Logic and property testing on graphs of bounded degree

Isolde Adler

Joint work with Polly Fahey, Frederik Harwath, Noleen Köhler, Pan Peng

Panhellenic Logic Symposium
2.7.2024

CV overview

- Student & PhD in Mathematics: Göttingen, **Thessaloniki**, Freiburg
- Postdoc & Substitute Prof., *Logic in CS*: HU Berlin, Bergen
- Juniorprofessor *Theory of CS*: Goethe University Frankfurt
- Lecturer, Associate Prof. *Algorithms & Complexity*: Leeds
- Since 2022: Chair of *Algorithms and Complexity Theory*: Otto-Friedrich University Bamberg

Algorithms & Complexity @ Bamberg (Bavaria, Germany)

Contents

Graphs

- We consider finite, simple, undirected graphs $G = (V(G), E(G))$.
- Degree of a vertex $:=$ number of its neighbours.
- Degree of a graph $G :=$ max degree of its vertices.
- Class $\mathcal{C}$ of graphs has bounded degree, if there is a constant $d \in \mathbb{N}$ such that all graphs in $\mathcal{C}$ have degree $\leq d$.
- We use n to denote the number of vertices of G.
- All graph classes are closed under isomorphism, and a graph class is sometimes called a property.

Remark
All techniques and results can be generalised to relational structures.

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Many classical algorithmic problems are NP-hard in general.
However, they need to be solved in practice. [1]

Example

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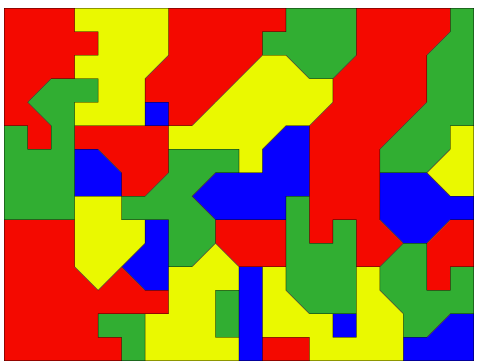
Input: Graph G, $k \in \mathbb{N}$
Question: Does G have a k-colouring?



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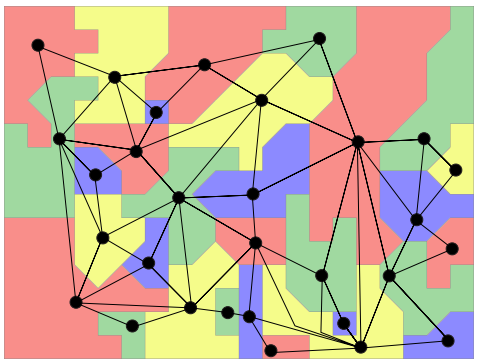

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Complexity of the problems

Theorem (Karp 1972)

COL *is* NP*-complete.*

Observation

On trees, COL *and many other problems can be solved in linear time.*

Goal

Find large classes of input graphs, on which many problems can be solved efficiently!

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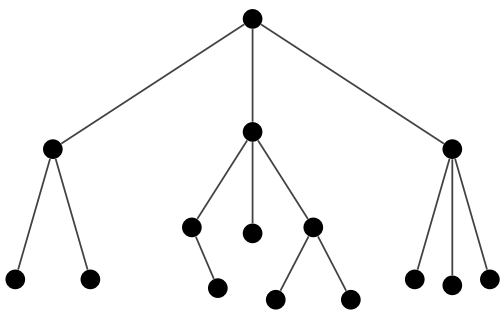

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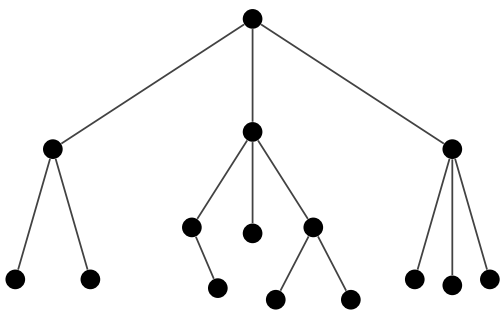

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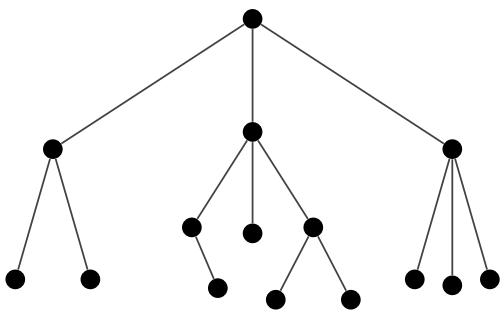

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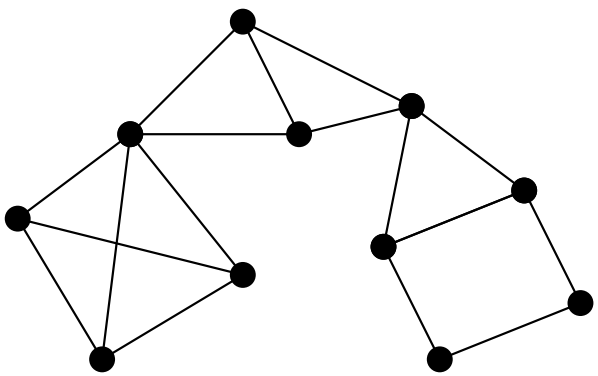

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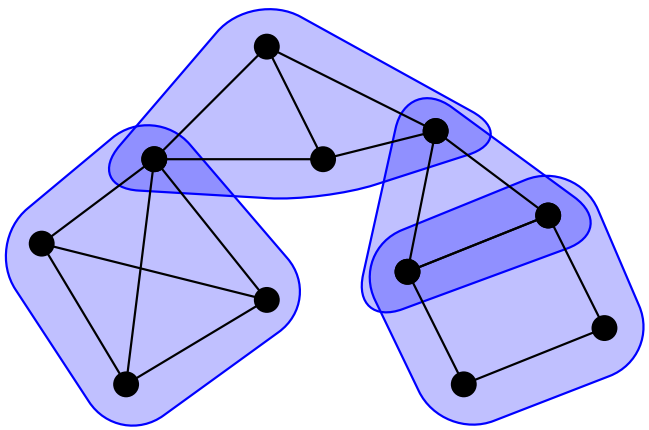

