Classicism, Free Classicism, and Necessitsm

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Based on joint work with Andrew Bacon, Peter Fritz, and Ethan Russo

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These slides available at `https://tinyurl.com/PLSDorr`

Background: classical higher-order logic

Type system

We'll use *simple relational types* to track the syntactic categories of our language.

Simple relational types

T (the set of **terminal types**) is the smallest set such that $t \in T$ and $(\sigma \to \tau) \in T$ whenever $\sigma \in (T \cup \{e\})$ and $\tau \in T$.

R (the set of **relational types**) is $T \cup \{e\}$.

- t (aka. o, Prop, $\mathbb{P}$) is 'the type of propositions'; terms of this type are called *sentences*.
- e (aka. ι) is "the type of individuals"; terms of this type are called *singular* terms. We won't have terms with types like $e \to e$ (since their treatment in the logic raises some distracting choice points).

Language

We use variables with built-in R-types (indicated with a superscript when necessary).

The language $\mathcal{L}$

$\mathcal{L}$ is the smallest R-typed collection : such that:

$$v^\sigma : \sigma$$

$$(AB) : \tau \text{ whenever } A : \sigma \to \tau \text{ and } B : \sigma$$

$$(\lambda v^\sigma . A) : \sigma \to \tau \text{ whenever } A : \tau \text{ and } \tau \neq e$$

$$\to : t \to t \to t$$

$$\forall_\sigma : (\sigma \to t) \to t$$

$$= : \sigma \to \sigma \to t$$

- We can also add a signature Σ of nonlogical constants to define a language $\mathcal{L}_\Sigma$.
- An $\mathcal{L}_\Sigma$-**sentence** is a term P with $P :_{\mathcal{L}_\Sigma} t$. An $\mathcal{L}_\Sigma$-**theory** is a set of $\mathcal{L}_\Sigma$-sentences.

Some shorthands

- $\forall v^\sigma.P$ for $\forall_\sigma(\lambda v^\sigma.P)$.
- $\bot$ for $\forall p^t.p$, $\neg$ for $\lambda p.p \to \bot$, and $\top$ for $\neg\bot$.
- $\wedge$ for $\lambda pq.\forall_t r.(p \to q \to r) \to r$
- $\vee$ for $\lambda pq.\forall_t r.(p \to r) \to (q \to r) \to r$
- $\exists_\sigma$ for $\lambda X^{\sigma\to t}.\forall p^t.(\forall y^\sigma.Xy \to p) \to p$ and $\exists v^\sigma.P$ for $\exists_\sigma(\lambda v^\sigma.P)$.
- $\Box$ for $\lambda p.p =_t \top$.
- $\leq_{\sigma_1\to\ldots\to\sigma_n\to t}$ for $\lambda XY.X =_{\sigma_1\to\ldots\to\sigma_n\to t} (\lambda z_1^{\sigma_1}\ldots z_n^{\sigma_n}.Xz_1\ldots z_n \wedge Yz_1\ldots z_n)$

Classical higher-order logic

The logic H

H is the smallest $\mathcal{L}$-theory $\vdash$ containing all instances of the following schemas:

Luk $\vdash (P \to Q \to R) \to (R \to P) \to S \to P$ (where $P, Q, R, S : t$)

$\beta\eta$ $\vdash P \to Q$ whenever P and Q are $\beta\eta$-equivalent.

UI $\vdash \forall_\sigma F \to FA$ (where $F : \sigma \to t$ and $A : \sigma$)

UD $\vdash (\forall v^\sigma . p^t \vee Q) \leftrightarrow (p^t \vee \forall v^\sigma . Q)$

REF $\vdash A =_\sigma A$

LL $\vdash A =_\sigma B \to (FA \to FB)$

and closed under the following two rules:

MP If $\vdash P \to Q$ and $\vdash P$ then $\vdash Q$.

Gen If $\vdash P$ then $\vdash \forall v.P$.

Classicism

Classicism

H is pretty weak. For example it fails to settle a host of identity questions such as whether $\forall p.p = \neg\neg p$.

A natural idea for strengthening it to answer questions like that is to close under the rule of *substitution of logical equivalents*:

Substitution If $\vdash P \leftrightarrow Q$, then $\vdash R[P/x] \rightarrow R[Q/x]$

We call the resulting theory **C** ('Classicism').

- Here, $R[P/x]$ is the result of replacing *all* occurrences of x in R with P, including bound occurrences. So given $\vdash p \leftrightarrow \neg\neg p$, we even get $(\lambda p.p) = (\lambda p.p) \rightarrow (\lambda p.\neg\neg p) = (\lambda p.p)$.

