Forcing Axioms, $(*)$, and the Continuum Problem

David Asperó

University of East Anglia

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Cantor's paradise

Set theory is the study of sets (of mathematical objects) under the membership relation $\in$. It provides a particularly simple language to model (all?) mathematical notions.

It also provides a nontrivial theory of the infinite: For example, in 1873, Georg Cantor famously proved that $|\mathbb{N}| < |\mathbb{R}|$.

A more general version of this is of course:

Theorem

(Cantor's Theorem) For every set X, $|X| < |\mathcal{P}(X)|$, where $\mathcal{P}(X) = \{Y : Y \subseteq X\}$.

Proof.

Clearly $|X| \leq |\mathcal{P}(X)|$ (take the function sending $y \in X$ to $\{y\} \in \mathcal{P}(X)$).

To see that there is no bijection $f : X \longrightarrow \mathcal{P}(X)$, suppose $f : X \longrightarrow \mathcal{P}(X)$ is a function. Let

$$A = \{y \in X : y \notin f(y)\}$$

If $A = f(y)$, then

$$y \in f(y) \iff y \notin f(y)$$

Which is absurd. Hence $A \notin \mathrm{range}(f)$ and f cannot be a bijection. □

The Continuum Problem: $2^{\aleph_0} = \aleph_1$? $2^{\aleph_0} = \aleph_2$? In general, which is the exact cardinality of $\mathbb{R}$? In other words: Exactly how many reals are there?

$\aleph_0 < 2^{\aleph_0}$ by Cantor's Theorem, and Cantor conjectured that $2^{\aleph_0}$ is the least possible value compatible with this inequality.

Cantor's Continuum Hypothesis (CH) is the statement $2^{\aleph_0} = \aleph_1$.

Deciding the truth value of the Continuum Hypothesis, and in general solving the Continuum Problem, was $\sharp 1$ on Hilbert's famous list of open problems for the ICM meeting in 1900.

A more fundamental question:

Question

What counts as a solution of the Continuum Problem?

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What counts as a solution of the Continuum Problem?

The first order theory known as Zermelo-Fränkel set theory with the axiom of Choice, a.k.a. ZFC, soon became the standard axiomatization of set theory.

It was hoped that ZFC would prove that $2^{\aleph_0} = \aleph_1$ or that it would prove that $2^{\aleph_0} > \aleph_1$. This would arguably constitute a solution of the Continuum Problem.

However:

- In 1938, Kurt Gödel proved that if ZFC is consistent, then ZFC $+ 2^{\aleph_0} = \aleph_1$ is also consistent.
- In 1963, Paul Cohen proved that if ZFC is consistent, then ZFC $+ 2^{\aleph_0} = \aleph_2$ is also consistent.

To prove the consistency of ZFC $+ 2^{\aleph_0} = \aleph_1$ (assuming the consistency of ZFC), Gödel showed that the constructible universe, L, satisfies ZFC $+ 2^{\aleph_0} = \aleph_1$.

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To prove the consistency of $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_2$, Cohen devised the method of *forcing*.

This is a very general method which, given a model M of (enough axioms of) ZFC, builds an outer universe $M[G] \supseteq M$ of M satisfying (enough axioms of) ZFC. $M[G]$ is called a *generic extension of M*.

If we choose the *generic object G* carefully, we may be able to arrange that $M[G]$ satisfies some interesting statement, like $2^{\aleph_0} = \aleph_2$.

Using the method of forcing one can show, given the consistency of ZFC, that each of the following are consistent:

- $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_2$
- $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_3$
- $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_{273453453667889}$
- $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_{\omega+1}$
- $\mathsf{ZFC} + 2^{\aleph_0} = \aleph_{\omega_1}$
- ...

Is this the end of the story? Do these consistency results show that the Continuum Problem is in fact a pseudo-problem?

A formalist, who takes our official theory ZFC as the only source of set-theoretic “truth”, would in fact answer Yes.

