Proceedings of the 14th Panhellenic Logic Symposium

July 1-5, Thessaloniki, Greece

Edited by Alex Kavvos and Vassilis Gregoriades.

These proceedings are available through the conference website:

http://panhellenic-logic-symposium.org/14/

Preface

For 27 years the Panhellenic Logic Symposium has been a pillar of the logic community in southeast Europe. It is our great joy and honour to be the ones preparing the 14th edition of this established meeting. This year's meeting is also the first one to not be encumbered by the coronavirus pandemic, which delayed the preceding one and forced us to meet on even years in the future.

Unfortunately, this year's meeting is also the first one since the untimely loss of Thanases Pheidas (1958–2023). Thanases was internationally recognised for his work on analogues of Hilbert's Tenth Problem. He was also a vibrant member of the Hellenic logic community, and had been actively involved with the Symposium since its inception in 1997. He was always the voice of conscience that reminded us to try to waive registration fees for young researchers and those in need, with a view to making logic as widely accessible as possible. We have tried to mitigate this deafening absence by dedicating a Special Session in his memory. The invited speakers include former and current colleagues and students, as well as others whose work followed in his footsteps.

Moreover, we have made great efforts to further bolster the Symposium's international character by inviting distinguished speakers from the four corners of the world. We have also tried to further extended the thematic breadth of the Symposium, by including categorical logic as a separate topic of interest in the Call for Papers. Finally, we have explicitly asked the speakers to keep their presentations accessible to a wider audience. We hope that this will faciliate the fruitful interchange between all the subjects upon which logic impinges—including Mathematics, Philosophy, and Computer Science. This is, after all, a Symposium—in the Hellenic sense.

This volume contains short abstracts of the invited talks and tutorials that were delivered during the first week of July 2024 in the Aristotle University Research Dissemination Center. Furthermore, it contains 20 abstracts of contributed talks, which were meticulously reviewed by the Symposium's Scientific Committee. We wish to extend our gratitude, both to the Committee and also the external reviewers that assisted them: Dimitra Chompitaki, Ioannis Eleftheriadis, Iosif Petrakis, Xinxin Liu, Benjamin Rossman, and Wenhui Zhang.

We would also like to thank our sponsors: the Aristotle University of Thessaloniki, and especially its Faculty of Sciences and the Research Committee; the Association for Symbolic Logic; the European Mathematical Society; the Foundation Compositio Mathematica; and the University of Bristol. This Symposium would not have been possible without their generous financial contributions.

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Alex Kavvos and Vassilis Gregoriades

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Conference Program

Logic and Property Testing on Graphs of Bounded Degree

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Property testing (for a property P) asks for a given graph, whether it has property P, or is "structurally far" from having that property. A "testing algorithm" is a probabilistic algorithm that answers this question with high probability correctly, by only looking at small parts of the input. Testing algorithms are thought of as "extremely efficient", making them relevant in the context of large data sets.

In this talk I will introduce property testing and present recent positive and negative results about testability of properties definable in first-order logic and monadic second-order logic on classes of bounded-degree graphs.

This is joint work with Polly Fahey, Frederik Harwath, Noleen Köhler, and Pan Peng.

Gaps in Hardy Fields

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Hardy fields are an algebraic setting for a tame part of asymptotic analysis. In this talk, after an introduction into this area, I will explain what we know and don't know about gaps in Hardy fields, with particular focus on analytic Hardy fields, which are those that mainly arise in practice. (Joint work with L. van den Dries and J. van der Hoeven.)

Forcing Axioms, (∗), and the Continuum Problem

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In this talk I will survey old and new results concerning the role that forcing axioms, and other principles in their region, play in the ongoing search for natural axioms supplementing ZFC.

Synthetic Computability Theory without Choice

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Mathematical practice in most areas of mathematics is based on the assumption that proofs could be made fully formal in a chosen foundation in principle. This assumption is backed by decades of formalising various areas of mathematics in various proof assistants and various foundations. An area that has been largely neglected for computer-assisted and machine-checked proofs is computability theory. This is due to the fact that making computability theory (and its sibling complexity theory) formal is several orders of magnitude more involved than formalising other areas of mathematics, due to the – citing Emil Post – "forbidding, diverse and alien formalisms in which this [...] work of Gödel, Church, Turing, Kleene, Rosser [...] is embodied.". For instance, there have been various approaches of formalising Turing machines, all to the ultimate dissatisfaction of the respective authors, and none going further than constructing a universal machine and proving the halting problem undecidable. Professional computability theorist and teachers of computability theory thus rely on the informal Church Turing thesis to carry out their work and only argue the computability of described algorithms informally.