An alternative axiomatization

Commutativity	$(\lambda pq.p \wedge q) =_{t\to t\to t} (\lambda pq.q \wedge p)$
Absorption	$(\lambda pq.(p \wedge q) \vee p) =_{t\to t\to t} (\lambda pq.p)$
Distribution	$(\lambda pqr.p \vee (q \wedge r)) =_{t\to t\to t\to t} (\lambda pqr.(p \vee q) \wedge (p \vee r))$
Complementation	$(\lambda pq.p \wedge \neg p) =_{t\to t\to t} (\lambda pq.q \wedge \neg q)$
Involution	$(\lambda p.\neg\neg p) =_{t\to t} (\lambda p.p)$
∀-Absorption	$(\lambda Xy.Xy \vee \forall_\sigma X) =_{(\sigma\to t)\to\sigma\to t} (\lambda Xy.Xy)$
∀-Distribution	$(\lambda Xp.p \vee \forall_\sigma X) =_{(\sigma\to t)\to t\to t} (\lambda Xp.\forall y^\sigma.p \vee Xy)$
Self-identity	$(\lambda x.x =_\sigma x) =_{\sigma\to t} (\lambda x.\top)$
Leibniz	$(\lambda Zxy.x =_\sigma y \wedge Zx) =_{(\sigma\to t)\to\sigma\to\sigma\to t} (\lambda Zxy.x =_\sigma y \wedge Zy)$

Two non-theorems and two theorems

The best-known systems of classical higher-order logic (often used for formalizing mathematics) also include the following two principles, neither of which is in **C**:

Fregean Axiom $(p \leftrightarrow q) \rightarrow (p =_t q)$ ('Propositional Extensionality').

Functionality $(\forall_\sigma x.Fx =_\tau Gx) \rightarrow F =_{\sigma\rightarrow\tau} G$ ('Functional Extensionality').

Adding these to **H** gives a system 'Extensionalism' **E**, even stronger than **C**.

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C does however contain the following (recall that $\Box := (\lambda p.p = \top)$):

Modalized Fregean Axiom $\Box(p \leftrightarrow q) \to (p =_t q)$

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- NB: **C**—indeed, even the quantifier-free fragment of **C**—implies that $\Box$ has an S4 modal logic.

Broad Necessitism

Here's a very controversial theorem of **C**:

Broad Necessitism $\forall x^\sigma.\Box\exists y^\sigma.x =_\sigma y$

This implies

$$\forall X^{t\to t}.(X\top \to \forall y^\sigma.X(\exists z^\sigma.z =_\sigma y))$$

Since *metaphysical necessity* uncontroversially satisfies $X\top$, Classicism implies Necessitism (Williamson, 2013).

m.$(\forall x. x \neq y) =_t (\forall x. x \neq y) \wedge (y \neq y)$

n-1.(y$\neq y) =_t \bot$Self-identity

n-1.$((\forall x. x \neq y) \wedge y \neq y) =_t \bot$

n. $(\forall x. x \neq y) =_t \bot$

Some regard Necessitism as obviously false. (E.g. on the grounds that Saul Kripke wouldn't have been identical to anything if Meyer and Dorothy Kripke had never met.)

Others (Fine, 2017) regard it as obviously *true*.

Like Williamson, we think it's a hard theoretical question, in the intersection of logic and metaphysics.

Free Classicism and Free Quantifiers

Free higher-order logic

Let **FH** (free higher order logic) be the result of weakening **H** by replacing UI with the following schema:

KQ $$(\forall x^{\sigma}.Fx \rightarrow Gx) \rightarrow (\forall_{\sigma} F \rightarrow \forall_{\sigma} G)$$

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And let Free Classicism (**FC**) be the result of closing Free **H** under Substitution.

This is a very natural (though not the only) weakening of **C** for contingentists (those who reject Necessitism) to explore.

An alternative axiomatization of FC

Commutativity	$(\lambda pq.p \wedge q) =_{t\to t\to t} (\lambda pq.q \wedge p)$
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Involution	$(\lambda p.\neg\neg p) =_{t\to t} (\lambda p.p)$
$\forall$-Commutativity	$(\lambda XY.\forall z^{\sigma}.Xz \wedge Yz) =_{(\sigma\to t)\to(\sigma\to t)\to t} (\lambda XY.\forall_{\sigma} X \wedge \forall_{\sigma} Y)$
$\forall$-Distribution	$(\lambda Xp.p \vee \forall_{\sigma} X) =_{(\sigma\to t)\to t\to t} (\lambda Xp.\forall y^{\sigma}.p \vee Xy)$
Self-identity	$(\lambda x.x =_{\sigma} x) =_{\sigma\to t} (\lambda x.\top)$
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Existence Closure

Free Classicism is very weak—it doesn't even imply $\exists x^\sigma.\top$ for any type σ! We can remedy this somewhat by adding

Existence Closure $\Box\forall x_1 \ldots x_n.\exists_\sigma y.y =_\sigma A(x_1, \ldots, x_n)$, where A has no free variables besides $x_1, \ldots, x_n$ (and no non-logical contants).