But this is not the only feasible position. In fact, the formalist position faces very serious problems. Here is one way to see this:

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Trouble with the formalist position

Taking a question like "How many real numbers are there?" to have no answer commits the formalist to denying that there is any such thing as "the set-theoretic universe $\bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$".

Presumably, for the formalist, all there is to say is things like

$$\text{“}T \text{ proves } \varphi\text{”}$$

or

$$\text{“If } T \text{ is consistent, then it does not prove } \varphi\text{”},$$

for $T = \mathrm{ZFC}$ or T being some given extension of ZFC, and for some sentence φ.

These proof-theoretic statements are ultimately just arithmetical statements. So the formalist seems to be at least committed to the existence of $V_\omega = \bigcup_{n \in \omega} V_n$.

But arithmetic is in fact dependent on the higher V_α's. For example, whether or not the Paris-Harrington theorem holds depends on whether or not there are infinite sets. And analytic number theory has a lot to say about arithmetic.

So let's throw in $V_{\omega+1}$ in our ontology after all.

But the properties of $V_{\omega+1}$ depend crucially on what happens at V_α for higher α's. For example, Borel Determinacy holding depends on the existence of V_α for all $\alpha < \omega_1$.

So the universe should in fact contain V_α for all $\alpha < \omega_1$. But the statement "There is no $X \subseteq \mathbb{R}$ with $\aleph_0 < |X| < |\mathbb{R}|$" (which is equivalent to CH in ZFC) lives already in $V_{\omega+2}$, so it should then have an answer!

Where should we stop and why?

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The realist position: natural axioms

If the set-theoretic universe V is real, then the Continuum Problem has a definite solution. The fact that ZFC does not solve the Continuum Problem means that we need to supplement ZFC with *natural* axioms solving this problem.

The search for natural axioms supplementing ZFC is also known as *Gödel's programme*. Gödel indeed made the above point. Although he had proved the consistency of CH, he suspected CH to be false *in the real world* and was hoping that natural axioms would eventually settle the issue.

Adopting $2^{\aleph_0} = \aleph_1$, or $2^{\aleph_0} = \aleph_{27}$, as a new axiom seems surely dogmatic, therefore unnatural. How do we decide if an axiom is natural?

We need general criteria to assess the naturalness of axioms. They should address the question: What virtues do we want our theory of sets to have?

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Strong axioms of infinity

Large cardinal (LC) axioms are an (open-ended) hierarchy of axioms asserting the existence of very high cardinals with strong properties. They realize the idea that "the universe is large".

They form a hierarchy of stronger and stronger theories: Given an LC axiom A, $\mathsf{ZFC} + A$ proves the consistency of ZFC, and in fact of $\mathsf{ZFC} + A'$ for any weaker large cardinal axiom A'.

The stronger A is, the more daring $\mathsf{ZFC} + A$ is (i.e., more likely to be inconsistent). We can prove the consistency of LC axioms only by working in a strictly stronger LC theory.

Arguably, the weakest large cardinal axiom is "ω exists". The next natural large cardinal axiom is "There is an inaccessible cardinal".

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Some classical large cardinal axioms:

ω < inaccessible < weakly compact < measurable < strong < Woodin < superstrong < < supercompact < huge < 2-huge < ...

An empirical fact: Every mathematical theory T ever considered can be interpreted relative to $\mathsf{ZFC} + A$ for some LC axiom A. In fact, in many cases we can show T to be *equiconsistent* with $\mathsf{ZFC} + A$ for some LC axiom A.

A natural axiom should therefore be compatible with all consistent large cardinal axioms.

Invariance w.r.t. forcing

Forcing is our prime method for proving the independence of some given statement φ from some base set theory; in other words, to prove that if T is consistent, then φ and its negation $\neg\varphi$ are both consistent with T.

Therefore, if we want our set theory T to be strong, we better have that T neutralizes the effects of forcing as much as possible, in the sense of proving, for as many sentences φ as possible, that the truth value of φ cannot be changed by forcing.

The axiom $V = L$ has this effect for silly reasons: Nontrivial generic extension do not satisfy $V = L$.

