Gaps in Hardy fields

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The boundary between convergence and divergence

Comparison with Bertrand's series (a.k.a. Abel's series)

Can the convergence/divergence of all series with positive terms be settled by comparison with a real multiple of a series of the form

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Paul du Bois-Reymond showed (1873) that the answer is "no", in the process inventing the "diagonal argument" a bit earlier than Cantor.

The boundary between convergence and divergence

He introduced the following useful notations, for (eventually non-vanishing) functions $f, \varphi\colon (a, +\infty) \to \mathbb{R}$ ($a \in \mathbb{R}$):

$$f \prec \varphi \quad :\Longleftrightarrow \quad \lim_{t \to +\infty} \frac{f(t)}{\varphi(t)} = 0,$$

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So for example, with real constants c, p,

$$\log x \prec x \prec \mathrm{e}^x \prec \mathrm{e}^{\mathrm{e}^x}, \quad x \prec x^p \; (p > 1), \quad cx^p \asymp x^p \; (c \neq 0),$$

but

$$f \not\prec \varphi, \quad f \not\asymp \varphi, \quad \varphi \not\prec f \quad \text{for } f = x(2 + \sin x), \varphi = x.$$

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Theorem (du Bois-Reymond)

Let $\varphi_i \colon [a, +\infty) \to \mathbb{R}^{\geqslant}$ *be continuous and strictly increasing and*

$$1 \prec \cdots \prec \varphi_{i+1} \prec \varphi_i \prec \cdots \prec \varphi_1 \prec \varphi_0.$$

There is a continuous $f \colon [a, +\infty) \to \mathbb{R}^{\geqslant}$ *with* $1 \prec f \prec \varphi_i$ *for each* i.

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If there were $C > 0$, i, and $p > 1$ such that $1/f(n) \leqslant C/\varphi_i(n)$ eventually, then $\varphi_i(n)/f(n) \leqslant C$ eventually $\lightning$.

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To prove the theorem it is convenient to replace the φ_i by their compositional inverses f_i and show a "dual" version:

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Set $M_i^n := \max\limits_{a \leqslant t \leqslant a+n} f_i(t)$, so $0 \leqslant M_i^0 \leqslant M_i^1 \leqslant M_i^2 \leqslant \cdots$.

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$$\sum_i \varepsilon_i M_i^n = \sum_{i=0}^{n} \varepsilon_i M_i^n + \sum_{i>n} \varepsilon_i M_i^n \leqslant \sum_{i=0}^{n} \varepsilon_i M_i^n + \sum_{i>n} \varepsilon_i M_i^i < \infty.$$

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Thus $\sum_i \varepsilon_i f_i$ converges uniformly on each set $[a, a+n]$, defining a continuous function on $[a,\infty)$, with $\sum_i \varepsilon_i f_i \geqslant \varepsilon_{n+1} f_{n+1} \succ f_n$. $\square$

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  In 2, is there an analytic f such that $f_i \prec f \prec g_j$ for all i, j?

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All this is joint work with (one or both of) *Lou van den Dries* and *Joris van der Hoeven*.

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All these inclusions are proper, but this is not obvious.

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Most Hardy fields that occur "in nature" are analytic. Easy examples:

$$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{R}(x) \subseteq \mathbb{R}(x, \mathrm{e}^x) \subseteq \mathbb{R}(\log x, x, \mathrm{e}^x)$$

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- f is eventually monotonic, and

$$\lim_{t \to +\infty} f(t) \in \mathbb{R} \cup \{\pm\infty\} \qquad \text{exists.}$$

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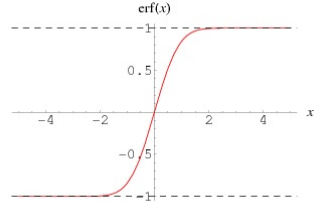
Example (for what Rosenlicht meant)