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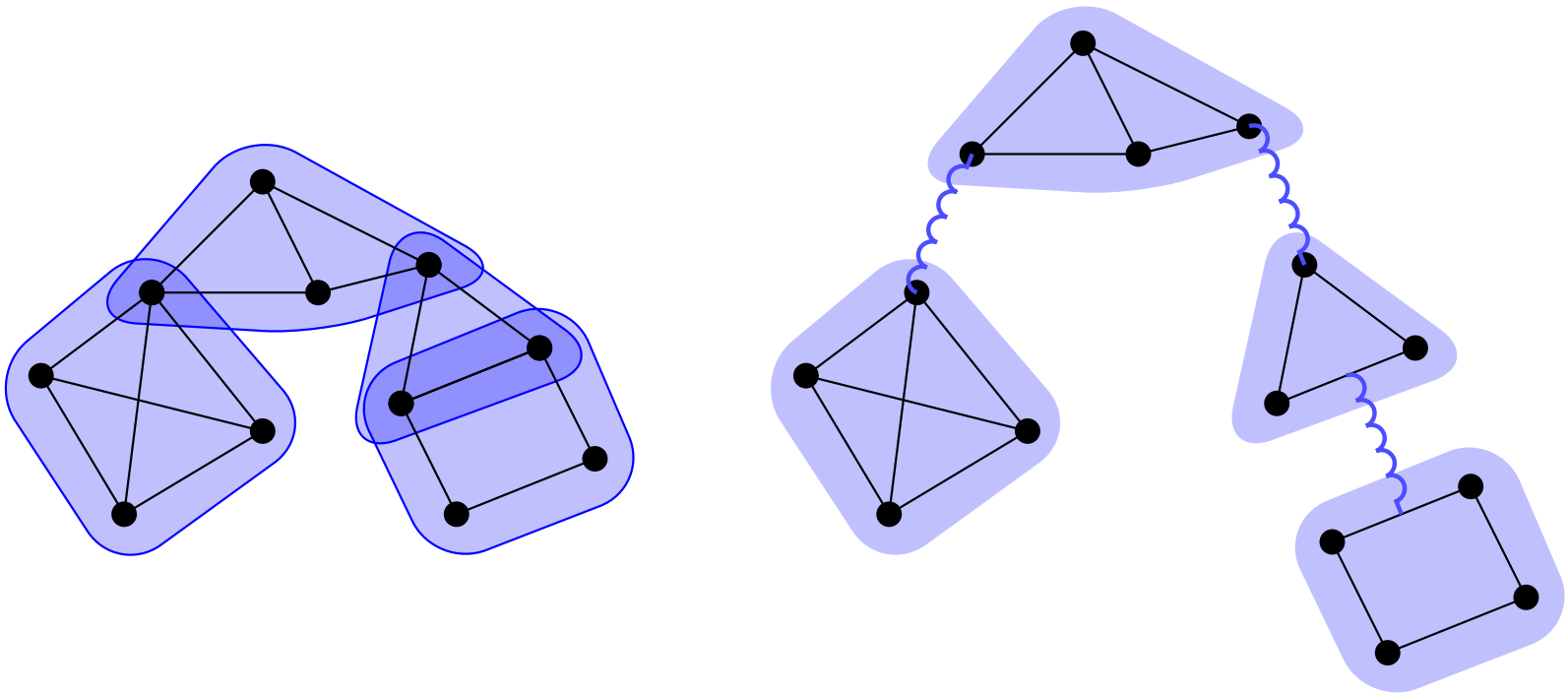

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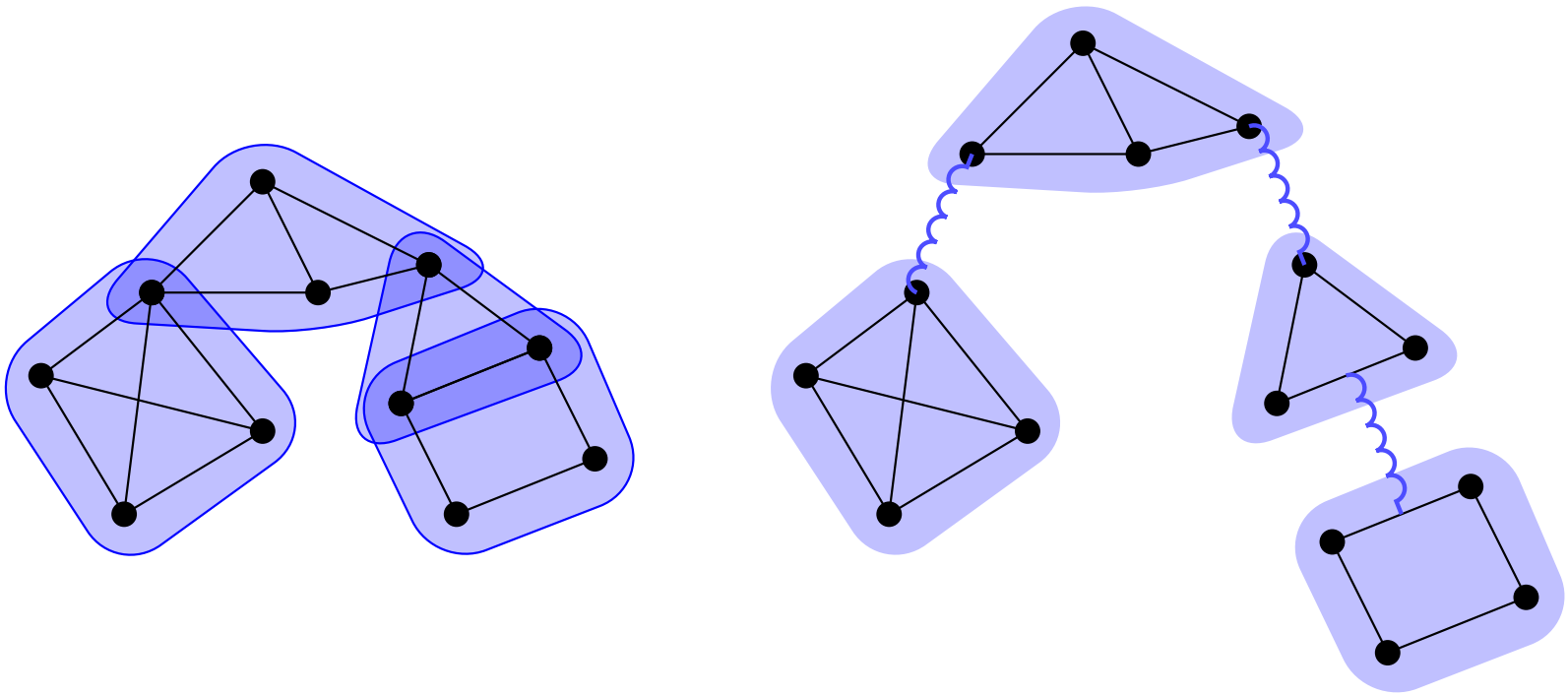

