thing)

In general, a realist philosophical account of arithmetic will eventually have to take a stand on whether the natural numbers are individuals or instead properties of properties.

§2.2. Fregean and neoFregean logicisms

Frege's and neoFregean forms of logicism are the most popular forms of logicism.

An underlying assumption of these views is that the natural numbers are individuals.

Frege's logicism (1884, 1893): Arithmetic is *analytically* true owing to being derivable . . .

. . . in classical second-order logic + BASIC LAW V;

… from "individualist", purely logical characterizations of the arithmetical primitives.

Basic Law V: Properties F and G have the same extension if and only if every individual is an F if and only if it is a G.

As it turns out, Frege's logicism is unviable.

For Russell (1902) showed that BASIC LAW V is inconsistent in the context of classical second-order logic.

NeoFregean logicism (Hale & Wright 2001): Arithmetic is *analytically* true owing to being derivable. . .

. . . in classical logic + Hume's principle;

. . . from “individualist”, purely logical characterizations of the arithmetical primitives.

Hume's principle: The number of F is identical to the number of G if and only if F and G are equinumerous (i.e., can be put in 1-1 correspondence).

Hume's principle is an abstraction principle.

I.e., it is a principle of the form

$\mathrm{Abs}_{\equiv}(F) = \mathrm{Abs}_{\equiv}(G)$ if and only if $F \equiv G$,

where $\equiv$ is an equivalence relation and $\mathrm{Abs}_{\equiv}(\cdot)$ is a function mapping the entities in its domain to individuals

(e.g., $\mathrm{Abs}_{\equiv}(F)$ is *the abstract of F with respect to the equivalence relation* $\equiv$).

NeoFregean logicism is plagued with difficulties stemming from its commitment to Hume's principle being analytically true in virtue of it being an abstraction principle.

– E.g., the *bad company objection*.

Bad company objection (Boolos 1990, 1997):
How can Hume's principle, as an *abstraction principle*, be *analytic*, if other abstraction principles are *inconsistent* with it.

For instance, the NUISANCE PRINCIPLE (Boolos 1990, Wright 1997) is an abstraction principle inconsistent with HUME'S PRINCIPLE:

Nuisance principle: The nuisance of F is identical to the nuisance of G if and only if F and G differ with respect to at most finitely many instances.

§2.3. Russell’s logicism

By contrast with Frege’s and neoFregean logicisms, Russell’s logicism was based on the assumption that the natural numbers are properties of properties

– Specifically, on the assumption that they are the finite cardinalities;

(i.e., n is the property of having being exactly n instances)

Russell's logicism (Whitehead and Russell 1910): Arithmetic is logically true owing to being derivable . . .

. . . in type theory + AXIOM OF INFINITY;

. . . from “higher-orderist”, purely logical characterizations of the arithmetical primitives.

Axiom of infinity: There are infinitely (i.e., not inductively finitely) many individuals.

By contrast with Frege’s and neoFregean logicisms, Russell’s is not plagued by Russell’s paradox, nor by problems such as the bad company problems.

(Indeed, higher-order versions of abstraction principles are logical theorems in type theory)

Major difficulty for Russell's logicism: reliance on the AXIOM OF INFINITY.

As Russell acknowledged, the AXIOM OF INFINITY seems to be neither a priori knowable nor necessarily true.

So, there is reason to think that it is not a truth of logic.

In such a case Russell's reduction is not a reduction of arithmetic to logic.

State of play:

Owing to the failures of Frege's, Russell's, and neoFregean logicisms, the lore of the land is that logicism is a dead-end.

Still, as Klement (2012) put it:

'By comparison to the litany of problems that continue to plague Abstractionist-style neo-logicisms, [a Russellian or neo-Russellian logicism] actually seems like a far less daunting route for a twenty-first century logicist to explore'.

§3. Upper Logicism

UPPER LOGICISM about arithmetic is the conjunction of:

(1) HIGHER-ORDERISM;

(2) HIGHER-TYPE ARITHMETICAL ONTOLOGY;

(3) LOGICALITY;

(4) NECESSITY OF FINITARY PLENITUDE;

(5) INTERPRETABILITY OF ARITHMETIC;

(6) NECESSITY OF ARITHMETIC.