Call the resulting theory **FC**$+$.

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Call the resulting theory **FC+**. FC+ includes all *closed* theorems of H and is closed under necessitation.

If you want, you can also add all *open* theorems of H; but if you closed *that* under necessitation you'd be back to **C**.

General theory of (universal) quantifiers

Definitions

For $Q, Q' : (\sigma \to t) \to t$ and $F : \sigma \to t$:

- **Comm** Q ('Q commutes with $\wedge$') $:= (\lambda Xp.p \vee QX) = (\lambda Xp.Q(\lambda y.p \vee Xy))$.
- **Dist** Q ('Q is distributive') $:= (\lambda Xp.p \vee QX) = (\lambda Xp.Q(\lambda y.p \vee Xy))$.
- **Abs** Q ('Q is absorptive') $:= (\lambda Xy.Xy \vee QX) = (\lambda Xy.Xy)$.

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- **Quant** Q ('Q is a quantifier') $:=$ **Comm** $Q \wedge$ **Dist** Q.
- **UQuant** Q ('Q is an absolutely unrestricted quantifier') $:=$ **Abs** $Q \wedge$ **Dist** Q.

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- **Abs** Q (‘Q is absorptive’) $:= (\lambda Xy.Xy \vee QX) = (\lambda Xy.Xy)$.
- **Quant** Q (‘Q is a quantifier’) $:=$ **Comm** $Q \wedge$ **Dist** Q.
- **UQuant** Q (‘Q is an absolutely unrestricted quantifier’) $:=$ **Abs** $Q \wedge$ **Dist** Q.
- $Q \restriction F$ (‘the restriction of Q to F’) $:= \lambda X.Q(\lambda y.Fy \to Xy)$.
- **E** Q (‘Q-existence’) $:= \lambda y.\neg Q(\lambda x.x \neq_\sigma y)$.

Theorems

These definitions are justified by theorem-schemas of FC:

- $\mathbf{UQuant}\, Q \to \mathbf{Quant}\, Q$.
- $\mathbf{UQuant}\, Q \to \mathbf{Quant}(Q \restriction F)$.
- $(\mathbf{Quant}\, Q \wedge \mathbf{UQuant}\, Q') \to Q = Q' \restriction \mathbf{E}\, Q$.
- $(\mathbf{UQuant}\, Q \wedge \mathbf{UQuant}\, Q') \to Q = Q'$.

Theorems

These definitions are justified by theorem-schemas of FC:

- $\textbf{UQuant}\, Q \to \textbf{Quant}\, Q$.
- $\textbf{UQuant}\, Q \to \textbf{Quant}(Q \upharpoonright F)$.
- $(\textbf{Quant}\, Q \wedge \textbf{UQuant}\, Q') \to Q = Q' \upharpoonright \mathbf{E}\, Q$.
- $(\textbf{UQuant}\, Q \wedge \textbf{UQuant}\, Q') \to Q = Q'$.

Classicism thus characterizes $\forall_\sigma$ in a way that *uniquely singles it out*.

By contrast, Free Classicism by contrast says nothing about $\forall_\sigma$ that distinguishes it from arbitrary restricted quantifiers.

Key choices for Free Classicists

A central question we can ask in Free Classicism: *are there any* absolute quantifiers (in type $(\sigma \to t) \to t$)? I.e., whether to accept the following schema:

Absolute Quantifier Existence $\exists X^{(\sigma \to t) \to t}. \, \mathbf{UQuant}\, X$

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Absolute Quantifier Existence $$\exists X^{(\sigma \to t) \to t}.\, \mathbf{UQuant}\, X$$

In **FC**, Absolute Quantifier Existence entails a witnessing sentence:

$$\mathbf{UQuant}\, \forall^{+}_{\sigma}$$

where $$\forall^{+}_{\sigma} := \lambda X. \forall Q^{(\sigma \to t) \to t}.\, \mathbf{Quant}(Q) \to QX$$

Absolute Quantifier Existence in the literature

Many contingentists (e.g. Prior and Fine, 1977) have wanted to define a so-called "outer" or "possibilist" quantifier Π, so as to charitably reinterpret certain ordinary claims which would conflict with contingentism if taken at face value.

One of their desiderata is that Necessitism should become true when $\forall$ is replaced by Π. This would make it natural for them to accept that Π is an absolute quantifier.

If they also accept Existence Closure, they will thereby be committed to Absolute Quantifier Existence.