However, $V = L$ is **not** compatible with LC axioms. In fact, if $V = L$, then there are no measurable cardinals.

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Large cardinals in the region of Woodin cardinals also have exactly this effect (and are of course compatible with all LC axioms):

Theorem

(Woodin, mid 1980's) Suppose there are arbitrarily large Woodin cardinals. Then the following are equivalent for every sentence φ in the language of set theory.

(1) *φ is true in $L(\mathbb{R})$.*

(2) *It can be forced that φ is true in $L(\mathbb{R})$.*

$L(\mathbb{R})$ is the $\subseteq$-minimal subuniverse of ZF (= ZFC without the Axiom of Choice) containing all the reals and all the ordinals. $L(\mathbb{R})$ is where all of classical analysis takes place.

Hence, if there are arbitrarily large Woodin cardinals, classical analysis is immune to the forcing method.

Maximality w.r.t. forcing: Forcing axioms

$\mathbb{P} = (P, \leq_\mathbb{P})$ is a *partial order* if P is a set and $\leq_\mathbb{P}$ is a relation on P which is transitive, anti-symmetric, and reflexive on P.

Given a partial order $\mathbb{P} = (P, \leq_\mathbb{P})$ (a.k.a. *forcing notion*), $D \subseteq P$ is a *dense* subset of $\mathbb{P}$ if for every $p \in P$ there is some $q \in D$ such that $q \leq_\mathbb{P} p$.

$G \subseteq P$ is a *filter* if

- for every $q \in G$ and $p \in P$, if $q \leq_\mathbb{P} p$, then $p \in G$;
- for all $q_1, q_2 \in G$ there is some $q \in G$ such that $q \leq_\mathbb{P} q_1$ and $q \leq_\mathbb{P} q_2$.

If M is some model such that $\mathbb{P} \in M$, we say that G is $\mathbb{P}$*-generic over* M if $G \cap D \neq \emptyset$ for every dense subset D of $\mathbb{P}$ such that $D \in M$.

Theorem

(Cohen) Suppose M is a transitive model of (enough of) ZFC. Let $\mathbb{P} = (P, \leq_{\mathbb{P}}) \in M$ be a forcing notion and let $G \subseteq P$ be $\mathbb{P}$-generic over M. Then:

(1) *If $\mathbb{P}$ is non-atomic, meaning that for every $p \in \mathbb{P}$ there are $q_0 \leq_{\mathbb{P}} p$ and $q_1 \leq_{\mathbb{P}} p$ such that q_0 and q_1 are $\mathbb{P}$-incompatible (i.e., there is no $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} q_0$ and $q \leq_{\mathbb{P}} q_1$), then $G \notin M$.*

(2) *There is a $\subseteq$-minimal model $M[G]$ of (enough of) ZFC such that*

   (a) *$M \subseteq M[G]$ and*

   (b) *$G \in M[G]$.*

An example:

Let $\mathbb{P}$ be the set of finite tuples $(x_0, x_1, \ldots, x_{n-1})$ of 0's and 1's. Given tuples $\vec{\sigma}_1$, $\vec{\sigma}_2$, let us set $\vec{\sigma}_2 \leq_{\mathcal{P}} \vec{\sigma}_1$ iff $\vec{\sigma}_1$ in an initial segment of $\vec{\sigma}_2$.

Let G be a filter of $\mathbb{P}$. If G meets $D_n = \{\vec{\sigma} \in \mathbb{P} : \text{length}(\vec{\sigma}) > n\}$ for every $n \in \mathbb{N}$, then $c = \bigcup G$ is an infinite sequence $(x_n : n \geq 0)$ of natural numbers.

If G meets $\{\vec{\sigma} \in \mathbb{P} : \text{there are } n, n+2 < \text{length}(\vec{\sigma}), \vec{\sigma}(n) \neq \vec{\sigma}(n+2)\}$, then it is not the case that all even entries of c are equal.