(1) Higher-orderism:

Predicates have a semantic role distinct from those of individual (and of plural) terms.

Quantification into predicate position is legitimate and irreducible to first-order singular (or to plural) quantification.

On higher-orderism:

The adoption of higher-orderism lends support to simple type-theory's underlying type distinctions, and to its axioms' logicality.

Otherwise, these would seem unmotivated at best.

(2) Higher-type arithmetical ontology:

Natural numbers are finite cardinalities, and the Russellian characterization of arithmetic's primitives – *natural number*, *zero* and *successor* – are all true.

On higher-type arithmetical ontology:

Gets motivation from:

(a) The use of number words as determiners;

(b) *Frege's constraint*.

According to *Frege's constraint* (Wright 2000), the canonical applications of mathematical entities to the characterization of the world must be contained within their nature.

Finite cardinalities presumably contain the canonical application of natural numbers – specifically, to counting – within their nature.

After all, to count is to attribute a cardinality property.

(3) LOGICALITY:

The metaphysically necessary truths formulated in a pure modal and type-theoretic language are all logically true.

On logicality:

It is a consequence of the view (Shapiro 1998, 2005) that the logical truths are those necessary truths expressible solely in terms of logical vocabulary, provided that the necessity operator is taken to be logical.

Insofar as Russell did not make room for modal resources in his logical framework, LOGICALITY constitutes one important difference between UPPER and RUSSELL'S LOGICISM.

The appeal to modal resources nicely dovetails with HIGHER-ORDERISM.

For instance, in general, values of predicate variables are distinct from sets and values of plural variables insofar as the latter, but not the former, have their members essentially.

Similarly, the appeal to modal resources also dovetails nicely with HIGHER-TYPE ARITHMETICAL ONTOLOGY.

For instance, it seems that it is contingent whether a property falls under a finite cardinality.

E.g., though Mars (actually) has two moons, it could have had more.

(4) Necessity of FP:

Finitary Plenitude is necessarily true.

Finitary plenitude: Every finite cardinality property could have been instantiated.

On necessity of FP:

It is another commitment of upper not shared with Russell's logicism.

By contrast with the AXIOM OF INFINITY, the NECESSITY OF FP is consistent with it being necessary that there are only finitely many individuals.

Also, note that, given LOGICALITY, if NECESSITY OF FP can be shown, then its logicality will have been established.

Later on we will say more in defense of NECESSITY OF FP.

(5) Interpretability of arithmetic:

There is a deductive system, formulated in a pure modal and type-theoretic language and including FP among its axioms, whose theorems are necessary truths, and include the higher-type arithmetical ontology-respecting translations of all theorems of $\mathtt{PA}_2$.

On interpretability of arithmetic:

I'll later present a result that arguably establishes interpretability of arithmetic.

The adoption of modal resources brings with it a novel challenge:

– To account for the interaction between modality and quantification in a way which is relatively uncontroversial.

The system in which the result is proven is neutral on the status of highly controversial theses such as the following:

For each type τ:

Barcan formula$_\tau$: "Every possible x_τ actually exists."

Converse Barcan formula$_\tau$: "Every x_τ necessarily exists."

Necessitism$_\tau$: Necessarily, every x_τ necessarily exists.

In addition, the propositional modal logic required for the formal result supporting INTERPRETABILITY OF ARITHMETIC is just K, the weakest propositional modal logic.

– It is uncontroversially true of metaphysical necessity.

(6) NECESSITY OF ARITHMETIC:

The purely arithmetical truths expressible in the language of $\mathtt{PA}_2$ are all metaphysically necessary truths also expressible in a pure modal type theory.

On necessity of arithmetic:

Gödel's incompleteness theorems pose an important limitation to Frege's, neoFregean and Russell's logicisms.

For any given effectively axiomatizable system of modal type theory, there will be arithmetical truths which are not derivable in it.

But these arithmetical truths may nonetheless still be logical.

Indeed, and given LOGICALITY and HIGHER-TYPE ARITHMETICAL ONTOLOGY, they will be – provided that the formulas of pure modal type theory which also express them are necessarily true.

I'll later present a result that arguably establishes NECESSITY OF ARITHMETIC.