A way out was proposed in the 1980s by Fred Richman and developed during the last decade by Andrej Bauer: Synthetic computability theory, where one assumes axioms in a constructive foundation which essentially identify all (constructively definable) functions with computable functions. A drawback of the approach is that assuming such an axiom on top of the axiom of countable choice - which is routinely assumed in this branch of constructive mathematics and computable analysis - is that the law of excluded middle, i.e. classical logic, becomes invalid. Computability theory is however dedicatedly classical: Almost all basic results are presented by appeal to classical axioms and even the full axiom of choice.

We observe that a slight foundational shift rectifies the situation: By basing synthetic computability theory in the Calculus of Inductive Constructions, the type theory underlying amongst others the Coq proof assistant, where countable choice is independent and thus not provable, axioms for synthetic computability are compatible with the law of excluded middle.

I will give an overview over a line of research investigating a synthetic approach to computability theory in constructive type theory, discussing, if time allows, suitable axioms, a Coq library of undecidability proofs, results in the theory of reducibility degrees, a synthetic definition of Kolmogorov complexity, constructive reverse analysis of theorems, and synthetic oracle computability.

Parts of results are in collaboration with Dominik Kirst, Gert Smolka, Felix Jahn, Fabian Kunze, Nils Lauermann, Niklas Mück, Haoyi Zeng, and the contributors of the Coq Library of Undecidability Proofs.

Broad Infinity and Generation Principles

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We introduce Broad Infinity, a new set-theoretic axiom scheme that may be considered plausible. It states that three-dimensional trees whose growth is controlled by a specified class function form a set. Such trees are called "broad numbers".

Assuming the axiom of choice, or at least the weak version known as WISC, we see that Broad Infinity is equivalent to Mahlo's principle, which states that the class of all regular limits is stationary. Broad Infinity also yields a convenient principle for generating a subset of a class using a "rubric" (family of rules). This directly gives the existence of Grothendieck universes, without requiring a detour via ordinals.

In the absence of choice, Broad Infinity implies that the derivations of elements from a rubric form a set. This yields the existence of Tarski-style universes.

Additionally, we reveal a pattern of resemblance between "Wide" principles, that are provable in ZFC, and "Broad" principles, that go beyond ZFC.

Incompleteness Theorems for Observables in General Relativity

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One of the biggest open problems in mathematical physics has been the problem of formulating a complete and consistent theory of quantum gravity. Some of the core technical and epistemological difficulties come from the fact that General Relativity (GR) is, fundamentally, a geometric theory and, as such, it ought to be 'generally covariant,' i.e., invariant under change of coordinates by the arbitrary diffeomorphism of the ambient manifold. The Problem of Observables is a famous instance of the difficulties that general covariance brings into quantization: no non-trivial diffeomorphism-invariant quantity has ever been reported on the collection of all spacetimes. It turns out that there is a good reason for this. In this talk, I will present my recent joint work with Marios Christodoulou and George Sparling, where we employ methods from Descriptive Set Theory (DST) in order to show that, even in the space of all vacuum solutions, no complete observables for full GR can be Borel definable. That is, the problem of observables is to 'analysis' what the Delian problem is to 'straightedge and compass.'

TUTORIAL
Degrees of Unsolvability: A Realizability-theoretic Perspective

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The theories of degrees of unsolvability and realizability interpretation both have long histories, having both been born in the 1940s. S. C. Kleene was a key figure who led the development of both theories. Despite having been developed by the same person, there seems to have been little deep mixing of these theories until recently. In this tutorial, we will reconstruct the theory of degrees of unsolvability from the perspective of realizability theory.

Formalising Mathematics with Proof Assistants

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Part I: Formalising mathematics with proof assistants

The first part of this tutorial will involve a general introduction to the area of formalisation of mathematics using proof assistants (interactive theorem provers). I will discuss the state of the art and potential of the area.