If G meets $\{\vec{\sigma} \in \mathbb{P} : \text{there are } n+1, n+3 < \text{length}(\vec{\sigma}), \vec{\sigma}(n+1) \neq \vec{\sigma}(n+1)\}$, then it is not the case that all odd entries of c are equal.

If G meets $\{\vec{\sigma} \in \mathbb{P} :$ $\text{there is } n \text{ with } n + 10000 < \text{length}(\vec{\sigma}) \text{ such that } \vec{\sigma}(n) = \vec{\sigma}(n+1) = \ldots = \vec{\sigma}(n+10000) = 0\}$, then c has somewhere 10001 consecutive entries all taking value 0.

...

If G is sufficiently generic, then c is a very chaotic sequence. In fact, if G is generic over some model M, then c avoids every regularity pattern expressible within M.

$\mathbb{P}$ is the simplest possible non-atomic forcing. It is called *Cohen forcing*. And if G is $\mathbb{P}$-generic over a model M, c is a *Cohen real over* M.

By (1) in Cohen's theorem, there are no $\mathbb{P}$-generic filters over V (whenever $\mathbb{P}$ is non-atomic).

On the other hand:

Fact

(ZFC) If $|M| = \aleph_0$, then for every forcing notion $\mathbb{P} \in M$ there is a $\mathbb{P}$-generic filter G over M.

Proof.

Let $(D_n : n \in \mathbb{N})$ enumerate all dense subsets of $\mathbb{P}$ in M. Let $q_0 \in D_0$. Since D_1 is dense, we may find $q_1 \in D_1$ such that $q_1 \leq_{\mathbb{P}} q_0$. In general, since D_{n+1} is dense, we may find $q_{n+1} \in D_{n+1}$ such that $q_{n+1} \leq_{\mathbb{P}} q_n$. Then

$$G = \{p \in P : q_n \leq_{\mathbb{P}} p \text{ for some } n\}$$

is a $\mathbb{P}$-generic filter over M. □

Is it possible to find strengthenings of this Fact applying to larger collections of dense sets?

It turns out that it is consistent with ZFC to have the answer to be Yes in some cases; for example if we restrict the class of forcings in some suitable way and if, for example, “larger” is interpreted as “ of size $\aleph_1$”.

These assertions are known as *forcing axioms*. Their consistency was traditionally shown by forcing “many times” with forcing notions in the relevant class.

For forcing axioms for meeting $\aleph_1$-many dense sets: Given a class Γ of forcing notions, $\mathrm{FA}(\Gamma)$ is the following assertion:

Suppose $\mathbb{P}$ is a forcing notion in Γ. If $\mathcal{D}$ is a collection of $\aleph_1$-many dense subsets of $\mathbb{P}$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

The first forcing axiom (1970) was Martin's Axiom at ω_1, MA_{ω_1} (Solovay-Tennenbaum, Martin). MA_{ω_1} is

$$\mathsf{FA}(\{\mathbb{P} : \mathbb{P} \text{ has the countable chain condition}\})$$

The strongest possible forcing axiom of this sort (i.e., applying to the widest possible class of forcing notions), known as Martin's Maximum (MM), was isolated and proved consistent by Foreman-Magidor-Shelah in 1984 assuming the consistency of ZFC+ "There is a supercompact cardinal". MM is

$$\mathsf{FA}(\{\mathbb{P} : \text{ forcing with } \mathbb{P} \text{ preserves all stationary subsets of } \omega_1\})$$

The existence of “partially generic” filters given by forcing axioms ensures that many of the facts that would hold in the corresponding generic extensions actually hold in V.

Thus, forcing axioms realize the following “maximality idea”:

Any statement (of the right syntactical form) that could possibly hold (by forcing with a forcing notion in the relevant class) is actually true.

Also, the wider the class Γ is, the more “democratic” $FA(\Gamma)$ is (in the sense of not discriminating between possible generic extensions).