[Larisch, Salfelder, Implementation `https://www.algok.uni-bamberg.de/freetdi.html`, 1. Prize, PACE 2017]

Logics

FO := First-order logic

MSO := Monadic second-order logic =
first-order logic + quantification over subsets of the universe

CMSO = MSO + modulo counting $\quad$ '$\exists_{k \bmod m} x\, \varphi(x)$'

Example

Graph properties expressible in CMSO

- (non-)existence of: k-clique, k-vertex cover, k-independent set, k-dominating set, a fixed subgraph H
- planarity, bounded genus, excluded minor
- connectivity, transitive closure,
- colorability, hamiltonicity,
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MSO: example

$$\varphi := \exists X_1 \exists X_2 \exists X_3 \forall y \forall z \Big(\bigvee_{1 \leq i \leq 3} (y \in X_i) \wedge \bigwedge_{1 \leq i < j \leq 3} \neg(y \in X_i \wedge y \in X_j) \wedge \bigwedge_{1 \leq i \leq 3} (Eyz \rightarrow \neg(y \in X_i \wedge z \in X_i)) \Big)$$

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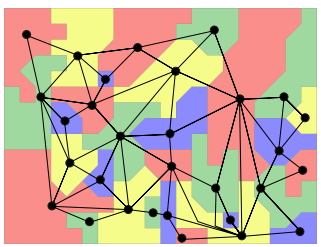

Algorithmic Meta-Theorems

$\mathcal{C}$: Class of graphs
φ: Formula of some logic

φ-MODELCHECK($\mathcal{C}$)	
Input:	$G \in \mathcal{C}$.
Question:	Does G satisfy φ?

Applications:

- Model checking
- Graph algorithms
- Verification
- Database query evaluation
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$\mathcal{C}$ = bounded degree, $\varphi \in$ FO: in time $\mathcal{O}(n)$.

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Theorem (Grohe, Kreutzer, Siebertz, 2014)
$\mathcal{C}$ = nowhere dense, $\varphi \in$ FO: f. a. $\varepsilon > 0$ in time $\mathcal{O}(n^{1+\varepsilon})$.

Theorem (Bonnet, Kim, Thomassé, Watrigant, 2020)
$\mathcal{C}$ = bounded twin-width, $\varphi \in$ FO:
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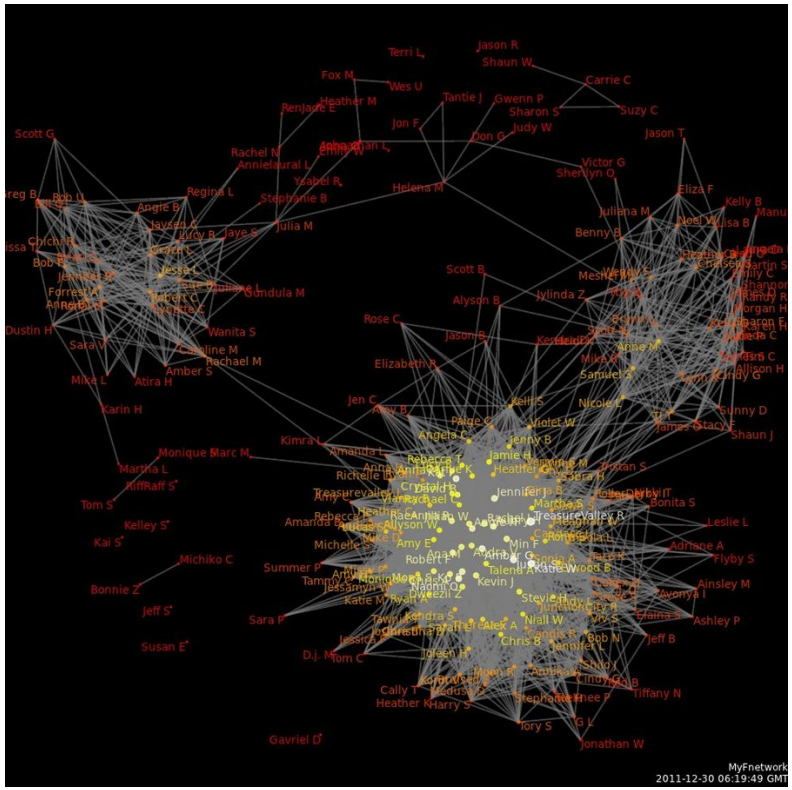
Property testing: motivation

‘Efficiency’ when the data set is huge:

Even reading the whole input just once can be too expensive.



Data visualization of Facebook relationships

Author: Kencf0618, License: Creative Commons Attribution-Share Alike 3.0 Unported

Hence: Algorithms with local access to the input.

Algorithms with oracle access

From now on: All graphs have degree $\leq d$.