Part II: Introduction to the proof assistant Isabelle/HOL & bonus example: Aristotle's Assertoric Syllogistic in Isabelle/HOL

During the second part, I will give some practical information for beginners on getting started with the proof assistant Isabelle/HOL. As an example, I will present a formalisation of Aristotle's Assertoric Syllogistic in Isabelle/HOL.

Special Session: Philosophy

Upper Logicism

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A very natural higher order logic, 'Classicism', can be axiomatized by adding the rule of substitution of logical equivalents to a basic higher order logic (comprising standard classical rules for connectives, quantifiers, and identity, together with lambda conversion rules). Perhaps its most controversial theorem is 'Broad Necessitism', according to which for every thing (in a given type), the proposition that there is something identical to that thing is identical to a tautology. For those who want to avoid this result, it is natural to retreat to 'Free Classicism', the variant of Classicism in which the quantifier rules are weakened to those of free logic. Within Free Classicism, we develop a general theory of what it is for a property of properties of things of some given type to be a (universal, perhaps-restricted) quantifier, as well as a notion of absolute unrestrictedness for quantifiers, which is uniquely instantiated (in a type) if instantiated at all. Using these tools we introduce a central question facing Free Classicists: are there any absolutely unrestricted quantifiers, and if not, is being an absolutely unrestricted quantifier even possible (in the sense of not being identical to a contradictory property)? We present an argument for the claim that there are absolutely unrestricted quantifiers, and a further argument, based on metasemantic considerations, that if there are such quantifiers, Classicism is true.

This talk will be based on a coauthored paper. The other authors are Andrew Bacon (USC), Peter Fritz (UCL), and Ethan Russo (NYU).

Special Session: Philosophy

Upper Logicism

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Russell strived to defend the view that arithmetic is nothing but logic by conceiving of the natural numbers as the finite cardinalities. The received view is that Russell's logicist project has failed. His reduction of arithmetic to logic presupposed the axiom of infinity's logicality. But the axiom of infinity doesn't appear to be a logical truth, as Russell himself acknowledged.

In this talk I present and partially defend Upper Logicism - a neoRussellian form of logicism based on higher-order modal logic. As I'll show, among its virtues is the fact that the Upper logicist reduction of arithmetic to logic does not rely on the axiom of infinity's truth or logicality. I'll conclude by comparing Upper Logicism with other, seemingly related, philosophies of arithmetic, and sketching how to extend it to a logicist reduction of set theory.

Special Session: Philosophy

Diagonalization and Paradox

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We aim to clarify the role of diagonalization in the derivation of important limitative results in higher-order logic. A familiar diagonal argument is generally involved in common derivations of the inconsistency of Frege's Axiom V and the Russell-Myhill theorem. These observations are often given a cardinality gloss, which presuppose a measure of impredicativity. We will look at further limitative results underwritten by the diagonal argument even in the presence of predicative strictures. These observations will place additional constraints on consistent implementations of structured views of propositions, even though they have nothing to do with cardinality.

Special Session
In Memoriam Thanases Pheidas

Decidability results of subtheories of polynomial rings and formal power series

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Hilbert's tenth problem (the tenth of the famous list of problems Hilbert proposed in 1900) was:

> H10($\mathbb{Z}$): Find a procedure (in modern terminology: an algorithm) which determines (in a finite number of steps) whether an arbitrary polynomial (for any degree and any number of variables), with integer coefficients, has or does not have integer zeroes.

When Hilbert proposed this problem in 1900, the notion of an algorithm was not yet formalized. The theory of recursive functions was developed about 30 years later. Hilbert's tenth problem was answered negatively by Y. Matiyasevich in 1970, after work by M. Davis, H. Putnam and J. Robinson. In modern terminology, the positive existential theory of the ring $\mathbb{Z}$ of the rational integers is undecidable.

Since then a number of similar problems have been solved over other domains of mathematical interest. However, some others remain open. For example, although the theory of a ring of power series $F[[z]]$ in a variable z over a field F with a decidable theory is decidable if the characteristic of F is zero, the similar problem for F of positive characteristic p is open problem.