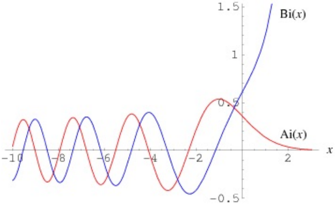
Suppose $0 \neq f, g \not\asymp 1$ are in a Hardy field. Then (l'Hôpital's Rule):

$$f \preccurlyeq g \quad \Longleftrightarrow \quad f' \preccurlyeq g'$$

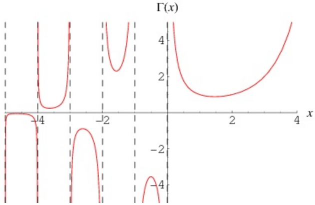
Examples of functions in Hardy fields



$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt$$



Ai, Bi are $\mathbb{R}$-linearly independent solutions to $y'' - xy = 0$



$$\Gamma(x) = \int_0^\infty t^{x-1}\,\mathrm{e}^{-t}\, dt$$

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- every *o-minimal expansion* of the ordered field of reals gives rise to a Hardy field; e.g. for the ordered field $\mathbb{R}$ itself one obtains

$$H = \{y \in \mathcal{C} : P(y) = 0 \text{ for some nonzero } P \in \mathbb{R}(x)[Y]\}.$$

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Hence $H(\mathbb{R})$ and $H(x)$ are Hardy fields, and for $h \in H$, so are

$$H(\textstyle\int h), \quad H(\mathrm{e}^h), \quad H(\log h) \text{ when } h > 0.$$

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($\Longrightarrow$ Hardy's field of LE-functions is indeed a Hardy field!)

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(Here, 2 is actually a special case of a more general Intermediate Value Property for differential polynomials over Hardy fields.)

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These are formal series (often divergent), involving exponential and logarithmic terms, which can be used to model the asymptotic behavior of germs in Hardy fields:

$$\operatorname{erf} \sim 1 - \frac{\mathrm{e}^{-x^2}}{\sqrt{\pi}} \left(x^{-1} - \frac{1}{2} x^{-3} + \frac{3}{4} x^{-5} \mp \cdots \right)$$

$$\operatorname{Ai} \sim \frac{\mathrm{e}^{-\xi}}{2\sqrt{\pi} x^{1/4}} \left(1 - \frac{5}{72} \xi^{-1} + \frac{385}{10368} \xi^{-2} \mp \ldots \right) \text{ where } \xi = \frac{2}{3} x^{3/2}$$

$$\log \Gamma(x) \sim \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log(2\pi) + \frac{1}{12} x^{-1} - \frac{1}{360} x^{-3} \pm \cdots$$

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Only recently we've been able to tackle the analytic/smooth cases:

Theorem (A., van den Dries)

The theorem above also holds with "$\mathcal{C}^\infty$-Hardy field" or "$\mathcal{C}^\omega$-Hardy field" in place of "Hardy field".

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- M also contains a *translogarithmic* germ g, that is,

$$\mathbb{R} < g < \cdots < \log\log\log x < \log\log x < \log x < x.$$

  (Answering a question of Boshernitzan.)

Some applications

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Corollary B

Let M be a maximal analytic Hardy field and N be a maximal Hardy field with $M \subseteq N$. Then $M \preccurlyeq_{\infty\omega} N$. (Likewise if M is a maximal smooth Hardy field.)

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The countability property relevant for 2: the ordered set $\mathbb{T}$ is *short*, i.e., every well-ordered or reverse well-ordered subset is countable.

A **gap** in a (partially) ordered set S is a pair $A < B$ of linearly ordered subsets of S such that $A < f < B$ for no $f \in S$, and the **character** of such a gap in S is the pair $\big(\mathrm{cf}(A), \mathrm{ci}(B)\big)$.

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The characters of gaps in M are

$$(\omega, \kappa),\ (\kappa, \omega),\ (\kappa, \kappa),\ (0, \lambda),\ (\lambda, 0),\ (1, \lambda),\ (\lambda, 1),$$

and if M is not complete, then also (λ, λ).

Hence under CH, the characters of gaps in M are

$$(\omega,\omega_1),\ (\omega_1,\omega),\ (\omega_1,\omega_1),\ (0,\omega_1),\ (\omega_1,0),\ (1,\omega_1),\ (\omega_1,1).$$

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Hausdorff showed (not assuming CH) that there is an (ω_1,ω_1)-gap in $(\mathcal{C},<_{\mathrm{e}})$, where

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Can check: also in $(\mathcal{C}^{\infty}, <_{\mathrm{e}})$ and in $(\mathcal{C}^{\omega}, <_{\mathrm{e}})$.