- Input: the number n of vertices of G, and
- Oracle access to G
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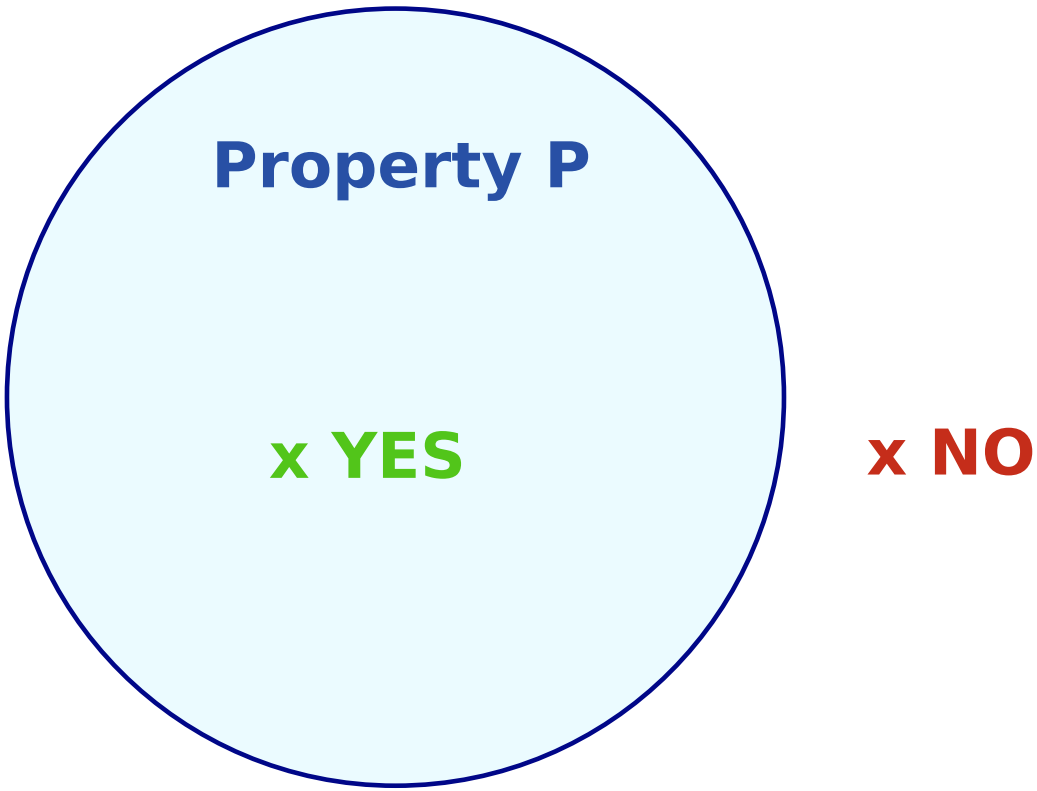
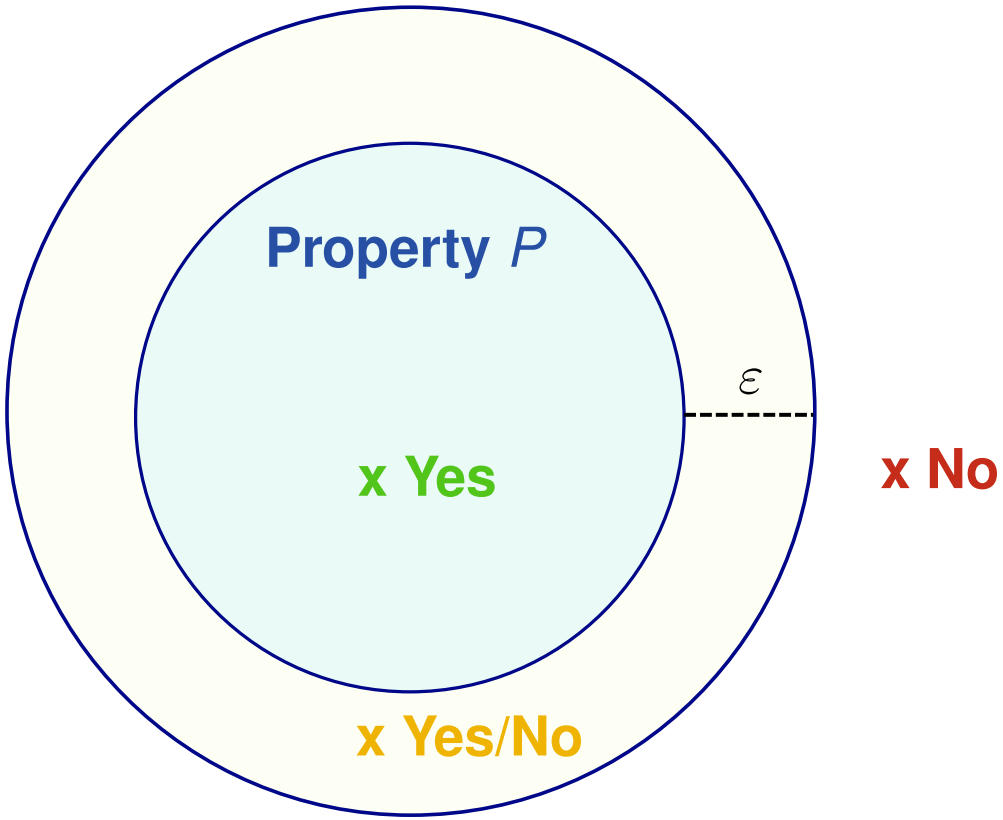


Property Testing = relaxation of decision problems



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On ε-far inputs: NO with probability at least $2/3$.

Aim: extremely efficient.