Thus, it is interesting at least to produce results which produce algorithms for deciding certain sub-theories of the full ring theory of $F[[z]]$, for F of positive characteristic. In this talk, we will survey decidability and undecidability results of the structures and substructures of polynomial rings and rings of formal power series.

Then we will focus on the structure of addition and localized divisibility in polynomial rings and the corresponding rings of formal power series and inter relations. In particular, we will show that a ring of polynomials over a prime field is an elementary substructure of the corresponding ring of formal power series in the language of addition, localized divisibility, equality and the constants 0 and 1.

Finally, the theories of these structures admit elimination of quantifiers. In addition, we will present some theorems that relate the positive and the zero characteristic cases.

Special Session
In Memoriam Thanases Pheidas

The Mathematics of Thanasis Pheidas: Explained for an audience that includes non-mathematicians

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Anticipating that a sizable portion of the audience will be non-mathematicians, I will give a presentation not of the novel technical results by Thanasis but of the wider area where these results belong and their significance aiming on the one hand to be comprehensible to the non-technically trained listeners and on the other to hold the interest of fellow logicians, especially the younger ones and students, working in other areas.

Special Session
In Memoriam Thanases Pheidas

Reflections on the work of T. Pheidas

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Hilbert's tenth problem is to devise an algorithm which decides whether a given polynomial equation in many variables has a solution in the integers. The celebrated DPRM theorem (after Davis, Putnam, J. Robinson and Matiyasevich) shows that in fact no such algorithm exists. It is also natural to consider variants of Hilbert's tenth problem, where one seeks for solutions in domains other than the integers. We will survey some of the most important results in that direction, which were obtained by Pheidas.

Special Session
In Memoriam Thanases Pheidas

The journey of Thanases Pheidas in the realm of analytic functions

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Though Thanases has worked in all the major open problems in the area of Hilbert's Tenth Problem, I believe that analytic functions had a special place in his heart. I will tell some things that I know about his contributions there, and how his 1995 failure to solving the problem for entire functions led him to a series of beautiful ideas.

Satisfiability-checking of modal logic with recursion via translations and tableaux*

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Abstract

In this talk proposal, we discuss a method based on tableau derivations that may not terminate, for achieving decidability and upper complexity bounds for a general family of modal logics with recursion. We show how to use this method to prove the decidability of certain modal logics that do not have a finite model property.

1 Introduction

In this talk we will be studying a family of multi-modal logics with fixed-point operators that are interpreted over restricted classes of Kripke models. The abstract is aimed to be a short summary of the methods and results that the authors recently gave in [2]. One can consider these logics as extensions of the usual multi-agent logics of knowledge and belief [10] by adding recursion to their syntax or of the μ-calculus [17] by interpreting formulas over different classes of frames and thus giving an epistemic interpretation to the modalities. We are concerned with the complexity of the satisfiability problem for these logics. Namely, given a formula φ, how much time it takes to determine whether there exists a model $\mathcal{M} \models \varphi$.

Modal logic comes in several variations [6]. Some of these, such as multi-modal logics of knowledge and belief [10], are of particular interest to Epistemology and other application areas. Semantically, the classical modal logics used in epistemic (but also other) contexts result from imposing certain restrictions on their models. On the other hand, the modal μ-calculus [17] can be seen as an extension of the smallest normal modal logic **K** with greatest and least fixed-point operators, νX and μX respectively. We explore the situation where one allows recursion (*i.e.* fixed-point) operators in a multi-modal language and imposes restrictions on the models.

Satisfiability for the μ-calculus is known to be EXP-complete [17]. For the modal logics between **K** and **S5** the problem is PSPACE-complete or NP-complete, depending on the presence of Negative Introspection [14,18]. We discuss the two main methods that we developed in [2] for proving complexity bounds for the satisfiability problem for different logics. Our first method is a *translation*, and it follows the natural intuition where a formula in one logic is projected to a formula in another logic, preserving its satisfiability. We demonstrate how this straightforward method can prove the known upper complexity bounds for modal logics without recursion.

We then present tableaux for the discussed logics, based on the ones by Kozen for the μ-calculus [17], and by Fitting and Massacci for modal logic [12,21]. For some of our logics with

*The work reported in this paper is supported by the project 'Mode(l)s of Verification and Monitorability' (MoVeMent) (grant no 217987) of the Icelandic Research Fund.