This second result requires a stronger propositional modal logic, `S5`.

– A system whose theorems, it is reasonable to think, are all true of metaphysical necessity.

It also requires some plausible principles of modal plural logic.

The resulting system is itself neutral on controversial theses such as the BARCAN FORMULA$_{\mathcal{T}}$, CONVERSE BARCAN FORMULA$_{\mathcal{T}}$ and NECESSITISM$_{\mathcal{T}}$.

§4. Interpretability of arithmetic

§4.1. Logical languages MT and PMT

Two languages:

(1) A pure modal type-theoretic language based on relational types;

(2) A plural extension extension PMT of MT.

Definition (Relational types)

The set of types is the smallest set such that:

(a) e is a type;

(the type of individuals)

Definition (Relational types)

(b) $\langle \tau^1, \ldots, \tau^n \rangle$ is a type, for every $n \in \mathbb{Z}^+$, if τ^1, …, τ^n are types;

(the type of n-ary relations between entities of, respectively, type τ^1, …, type τ^n).

Singular variables:

For each type θ, there are denumerably many singular variables $v_\theta^1, v_\theta^2, v_\theta^3, \ldots$.

Plural Variables:

For each type θ, there are denumerably many plural variables vv^1_θ, vv^2_θ, vv^3_θ, ...

Definition (PMT singular terms and formulae)

The sets of formulae and type θ-singular terms of PMT are the smallest sets such that:

(a) A singular variable subscripted with θ is a type θ-singular term;

Definition (PMT singular terms and formulae)

(b) $=_\theta$ is a type θ-singular term,

if $\theta = \langle \tau, \tau \rangle$;

Definition (PMT singular terms and formulae)

(c) $\beta(\alpha^1, \ldots, \alpha^n)$ is a formula,

if α^i is a type τ^i-singular term, for each $i \leqslant n \in \mathbb{Z}^+$, and β is a type $\langle\tau^1, \ldots, \tau^n\rangle$-singular term, for each $n \in \mathbb{Z}^+$;

Definition (PMT singular terms and formulae)

(d) $\neg\varphi$, $\varphi \wedge \psi$, $\Box\varphi$ are formulae,

if φ and ψ are formulae;

Definition (PMT singular terms and formulae)

(e) $\forall\delta\varphi$ is a formula,

if φ is a formula and δ is a singular or plural variable;

Definition (PMT singular terms and formulae)

(f) $\lambda v^1 \dots v^n \varphi$ is a type θ-singular term,

if φ is a formula and v^1, …, v^n are distinct singular variables of, respectively, types τ^1, …, τ^n, and $\theta = \langle \tau^1, \dots, \tau^n \rangle$, for each $n \in \mathbb{Z}^+$;

Definition (PMT singular terms and formulae)

(g) $\alpha \prec \beta$ is a formula,

if α and β are plural type τ-variables;

Definition (PMT singular terms and formulae)

(g) $\alpha = \beta$ is a formula,

if α and β are plural type τ-variables.

Definition (MT terms and formulae)

The terms of MT are the singular terms of PMT in which no plural variables occur.

The formulae of MT are those formulae of PMT in which no plural variables occur.

§4.2. Deductive system

Definition (Deductive system $\mathtt{KQL_C}$: I)

Axioms (in the language of MT):

(PL) Propositional tautologies

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Definition (Deductive system $\mathtt{KQL_C}$: II)

($\forall$1) $\forall v(\varphi \to \psi) \to (\forall v\varphi \to \forall v\psi)$

($\forall$2) $\varphi \to \forall v\varphi$,

provided that v occurs free nowhere in φ.

($\forall$E) $E[\alpha] \to (\forall v\varphi \to \varphi^v_\alpha)$,

where $E[\alpha] := \exists u(u = \alpha)$.

Definition (Deductive system $\mathtt{KQL}_{\mathtt{C}}$: III)

$(\forall{=})$ $\forall v(v = v \wedge E[v])$

(Ind) $\alpha = \gamma \rightarrow (\varphi \rightarrow \varphi')$,

where φ' is a formula which results from φ by having γ occur at some places where α occurs, re-lettering bound variables to ensure that no variables free in $\alpha = \gamma$ are bound in φ or in φ'.