Given a set X, $\mathrm{TC}(X) = X \cup \bigcup X \cup \bigcup\bigcup X \cup \ldots = X \cup \{a : a \in b \in X \text{ for some } b\} \cup \{a : a \in b \in c \in X \text{ for some } b, c\} \cup \ldots$.
Let

$$H(\omega_2) = \{X : |\mathrm{TC}(X)| < \aleph_2\}$$

Thus, $H(\omega_2)$ is the collection of all sets which are "small relative to $\aleph_2$".

$H(\omega_2) \models \mathrm{ZFC} \setminus \{\text{Power set Axiom}\}$

Many natural statement, like CH and $\neg$ CH, live in $H(\omega_2)$.

Forcing axioms have many consequences at the level of $H(\omega_2)$. In particular, MM implies all of the following.

(1) If $(L, \leq)$ is a Dedekind complete dense linear order such that $|\mathcal{I}| \leq \aleph_0$ whenever $\mathcal{I}$ is a collection of pairwise disjoint intervals of L, then $(L, \leq)$ is order-isomorphic to the real line with the usual order.

(2) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

(3) (Moore) There is a set of reals X and a Countryman line C such that whenever $(L, \leq)$ is a linear order such that $\aleph_0 < |L|$, L contains some suborder order-isomorphic to one of the following:
   - X
   - ω_1
   - ω_1^* (the reverse of ω_1)
   - C
   - C^* (the reverse of C)

The consistency proof of MM shows in fact that the following enhanced form of this axiom is consistent:

MM^{++}: For every forcing notion $\mathcal{P}$ preserving stationary subsets of ω_1, every collection $\mathcal{D}$ of $\aleph_1$-many dense subsets of $\mathcal{P}$ and every collection $\{\dot{S}_i : i < \omega_1\}$ of $\mathcal{P}$-names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{P}$ such that

- $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$ and
- for each i, $(\dot{S}_i)_G = \{\nu < \omega_1 : (\exists p \in G) p \Vdash_{\mathcal{P}} \nu \in \dot{S}_i\}$ is stationary.

MM^{++} in fact seemed to decide **all** questions about $H(\omega_2)$ modulo forcing.

The Axiom of Determinacy

Let A be a set of sequences $(x_i : i \geq 0)$ of natural numbers.

Consider the following game G_A between two players, I and II, who alternate picking natural numbers x_i:

I	x_0
II	

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Consider the following game G_A between two players, *I* and *II*, who alternate picking natural numbers x_i:

I	x_0		x_2		$\ldots$		
II		x_1		x_3		$\ldots$	

- player *I* wins this run of the game if $(x_i : i \geq 0) \in A$.
- player *II* wins if $(x_i : i \geq 0) \notin A$.

We say that A *is determined* if either player *I* or player *II* has a winning strategy in G_A (i.e., ensuring a win for that player no matter how the other player makes their moves).

The *Axiom of Determinacy* (AD) is the statement: "Every set A of sequences of natural numbers is determined."

AD contradicts the Axiom of Choice.

On the other hand, ZF+AD gives a remarkably rich theory. In particular, it provides a very fine analysis of $L(\mathbb{R})$. Also, ZF+AD arguably gives the correct structure theory for the definable sets of reals.

Theorem

(Martin-Steel, Woodin, mid 1980's) Suppose there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then AD *holds in* $L(\mathbb{R})$.

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Theorem

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A simple application of AD (due to Martin):

A sequence $(b_n)_{n\geq 0}$ *computes* another sequence $(a_n)_{n\geq 0}$ (written $(a_n)_{n\geq 0} \leq_{\text{Tu}} (b_n)_{n\geq 0}$) iff there is a computer program which outputs $(a_n)_{n\geq 0}$ upon input of $(b_n)_{n\geq 0}$.

$(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are *Turing equivalent* (written $(a_n)_{n\geq 0} \equiv_{\text{Tu}} (b_n)_{n\geq 0}$) if

- $(a_n)_{n\geq 0} \leq_{\text{Tu}} (b_n)_{n\geq 0}$ and
- $(b_n)_{n\geq 0} \leq_{\text{Tu}} (a_n)_{n\geq 0}$.