The metasemantic argument

Given how the symbols $\forall_\sigma$ were introduced—in particular, the stipulation that they are to be understood as *unrestricted* quantifiers—why doesn't '$\forall_\sigma$' end up expressing Π_σ?

In interpreting this stipulation, it is natural to take 'quantifier' to mean what we earlier called 'free quantifier', and to take 'Q_1 is a restriction of Q_2' to mean that Q_2 entails Q_1. It follows from this that if there is an absolute quantifier, all other quantifiers are restrictions of it. So in what sense could any of the other quantifiers count as 'unrestricted'?

Denying Absolute Quantifier Existence

Can Free Classicists avoid the metasemantic challenge by denying the existence of an absolute quantifier?

It's not so straightforward. For given contingentism, there are reasons to accept that a word might be meaning*ful*—in the sense of being capable of contributing in a discriminating fashion to the truth values of sentences—without there being anything (of the relevant type) that it means.

For example, consider Salmon's name 'Noman':

> *Let S be a particular male sperm cell of my father's and let E be a particular ovum of my mother's such that neither gamete ever unites with any other to develop into a human zygote. Let us name the (possible) individual who would have developed from the union of S and E, if S had fertilized E in the normal manner, 'Noman'. (Salmon, 1987, inspired by Kaplan 1973)*

Bacon (2013) argues that contingentists should think that 'Noman' refers to Noman, despite not referring to anything. This helps explain why, e.g., 'Noman is either wise or not wise' is true.

Similarly, even Free Classicists who deny Absolute Quantifier Existence might take themselves to understand a certain symbol Π_σ for which they accept $\mathbf{UQuant}(\Pi_\sigma)$. They will also face the metasemantic challenge.

Two ways in which Free Classicists might end up in this position:

1. They could accept '$\mathbf{UQuant}\,\Pi_\sigma$' for some Π_σ defined in terms of the logical constants (or other uncontroversially meaningful vocabulary?), while rejectin $\exists X^{(\sigma\to t)\to t}.X =_{(\sigma\to t)\to t} \Pi_\sigma$ (and thus also Existence Closure).
2. They could allow constants Π_σ to be introduced *without* explicit definition, just by reasoning with them in ways that assume $\mathbf{UQuant}(\Pi_\sigma)$. So long as this gives a conservative extension of their old theory, it is hard to see (Belnap, 1962) why they would have any in-principle objection to doing this.

Hardline contingentism

Of course, **UQuant** Π_σ won't be a conservative extension if the old theory included

Absolute Quantifier Impossibility **UQuant** $=_{((\sigma\to t)\to t)\to t} \lambda X.\bot$

This is arguably the most interesting/principled option for the contingentist.

- Result (E. Russo): Absolute Quantifier Impossibility is consistent with **FC**.
- Open question: is Absolute Quantifier Impossibility consistent with **FC**+?

A candidate “outer quantifier”

Super-universality

We can define a "super-universality" operator Π_σ: property X is super-universal iff it is entailed by *being such that p*, for some truth p:

$$\Pi_\sigma := \lambda X^{\sigma\to t}.\exists p(p \wedge (\lambda x.p) \leq_{\sigma\to t} X)$$

Classicism implies $\Pi_\sigma =_{(\sigma\to t)\to t} \forall_\sigma$.

FC implies that super-universality obeys Absorption-$\forall$, but does not imply that it obeys Π_σ obeys Distribution-$\forall$

The Truth Principle

However we *can* prove this—and hence that Π_σ is an absolute quantifier—if we add to Free Classicism the principle that *truth entails being entailed by some truth*:

Truth Principle $(\lambda p.p) \leq_{t \to t} (\lambda p.\exists q.q \wedge (q \leq_t p))$

Thus any Free Classicist who accepts the Truth Principle but not Classicism will face the metasemantic challenge.

It seems to us that the Truth Principle is quite compelling even from a contingentist starting point. (Evidence: Fine (1981) argues for a "World Actualism" principle that entails the Truth Principle.)

Illustration

For example, suppose that a certain electron e could have not been, and if it had not been, the propositions Pe (*e orbits a proton*) and $\neg Pe$ (*e doesn't orbit any proton*) would also have not been, although the former would have been false and the latter true.

Still, it is plausible that if there hadn't been e, there would have been a truth that *entailed* that e doesn't orbit any proton.

- Proponents of the *necessity of distinctness* can point to a natural candidate for such a truth: the disjunction (least upper bound) of all true predications of the form *x orbits some proton*.

And this isn't something special about *orbiting a proton*. Plausibly *every* property F is such that if there hadn't been e, then there would have been a truth which either entailed Fe or entailed $\neg Fe$.

Thanks.

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