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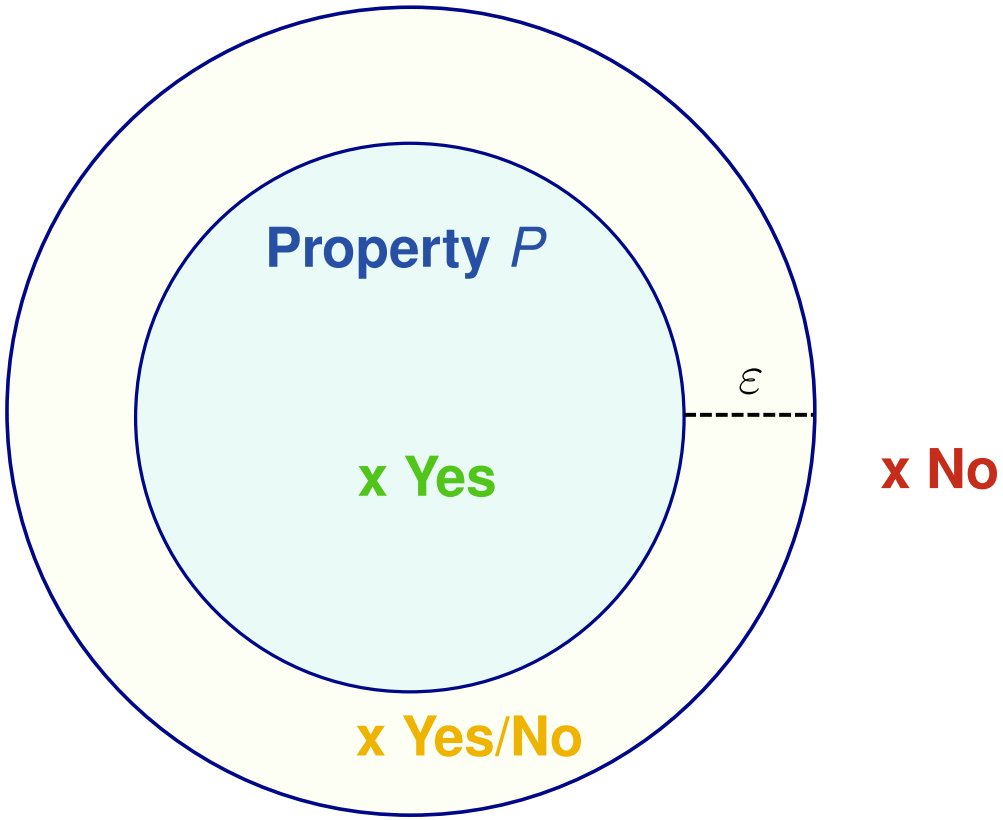

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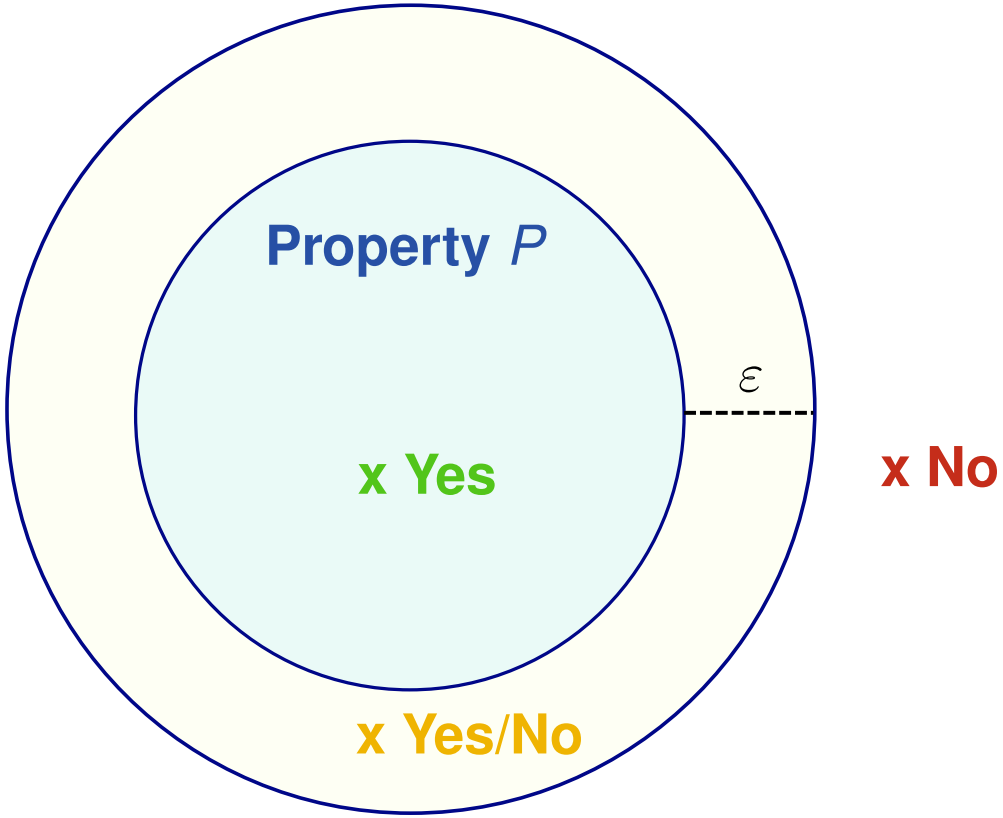

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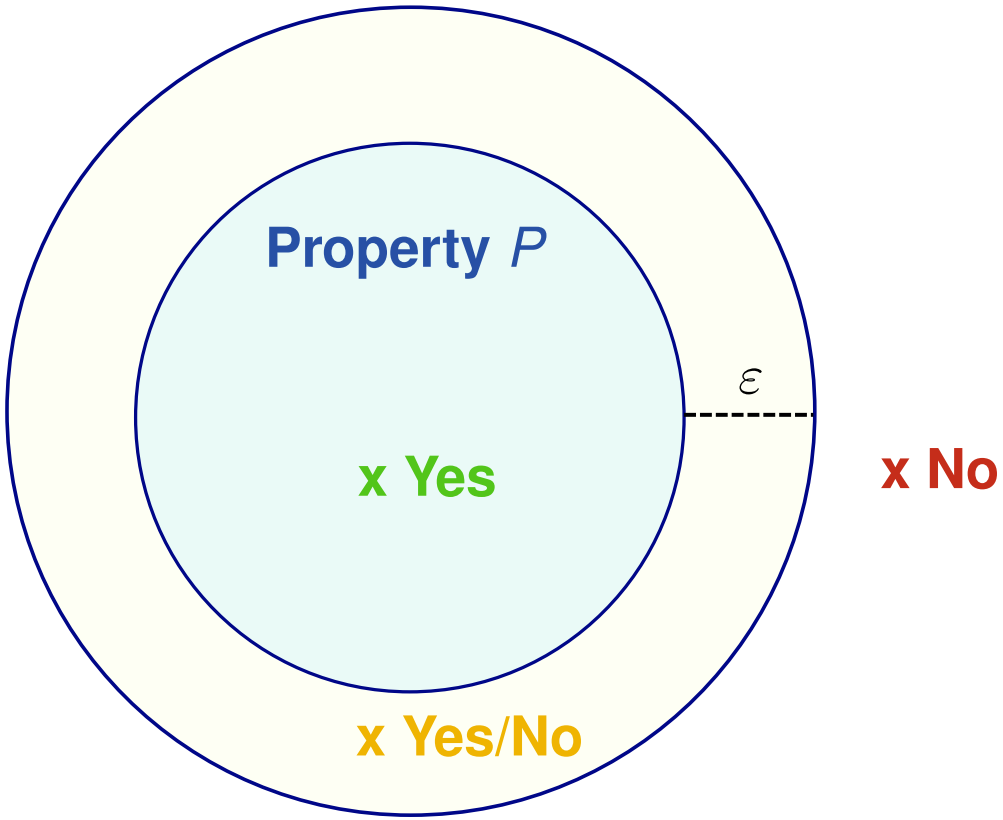