C is a *Turing cone* if there is some $(a_n)_{n\geq 0}$ such that

$$C = \{(b_n)_n \, : \, (a_n)_n \leq_{\text{Tu}} (b_n)_n\}$$

$(a_n)_{n\geq 0}$ is the *base of* C.

Now, let A be a *Turing invariant* set of sequences (i.e., if $(a_n)_{n\geq 0} \in A$ and $(b_n)_{n\geq 0} \equiv_{\mathrm{Tu}} (a_n)_{n\geq 0}$, then also $(b_n)_{n\geq 0} \in A$).

- If player I has a winning strategy σ in G_A and $\vec{a} = (a_n)_{n\geq 0}$ codes σ, then $C \subseteq A$, where C is the Turing cone with base $\vec{a}$. ($\vec{a} \leq_{\mathrm{Tu}} \vec{b} \Longrightarrow \vec{b} \equiv_{\mathrm{Tu}} \sigma * \vec{b} \in A \Longrightarrow \vec{b} \in A$.)
- If player II has a winning strategy τ in G_A and $\vec{a} = (a_n)_{n\geq 0}$ codes τ, then $C \cap A = \emptyset$, where C is the Turing cone with base $\vec{a}$. (Argue as above.)

$\sigma * \vec{b}$ is the play of G_A resulting from player I moving according to the strategy σ and player II playing the members of $\vec{b}$ in increasing order. (And we would define $\vec{b} * \tau$ similarly if τ is a strategy for player II.)

Hence, if AD holds, then every Turing invariant A is either large (i.e., it contains a Turing cone) or small (i.e., it is disjoint from a Turing cone).

Note: If C_1 is the Turing cone with base $(a_n)_{n\geq 0}$ and C_2 is the Turing cone with base $(b_n)_{n\geq 0}$, then $C \subseteq C_1 \cap C_2$, where C is the Turing cone with base $(a_0, b_0, a_1, b_1, a_2, b_2, \ldots)$.

This means that for any Turing invariant properties $P_1, \ldots, P_n$ of sequences there are choices $Q_i \in \{P_i, \neg P_i\}$, for $i \leq n$, such that the set of sequences satisfying Q_i for all i is large (in particular nonempty).

In fact this is true for any countable collection P_i (for $i \in \mathbb{N}$) of properties.

$(*)$

In the 1990's, Woodin defined and studied the following axiom $(*)$.

Definition

$(*)$ is the conjunction of (1) and (2).

(1) AD holds in $L(\mathbb{R})$. [This follows from large cardinals by earlier theorem.]

(2) There is a $\mathbb{P}_{\max}$-generic filter G over $L(\mathbb{R})$ such that $L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G]$.

Here, $\mathbb{P}_{\max}$ is a certain *homogeneous* forcing notion in $L(\mathbb{R})$ (i.e., with the property that $L(\mathbb{R})[G_1]$ and $L(\mathbb{R})[G_2]$ satisfy the same sentences whenever G_1 and G_2 are $\mathbb{P}_{\max}$-generic filters over $L(\mathbb{R})$).

Assuming the theory of $L(\mathbb{R})$ is frozen under forcing (which follows from large cardinals), it follows that the $L(\mathcal{P}(\omega_1))$'s of any two models of $(*)$ obtained by forcing satisfy exactly the same sentences.

Also, $(*)$ implies the following very strong maximality principle (relative to all generic extensions):

Theorem

(Π_2 maximality) (Woodin) Suppose $(*)$ holds and there are arbitrarily large Woodin cardinals. For every Π_2 sentence φ ($= (\forall x)(\exists y)\psi(x, y)$, where all quantifiers in $\psi(x, y)$ are restricted), the following are then equivalent.