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Remark: if M is a maximal analytic Hardy field and $N \neq M$ is a maximal Hardy field extension of M, then $(N, M) \equiv (\mathbb{T}, \mathbb{T}^c)$, where $\mathbb{T}^c =$ completion of $\mathbb{T}$.

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(Using part 2 of Corollary C one can obtain a pair (N_1, M_1) of analytic Hardy fields such that $(N_1, M_1) \cong (\mathbb{T}, \mathbb{T}^c)$.)

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Part (II) includes Hardy field versions of du Bois-Reymond-Hadamard's theorem from earlier:

given $f_0 \prec f_1 \prec \cdots$ in $M^>$ there is an $f \in M$ with $f_i \prec f$ for all i.

To tackle II we separate three cases.

Let $A < B$ be a countable gap in M, where $A, B \subseteq M^{>\mathbb{R}}$.

1. *The case $B = \emptyset$: obtain an $f \in M$ with $A < f$.*
2. $A < B$ *is **wide**: $A, B \neq \emptyset$ and $A, \exp A$ are cofinal.*
3. *$A, B \neq \emptyset$, and $A < B$ is not wide.*

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The $\mathcal{C}^\infty$-case of (1) was done by Sjödin; this adapts to general Hardy fields, and can also be extended to (2).

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Various results about the *asymptotic couple* of $H\langle y\rangle$ — that is, its value group equipped with the map $vf \mapsto v(f'/f)$ $(0 \neq f \not\asymp 1)$ — entail that such y has to have a specific form:

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To construct such y analytically is a bit delicate (and also involves a diagonalization argument).

The proof of the main results

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Let $f\colon [a,+\infty) \to \mathbb{R}$ *be* $\mathcal{C}^\infty$ *and* $\varepsilon\colon [a,+\infty) \to \mathbb{R}$ *be continuous with* $\varepsilon > 0$. *Then there exists an analytic* $g\colon [a,+\infty) \to \mathbb{R}$ *such that*

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This entails a useful version for germs:

Corollary

For any germs $f \in \mathcal{C}^{<\infty}$ *and* $\varepsilon \in \mathcal{C}$ *with* $\varepsilon >_{\mathrm{e}} 0$, *there exists a* $g \in \mathcal{C}^\omega$ *such that* $|(f-g)^{(n)}| <_{\mathrm{e}} \varepsilon$ *for all* n.

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Theorem (Whitney)

Let $f\colon [a,+\infty) \to \mathbb{R}$ *be* $\mathcal{C}^\infty$ *and* $\varepsilon\colon [a,+\infty) \to \mathbb{R}$ *be continuous with* $\varepsilon > 0$. *Then there exists an analytic* $g\colon [a,+\infty) \to \mathbb{R}$ *such that*

$$|(f-g)^{(n)}(t)| < \varepsilon(t) \qquad \text{for all } t \geqslant a \text{ and } n \leqslant 1/\varepsilon(t).$$

This entails a useful version for germs:

Corollary

For any germs $f \in \mathcal{C}^{<\infty}$ *and* $\varepsilon \in \mathcal{C}$ *with* $\varepsilon >_{\mathrm{e}} 0$, *there exists a* $g \in \mathcal{C}^\omega$ *such that* $|(f-g)^{(n)}| <_{\mathrm{e}} \varepsilon$ *for all* n.

This is the key approximation result that allows us to replace a germ in a Hardy field extension filling a given countable gap by an analytic germ with the same property.

Further work

At the moment we are developing a theory of analytic Hardy fields which includes information about the domain of convergence of the holomorphic extension. (Some early steps already done; related work by Tobias Kaiser, Patrick Speissegger, and Alex Wilkie, on the Hardy field of the o-minimal structure $\mathbb{R}_{\mathrm{an},\exp}$.)

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2. Is it possible that there is a maximal Hardy field M which is complete, as an ordered field? (In this case $M \not\cong \mathbf{No}(\omega_1)$.)

3. Is it possible that $\mathrm{cf}(M) \neq \mathrm{cf}(N)$ for some maximal Hardy fields M, N? Similarly for $\mathrm{ci}(M^{>\mathbb{R}})$ and $\mathrm{ci}(N^{>\mathbb{R}})$.

Thank you!