$[\![\mathtt{tt},\rho]\!] = W \quad [\![\mathtt{ff},\rho]\!] = \emptyset \quad [\![p,\rho]\!] = \{s \mid p \in V(s)\}$	$[\![\neg p,\rho]\!] = W \setminus [\![p,\rho]\!] \quad [\![X,\rho]\!] = \rho(X)$
$[\![[\alpha]\varphi,\rho]\!] = \{s \mid \forall t.\ sR_\alpha t \text{ implies } t \in [\![\varphi,\rho]\!]\}$	$[\![\varphi_1 \wedge \varphi_2,\rho]\!] = [\![\varphi_1,\rho]\!] \cap [\![\varphi_2,\rho]\!]$
$[\![\langle\alpha\rangle\varphi,\rho]\!] = \{s \mid \exists t.\ sR_\alpha t \text{ and } t \in [\![\varphi,\rho]\!]\}$	$[\![\varphi_1 \vee \varphi_2,\rho]\!] = [\![\varphi_1,\rho]\!] \cup [\![\varphi_2,\rho]\!]$
$[\![\mu X.\varphi,\rho]\!] = \bigcap\{S \mid S \supseteq [\![\varphi,\rho[X \mapsto S]]\!]\}$	$[\![\nu X.\varphi,\rho]\!] = \bigcup\{S \mid S \subseteq [\![\varphi,\rho[X \mapsto S]]\!]\}$

Table 1: Semantics of formulas on model $\mathcal{M} = (W, R, V)$, which we omit from the notation.

axiom 5 ($\langle\alpha\rangle\varphi \to [\alpha]\langle\alpha\rangle\varphi$), or B ($\varphi \to [\alpha]\langle\alpha\rangle\varphi$), the tableaux may not terminate, as these logics have no finite-model property [8]. We give a general satisfiability-preserving translation from each logic to the μ-calculus, using our tableaux, which describes a tableau branch with an exponentially larger μ-calculus formula, establishing a 2EXP-upper bound for all our logics.

2 Modal logics with recursion

We consider formulas constructed from the following grammar:

$$\varphi,\psi ::= p \quad |\ \neg p \quad |\ \mathtt{tt} \quad |\ \mathtt{ff} \quad |\ X \quad |\ \varphi\wedge\psi \quad |\ \varphi\vee\psi \quad |\ \langle\alpha\rangle\varphi \quad |\ [\alpha]\varphi \quad |\ \mu X.\varphi \quad |\ \nu X.\varphi,$$

where X comes from a countably infinite set of logical (or fixed-point) variables, LVar, α from a finite set of agents, Ag, and p from a finite set of propositional variables, PVar.

We interpret formulas on the states of a *Kripke model*. A Kripke model, or simply model, is a triple $\mathcal{M} = (W, R, V)$ where W is a nonempty set of states, $R \subseteq W \times \text{Ag} \times W$ is a transition relation, and $V : W \to 2^{\text{PVar}}$ determines the propositional variables that are true at each state. (W, R) is called a *frame*. We usually write $(u, v) \in R_\alpha$ or $uR_\alpha v$ instead of $(u, \alpha, v) \in R$.

Formulas are evaluated in the context of an *environment* $\rho : \text{LVar} \to 2^W$, which gives values to the logical variables. For an environment ρ, variable X, and set $S \subseteq W$, we write $\rho[X \mapsto S]$ for the environment that maps X to S and all $Y \neq X$ to $\rho(Y)$. The semantics for our formulas is given through a function $[\![-]\!]_{\mathcal{M}}$, defined in Table 1.

Without further restrictions, the resulting logic is the μ-calculus [17]. If $|\text{Ag}| = k \in \mathbb{N}^+$ and we allow no recursive operators and variables, we have the basic modal logic $\mathbf{K}_k$, and further restrictions on the frames can result in a variety of modal logics (see, for instance, [5]). We give names to the following frame conditions, or frame constraints, for an agent $\alpha \in \text{Ag}$:

- D**:** R_α is *serial* ($\forall s \exists t, sR_\alpha t$);
- T**:** R_α is *reflexive* ($\forall s, sR_\alpha s$);
- B**:** R_α is *symmetric*;
- 4**:** R_α is *transitive*;
- 5**:** R_α is *euclidean* (if $sR_\alpha t$ and $sR_\alpha r$, then $tR_\alpha r$).