(1) $H(\omega_2) \models \varphi$

(2) *In some forcing extension it holds that $H(\omega_2) \models \varphi$.*

It turns out that Π_2 maximality in the above sense, when conditioned to CH, is false:

Theorem

(Asperó–Larson–Moore) There are Π_2 sentences σ_1 and σ_2 such that:

(1) *There is a proper poset forcing $H(\omega_2) \models \sigma_1 \wedge CH$.*

(2) *If there is an inaccessible limit of measurable cardinals, then there is a proper poset forcing $H(\omega_2) \models \sigma_2 \wedge CH$.*

(3) *$H(\omega_2) \models \sigma_1 \wedge \sigma_2$ implies $2^{\aleph_0} = 2^{\aleph_1}$.*

In addition, $(*)$ implies that $L(\mathcal{P}(\omega_1))$ is obtained from $L(\mathbb{R})$ by adding to it any subset of ω_1 not in $L(\mathbb{R})$. Given that AD makes $L(\mathbb{R})$ into a 'canonical' subuniverse, this means that $(*)$ makes $L(\mathcal{P}(\omega_1))$ into a larger 'canonical' subuniverse.

The above facts rendered $(*)$ a very appealing axiom.

However, in order for $(*)$ to be truly natural, it would have to be compatible with all consistent LC axioms.

The main question was therefore:

Question: Is $(*)$ compatible with all consistent LC axioms? Is $(*)$ even forcible over V (assuming enough large cardinals)?

$\mathrm{MM}^{++} \Longrightarrow (*)$

In 2018, Ralf Schindler and I answered this question.

Theorem

(Asperó-Schindler) MM^{++} implies $(*)$.

In particular, if there is a supercompact cardinal κ, then there is a forcing notion of cardinality κ forcing MM^{++}, and therefore forcing $(*)$. All large cardinals there might be above κ are preserved in the generic extension.

This theorem renders $(*)$ a truly natural axiom. And it makes the observed completeness of MM^{++} for the theory of $L(\mathcal{P}(\omega_1))$ under forcing into a mathematical fact, thus rendering MM^{++} even more natural.

And both $(*)$ and MM^{++} were known to imply $2^{\aleph_0} = \aleph_2$.

There are many ways to see this. For example:

1. Already MA_{ω_1} implies $2^{\aleph_0} = 2^{\aleph_1}$: Given a sequence $(a_\xi : \xi < \omega_1)$ of pairwise almost disjoint subsets of ω and $A \subseteq \omega_1$ there is $x \subseteq \omega$ such that for each $\xi < \omega_1$, $\xi \in A$ iff $x \cap a_\xi$ is infinite.

2. Both MM and $(*)$ imply ψ_{AC}: For every stationary $S \subseteq \omega_1$ and every stationary and co-stationary $T \subseteq \omega_1$ there is some $\alpha < \omega_2$ and some $\subseteq$-continuous $\subseteq$-increasing chain $(X_\nu : \nu < \omega_1)$ of countable subsets of α such that

- $\bigcup_{\nu<\omega_1} X_\nu = \alpha$ and
- for each ν, $X_\nu \cap \omega_1 \in S$ iff $\mathrm{ot}(X_\nu) \in T$.

3. ψ_{AC} implies $2^{\aleph_0} = 2^{\aleph_1}$: Let $(S_\xi : \xi < \omega_1)$ be a partition of ω_1 into stationary sets and T a stationary and co-stationary subset of ω_1. Given $A \subseteq \omega_1$, and application of ψ_{AC} to $S := \bigcup_{\xi \in A} S_\xi$ and T yields $\alpha_A < \omega_2$ coding A. But then the map sending $A \subseteq \omega_1$ to α_A is injective.

Σ_2 chaos

Σ_2 chaos: Suppose σ ($= (\exists\alpha \in \mathrm{Ord})\, V_\alpha \models \psi$) is a Σ_2 sentence and for every $\beta \in \mathrm{Ord}$ there is a forcing notion leaving V_β unchanged and forcing σ. Then σ is true.

So, Σ_2 chaos implies there is a cardinal κ such that $2^\kappa = \kappa^+$ and one such that $2^\kappa = \kappa^{+127}$ and one such that there are κ-Suslin trees and one such that there are no κ-Suslin trees, etc.