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Property Testing

- First introduced in the context of programme checking
  [R. Rubinfeld and M. Sudan. Robust characterizations of polynomials with applications to program testing. SIAM J. Computing, 1996]
- Property testing of dense graphs
  [O. Goldreich, O. Goldwasser and D. Ron. Property Testing and its Connection to Learning and Approximation. Journal of the ACM, 1998]

  [N. Alon, E. Fischer, I. Newman, A. Shapira. A combinatorial characterization of the testable graph properties: It's all about regularity. SIAM J. Computing, 2009]
- Property testing of regular languages
  [N. Alon, M. Krivelevich, I. Newman, and M. Szegedy. Regular languages are testable with a constant number of queries. SIAM J. Computing, 2000.]
- Property testing of sparse graphs
  [O. Goldreich and D. Ron. Property Testing in Bounded Degree Graphs. Algorithmica, 2002]

  [Ito, Khoury, Newman. On the characterization of 1-sided error strongly testable graph properties for bounded-degree graphs. J. Comput. Complex. 2020]

  [O. Goldreich and L. Tauber. On Testing Isomorphism to a Fixed Graph in the Bounded-Degree Graph Model. ECCC, 2023]

Property Testing

- Constraint Satisfaction and Satisfiability
  [H. Chen, M. Valeriote, Y. Yoshida. Testing Assignments to Constraint Satisfaction Problems. FOCS 2016]
  [Y. Yoshida. Optimal constant-time approximation algorithms and (unconditional) inapproximability results for every bounded-degree CSP.]
- Machine Learning
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Distance (d-bounded degree model)

- Let $\varepsilon \in [0,1]$.
  Graphs G and H, both on n vertices, are ε-close, if we can make them isomorphic by modifying up to εdn edges of G or H.

  Edge modification = insertion/deletion

- If G, H are not ε-*close*, then they are ε-far.

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Testability

Let $\mathcal{P}$ be a property.
An ε-tester for $\mathcal{P}$ is a probabilistic algorithm that, given oracle access to G and given $n := |V(G)|$ as input, does the following:

1. If $G \in \mathcal{P}$, then the tester accepts with probability $\geq \frac{2}{3}$,
2. if G is ε-far from $\mathcal{P}$, then the tester rejects with probability $\geq \frac{2}{3}$.

$\mathcal{P}$ is uniformly testable, if for each ε there is an ε-tester for $\mathcal{P}$ with constant query complexity.

$\mathcal{P}$ is (non-uniformly) testable, if for each ε and each n, there is a tester for $\mathcal{P}_n := \{G \in \mathcal{P} : |V(G)| = n\}$ with constant query complexity.

Remark
'Uniformly testable' is non-standard.

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Example: induced subgraph-freeness

Fix H.
$\mathcal{P} :=$ all graphs that omit H as induced subgraph.

Theorem
$\mathcal{P}$ is testable with constant query complexity and running time.

Proof: Assume H is connected and $|V(H)| > 1$.
Let $r :=$ diameter of H.
Given ε and oracle access to input G on n vertices, do:

1. Uniformly and independently sample $\alpha := \log_{1-\varepsilon} 1/3$ vertices.
2. Explore the r-neighbourhood of each sampled vertex.
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Running time: constant (r-neighbourhoods have constant size).

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   - Claim: $|X| > \varepsilon n$.
     Otherwise, remove all edges incident to X. Then the resulting graph G' contains no induced copy of H, so $G' \in \mathcal{P}$. But we removed $\leq \varepsilon dn$ edges, so G is ε-close to $\mathcal{P}$, a contradiction.
   - Hence the probability that out of the α elements sampled, no element of X appears, is less than $(1-\varepsilon)^{\alpha} = 1/3$.
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Uniformly testable with constant query complexity and running time:

- k-(edge-)connectivity
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Let $\mathcal{C}$ be a hyperfinite class of graphs of bounded degree.

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Let $0 \leq \varepsilon \leq 1$ and $k \in \mathbb{N}$.

- G is (ε, k)-hyperfinite if one can remove εn edges from G and obtain a graph whose connected components have size $\leq k$.
- Fix a function $\rho : \mathbb{R}^+ \to \mathbb{N}$.
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Hyperfinite graph classes include
- bounded tree-width graphs,
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Remark
The theorem does not say anything about the running time.
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CMSO *on bounded tw with sublinear running time*

Theorem (A., Harwath, STACS 2018)

Let $\mathcal{C}_d^t$ be the class of all t-bounded tree-width graphs of degree $\leq d$.

Every CMSO-definable property $\mathcal{P} \subseteq \mathcal{C}_d^t$ is uniformly testable with constant query complexity and polylogarithmic running time.