Definition (Deductive system $\mathtt{KQL_C}$: IV)

(Ab1) $\lambda\overline{u}\varphi(\overline{\delta}) \to \varphi^{\overline{u}}_{\overline{\delta}}$

(Ab2) $(\varphi^{\overline{u}}_{\overline{\delta}} \wedge \bigwedge_{1\leqslant i\leqslant n} E[\delta^i] \wedge E[\lambda\overline{u}\varphi]) \to \lambda\overline{u}\varphi(\overline{\alpha})$

Definition (Deductive system $\texttt{KQL}_\texttt{C}$: V)

(Comp$_\mathrm{C}$) $E_\beta \rightarrow E[\beta]$,

where, given a list $\mu^1, \ldots, \mu^n$ of all atomic terms (or plural variables, except $\ulcorner =_{\langle\tau,\tau\rangle} \urcorner$, for all types τ) free in β,
$E_\beta := E[\mu^1] \wedge \ldots \wedge E[\mu^n]$.

Definition (Deductive system $\mathtt{KQL_C}$: VI)

Inference rules of $\mathtt{KQL_C}$:

(MP) $(\vdash \varphi \;\&\; \vdash \varphi \rightarrow \psi) \Rightarrow \vdash \psi$

(Nec) $\vdash \varphi \Rightarrow \vdash \Box\varphi$

(Gen) $\vdash \varphi \Rightarrow \vdash \forall v\varphi$

§4.3. Interpretability

Russellian translations:

The τ-Russellian translation $(\varphi)^{\mathcal{R}}_{\tau}$ of each PA_2 formula φ is that formula of MT which results from φ by:

(a) replacing each primitive of PA_2 with its Russellian $\langle\langle\tau\rangle\rangle$-characterization, and

(b) indexing φ's quantified variables with the type $\langle\langle\tau\rangle\rangle$.

Observation:

Russellian translations reflect the fact that, according to UPPER, the mathematical primitives are typically ambiguous.

This is unsurprising. Higher-orderists see the logical notions recurring along the type-hierarchy.

E.g.

– There is a property of *having exactly one instance* which applies to properties of individuals.

– There is a property of *having exactly one instance* which applies to properties of properties of individuals.

– Etc.

Theorem 1 (BJ, Forthcoming)

$$\vdash_{\mathtt{KQL_c}} \mathrm{FP} \rightarrow (\varphi)^{\mathcal{R}}_{\tau},$$

for every closed theorem φ *of* $\mathtt{PA}_2$ *and every type* τ.

Significance of Theorem 1:

The theorem establishes INTERPRETABILITY OF ARITHMETIC, provided:

(a) the reasonable view that all theorems of $\mathtt{KQL}_{\mathtt{C}}$ are necessarily true; and

(b) that FINITARY PLENITUDE is necessarily true.

Moreover, INTERPRETABILITY OF ARITHMETIC, HIGHER-ORDER ARITHMETICAL ONTOLOGY and LOGICALITY jointly imply that the theorems of $\mathtt{PA}_2$ are all expressible by logical truths.

§5. Necessity of arithmetic

§5.1. Deductive system

Definition (Deductive system $\mathtt{S5PQL_C}$: I)

Axioms: Those of $\mathtt{KQL_C}$ (now stated in PMT) together with:

(T) $\Box\varphi \rightarrow \varphi$

(5) $\Diamond\varphi \rightarrow \Box\Diamond\varphi$.

Definition (Deductive system $\mathtt{S5PQL_C}$: II)

$(\prec \mathrm{I})$ $u \prec uu \to \Box(E[uu] \to (u \prec uu \wedge E[u]))$

$(=\mathrm{I_P})$ $(E[uu] \wedge \forall u(u \prec uu \leftrightarrow u \prec tt)) \to uu = tt.$

Definition (Deductive system $\mathtt{S5PQL_C}$: III)

(PluComp) $\exists u\varphi \to \exists uu \forall u(u \prec uu \leftrightarrow \varphi)$

($\Box\mathrm{E_P}$) $(E[uu] \wedge \forall u(u \prec uu \to \Box E[u])) \to \Box E[uu]$.