Σ_2 is thus a natural forcing maximality principle.

Woodin asked if it is consistent: Forcing any finite number of instances of the principle is trivial by definition. But it is not obvious if one can run the obvious forcing iteration in infinite length without, for example, adding new sets of integers.

Theorem

(Goldberg-Kaplan) If there is a Σ_2-correct strongly compact cardinal, then Σ_2 chaos can be forced. If there is a proper class of Σ_2-correct strongly compact cardinals, then boldface Σ_2 chaos can be forced.

A straightforward elaboration of their construction yields:

Theorem

Suppose there is a supercompact cardinal and a proper class of Σ_2-correct strongly compact cardinals. Then the following can be forced simultaneously.

(1) MM^{++}

(2) Σ_2 *chaos.*

High forcing axioms

Given the success of classical forcings (for meeting $\aleph_1$-many dense sets) culminating in MM^{++}, it is natural to enquire whether a comparable theory can be developed for higher forcing axioms (i.e., for meeting more than $\aleph_1$-many dense sets).

The answer is No. For example:

- Π_2 maximality for $H(\omega_3)$ fails: CH and $\neg$CH are both expressible over $H(\omega_3)$ by Π_2 sentences, and both can be forced.
- There are serious problems with building models of strong high forcing axioms. In fact, many forcing axiom candidates for meeting families of $\aleph_2$-many dense, even for fairly modest classes Γ of forcing notions, are just false.

The conclusion is that the success of MM^{++} cannot be replicated, at the level of high forcing axioms, to yield a natural axiom implying for example $2^{\aleph_0} = \aleph_3$.

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This completes the argument for $2^{\aleph_0} = \aleph_2$.

Still, there are interesting open questions in the higher forcing axioms camp.

Question: Do reasonable LC axioms imply that there is no partition $\mathcal{P} \in L(\mathbb{R})$ of $\mathbb{R}$ into $\aleph_3$-many pieces?

Question: Do reasonable LC axioms imply that one can force some Σ_2 statement A which is complete for the theory of $H(\omega_3)$ modulo forcing preserving A?

Question: Is there any Π_2 sentence σ such that the following holds?

(1) ZFC proves that if $H(\omega_3) \models \sigma$, then $2^{\aleph_0} = \aleph_3$.

(2) For some reasonable LC axiom A, ZFC+ A proves that it is forcible that $H(\omega_3) \models \sigma$.

A competing view

Woodin has championed an alternative view. This is the content of *the Ultimate-L programme*.

One of its main goals is to construct an L-like subuniverse, called Ultimate-L, accommodating all possible LC axioms.

The axiom $V = \text{Ultimate-}L$ would provide a complete picture modulo forcing for the entire universe. And it implies $2^{\aleph_0} = \aleph_1$.

On the down side, $V = \text{Ultimate-}L$ is a difficult axiom to work with. And it is at present not at all clear to what extent the programme can be implemented.

At any rate, the implications and ramifications of the programme are extremely deep.

Back to MM^{++}: A challenge

By our theorem, MM^{++} implies that $L(\mathcal{P}(\omega_1))$ is a homogeneous extension of the AD-model $L(\mathbb{R})$ and hence a canonical model. And by forcing with homogeneous forcing notions over stronger AD-models it is possible to obtain models of $\mathsf{MM}^{++}(2^{\aleph_0})$ and even stronger fragments of MM^{++}.

A major challenge nowadays is therefore:

Challenge:

(1) Obtain a model of full MM^{++}, or of stronger fragments of MM^{++}, by (homogeneous) forcing over strong models of the Axiom of Determinacy.

(2) Obtain a model of MM^{++} in which $(*)^{++}$ holds. $(*)^{++}$ says that there is $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ and a $\mathbb{P}_{\max}$-generic filter G over $L(\mathbb{R}, \Gamma)$ such that $\mathcal{P}(\mathbb{R}) \subseteq L(\mathbb{R}, \Gamma)[G]$.