- First algorithmic meta-theorem with sublinear running time.
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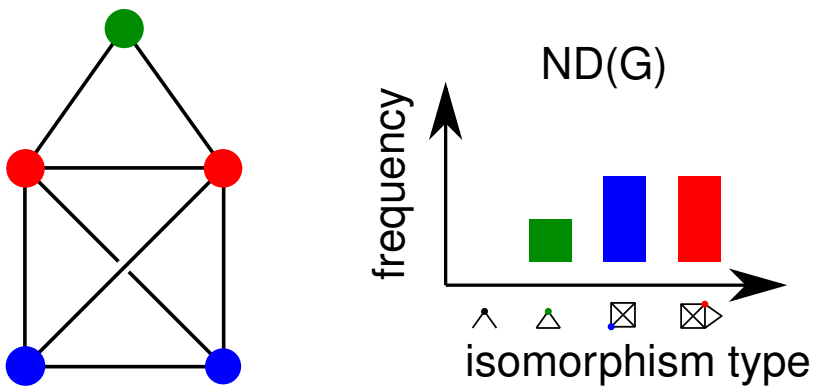

Algorithm: Fix ε. Given oracle access to $G \in \mathcal{C}_d^t$, and $n = |V(G)|$.

- $\bar{x} =$ estimation of neighbourhood distribution $\mathrm{ND}(G)$, via sampling
- Accept, if $\|\bar{x} - \mathrm{ND}(H)\|_1 \leq \lambda = \lambda(\varepsilon)$ is small, for some $H \in \mathcal{P}$ on n vertices, otherwise reject.

Correctness: Characterisation of testability; theorem from [Newman, Sohler, 2013].

Runtime: Show: histogram vectors of CMSO properties are semilinear, via [Fischer, Makowsky, 2003]; use [Lenstra, IP with a fixed number of variables, 1983]. □

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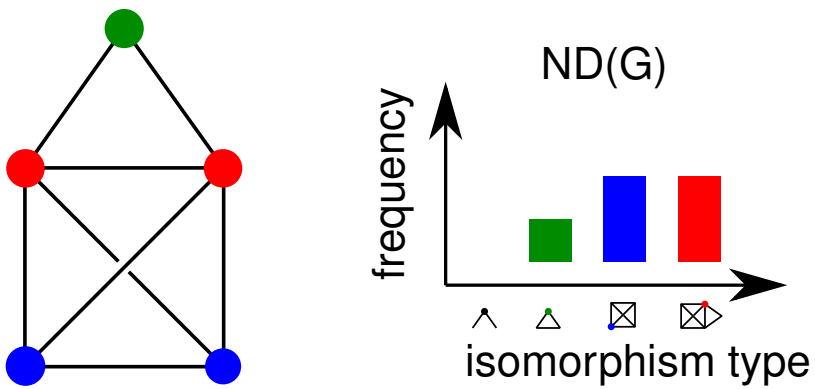

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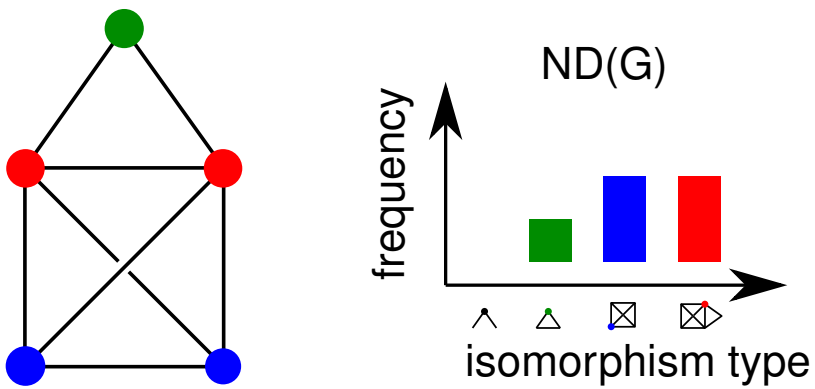

Algorithm: Fix ε. Given oracle access to $G \in \mathcal{C}_d^t$, and $n = |V(G)|$.

- $\bar{x} =$ estimation of neighbourhood distribution $\mathrm{ND}(G)$, via sampling
- Accept, if $\|\bar{x} - \mathrm{ND}(H)\|_1 \leq \lambda = \lambda(\varepsilon)$ is small, for some $H \in \mathcal{P}$ on n vertices, otherwise reject.

Correctness: Characterisation of testability; theorem from [Newman, Sohler, 2013].

Runtime: Show: histogram vectors of CMSO properties are semilinear, via [Fischer, Makowsky, 2003]; use [Lenstra, IP with a fixed number of variables, 1983]. □

Algorithm and proof idea

Theorem (A., Harwath, STACS 2018)

$\mathcal{C}_d^t :=$ *all* t*-bounded tree-width graphs of degree* $\leq d$*.*

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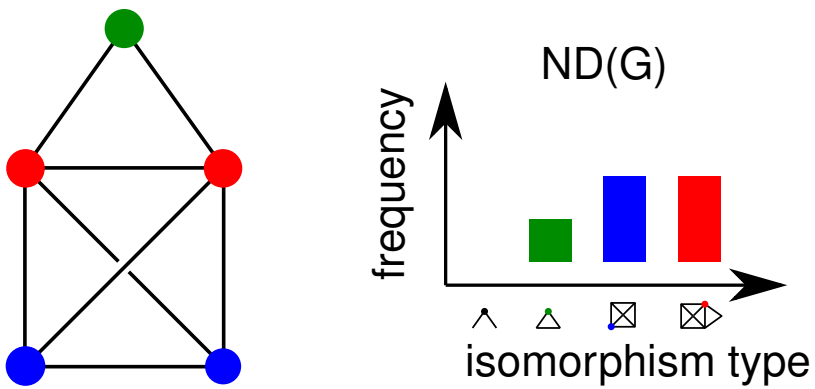

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Logic & PT on bounded degree

[Courcelle, 1990]
$\mathcal{C}$:= graph class of bounded treewidth

Input: $G \in \mathcal{C}$, $\varphi \in$ CMSO
Question: Does G satisfy φ?

Decidable in FPT: time $f(|\varphi|) \cdot \mathcal{O}(n)$

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Testability of FO*?*

Theorem (A., Köhler, Peng, SODA 2021, SIAM J. Computing 2024)

In the bounded degree model:

- *Every property that can be expressed by an* FO*-formula with quantifier prefix type* $\exists^*\forall^*$ *is testable with constant query complexity.*
- *There is an* FO*-property of quantifier prefix type* $\forall^*\exists^*$ *that is not testable.*

Every $\exists^*\forall^*$-property is testable (proof idea)

- On bounded-degree graphs, every $\exists^*\forall^*$-property is 'indistinguishable' from a $\forall^*$-property.