Definition (Deductive system $\texttt{S5PQL}_{\texttt{C}}$: IV)

Inference rules: The same as those of $\texttt{KQL}_{\texttt{C}}$ (now formulated in PMT).

§5.2. Necessity

Arithmetical translations:

The τ-arithmetical translation $(\varphi)^{\mathrm{A}}_{\tau}$ of a formula φ of $\mathtt{PA}_2$ is defined exactly as its τ-Russellian translation, except that the quantifiers are restricted to the finite cardinalities.

After all, we are interested in the truths about the natural numbers – not about any entities whatsoever.

Theorem 2 (BJ, Forthcoming)

$\vdash_{\mathtt{S5PQL_C}} (\varphi)^{\mathrm{A}}_{\tau} \to \Box(\varphi)^{\mathrm{A}}_{\tau}$,

for every closed formula φ *of* PA_2.

Significance of Theorem 2:

The theorem establishes NECESSITY OF ARITHMETIC, provided:

(a) that the theorems of $\mathtt{S5PQL}_{\mathtt{C}}$ are all true, as it is reasonable to think.

(b) that HIGHER-ORDER ARITHMETICAL ONTOLOGY is true.

Moreover, NECESSITY OF ARITHMETIC and LOGICALITY establish that the the truths of arithmetic are all logically true.

§6. Finitary Plenitude

§6.1. Necessary, if true

Lemma 3 (BJ, Forthcoming)

$\vdash_{\mathtt{S5PQL_C}} \mathrm{FP}_\tau \rightarrow \Box \mathrm{FP}_\tau.$

Significance of Lemma 3:

In order to establish the necessary truth of FP it suffices to establish its plain truth.

Given LOGICALITY, this means that FP is, if true, logically true.

§6.2. An easy argument

Easy argument for FP
(adapted from (Williamson 2013)):

Premise: There could have been n donkeys, for every finite cardinality n.

(The contradictory hypothesis – that the metaphysical laws imply the existence of an upper bound on the finite cardinality that possibly numbers *being a donkey* – is presumably absurd.)

$\therefore$ Every finite cardinality n could have been instantiated.

§6.3. Recombination and generation

Recombination (Lewis 1986): For any objects in any worlds, there is a world that contains any number of duplicates of all those objects, "size and shape permitting".

Recombination has been argued to lead to paradox.

Regardless, GENERATION is at least in the spirit of RECOMBINATION.

Generation: There is an attribute of individuals numbered by some finite cardinality which, necessarily, no matter what finite cardinality numbers it, its successor could have numbered it.

GENERATION is a consequence of the claim that, no matter how finitely-many duplicates of a chosen individual there are, there could have been one more.

Also, SERIOUS ACTUALISM seems a truism:

(For robust defenses, see (Stephanou 2007) and (Jacinto 2019))

For each type τ:

Serious actualism$_\tau$: "necessarily, standing in a relation implies being something."

But then:

Lemma 4 (BJ, Forthcoming)

$\vdash_{\texttt{S5PQL}_{\texttt{C}}} \left(\textsc{generation} \wedge \textsc{serious actualism}_{\tau}\right) \rightarrow \mathrm{FP}_{\tau}.$

That is, FINITARY PLENITUDE is true provided that GENERATION is, given SERIOUS ACTUALISM.

Hence, FINITARY PLENITUDE gets support from both:

(a) common sense considerations; and

(b) influential theoretical views on what is possible.

§7. Conclusion

§7.1. Contributions

(1) A characterization of UPPER, a neoRussellian form of logicism;

(2) Two technical results (Theorems 1 and 2) establishing an intimate relationship between arithmetic and modal type-theory;

(3) A partial defense of UPPER supported, among other things, by Theorems 1 and 2.

§7.2. Future work

– Philosophically perspicuous higher-order characterizations of other mathematical entities satisfying *Frege's constraint*;

(e.g.: sets and real numbers)

– Technical results supporting UPPER LOGICISM about other areas of mathematics;

(e.g., set theory and real analysis)

– Defenses of the remaining component theses of UPPER LOGICISM;

(e.g.: HIGHER-TYPE ARITHMETICAL ONTOLOGY; the logicality of metaphysical necessity)

– Historical investigation of the reasons underlying Frege’s and Russell’s skepticism about modality;

– Etc.

§8. References

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