- $\forall^*$-properties are testable with constant query complexity:

  - $\forall\overline{x}\varphi(\overline{x}) \equiv \neg\exists\overline{x}\psi(\overline{x})$, for $\psi(\overline{x}) := \neg\varphi(\overline{x})$.

  - '$\psi(\overline{x})$-freeness' is testable with constant query complexity and running time similar to induced subgraph freeness.

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Non-testable $\forall^*\exists^*$-property (proof idea)

Find $\forall^*\exists^*$-formula φ whose models are expander graphs.

$(G_i)_{i\in\mathbb{N}}$: zig-zag construction [Rheingold, Vadhan, Wigdersen, Ann. of Math., 2002]

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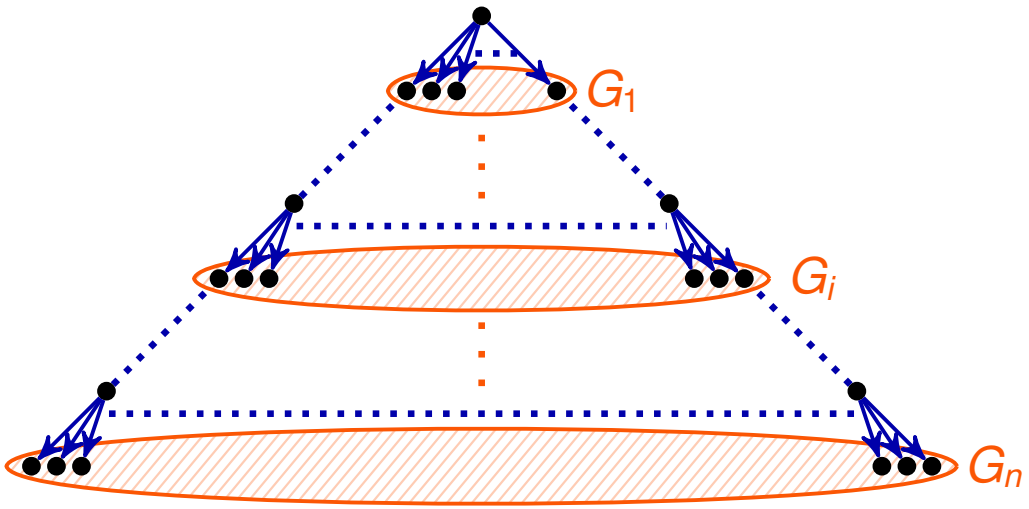

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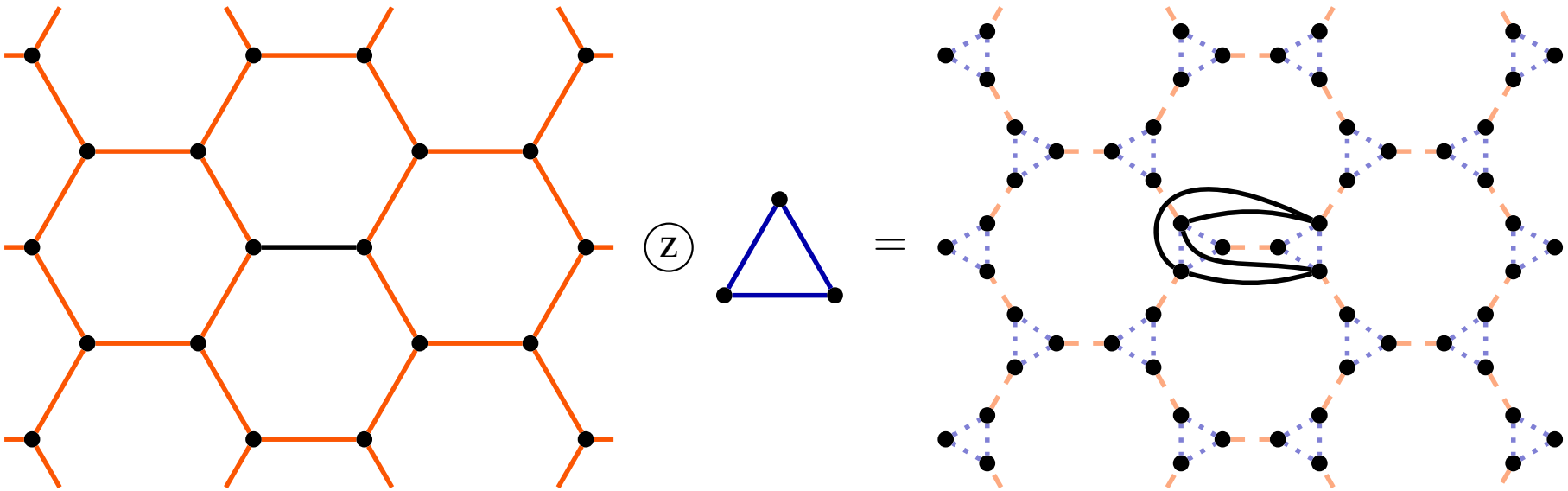
Avi Wigdersen: Turing Award recipient 2023

*1965: Israeli computer scientist and mathematician, school of mathematics, Princeton, USA

Avi Wigdersen

"For reshaping our understanding of the role of randomness in computation, and for decades of intellectual leadership in theoretical computer science."

Example: Zig-zag product of a 3-regular grid with a triangle



Non-testable $\forall^*\exists^*$-property (proof idea), cont'd

Theorem (Alon 2011)

On graphs of bounded degree:
For any graph G, radius r and ε, there exists a graph H whose size is independent of G such that $||ND_r(G) - ND_r(H)||_1 \leq \varepsilon$.

- Corollary: Tester cannot distinguish expander G from $H \dot\cup \ldots \dot\cup H$.
- $G \models \varphi$ and $H \dot\cup \ldots \dot\cup H$ is far from satisfying φ.

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Thank you!

CMSO *on bounded tw with constant running time?*

Theorem (A., Fahey, FSTTCS 2020)

Let $\mathcal{C}_d^t$ be the class of all t-bounded tree-width graphs of degree $\leq d$.

If we allow vertex deletions/additions in addition to edge modifications:
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