Each frame condition x for agent α is associated with an axiom ax^x_α, such that whenever a model has condition x, every substitution instance of ax^x_α is satisfied in all its states (see [5,6,10]):

$\mathsf{ax}^D_\alpha = \langle\alpha\rangle\mathtt{tt}$; $\quad \mathsf{ax}^T_\alpha = [\alpha]p \to p$; $\quad \mathsf{ax}^B_\alpha = \langle\alpha\rangle[\alpha]p \to p$;

$\mathsf{ax}^4_\alpha = [\alpha]p \to [\alpha][\alpha]p$; and $\quad \mathsf{ax}^5_\alpha = \langle\alpha\rangle[\alpha]p \to [\alpha]p$.

We consider logics interpreted over models that satisfy a combination of these constraints for each agent. For each logic $\mathbf{L}$ and agent α, $\mathbf{L}(\alpha)$ is the single-agent logic that includes exactly the frame conditions that $\mathbf{L}$ has for α.

3 Translations for Recursion-free formulas

Translations are functions that transform each formula to another. We require that translations preserve satisfiability, allowing us to transfer decidability results among logics. We also require that they only increment the size of the formula by a polynomial factor. This is because we will use the translations to prove complexity bounds as well. Finally, we want translations to be compositional. Compositionality ensures that we can apply a sequence of translation steps, resulting in a composite translation that has the above good properties.

We fix an order to the frame conditions: $D, T, B, 4, 5$. We present a straightforward and uniform translation for recursion-free logics. Let $\overline{\mathsf{sub}}(\varphi) = \{\psi, \neg\psi \mid \psi \text{ is a subformula of } \varphi\}$.

Translation 3.1 (One-step Translation)**.** Let $A \subseteq \text{AG}$ and let x be one of the frame conditions. For every formula φ, let $d = md(\varphi)$ if $x \neq 4, 5$, and $d = md(\varphi)|\varphi|$, if $x = 4$ or $x = 5$. We define:

$$\mathsf{F}^x_A(\varphi) = \varphi \wedge \bigwedge_{k \leq d} \bigwedge_{\alpha_1 \in \text{AG}} [\alpha_1] \cdots \bigwedge_{\alpha_k \in \text{AG}} [\alpha_k] \Bigg(\bigwedge_{\substack{\psi \in \overline{\mathsf{sub}}(\varphi) \\ \alpha \in A}} \mathsf{ax}^x_\alpha[\psi/p] \Bigg).$$

Theorem 1. *Let $A \subseteq \text{AG}$, x be one of the frame conditions, and let $\mathbf{L}_1, \mathbf{L}_2$ be logics without recursion operators, such that $\mathbf{L}_1(\alpha) = \mathbf{L}_2(\alpha) + x$ when $\alpha \in A$, and $\mathbf{L}_2(\alpha)$ otherwise. Assume that $\mathbf{L}_2(\alpha)$ only includes frame conditions that precede x in the fixed order of frame conditions. Then, φ is $\mathbf{L}_1$-satisfiable if and only if $\mathsf{F}^x_A(\varphi)$ is $\mathbf{L}_2$-satisfiable.*

In the above we see that indeed, a translation preserves satisfiability and is only changing the size of the formula by a small factor. The compositionality gives us that for a logic $\mathbf{L}$ with multiple frame restrictions, one would have to apply these translations in series to acquire a formula that is satisfiable over general frames if and only if the original was $\mathbf{L}$-satisfiable.

Corollary 1. *If $\mathbf{L}$ has no recursion operators, then $\mathbf{L}$-satisfiability is in* PSPACE*.*

4 Tableaux and General Upper Bounds

Our tableaux are based on the ones given by Kozen in [17], and are extended similarly to the tableaux of Fitting [12] and Massacci [21] for taking into account the different conditions of modal logic. Intuitively, a tableau attempts to build a model that satisfies the given formula. When it needs to consider two possible cases, it branches, and thus it may generate several branches. Each successful branch represents a corresponding model.

Our tableaux use *prefixed formulas*, that is, formulas of the form $\sigma\ \varphi$, where $\sigma \in (\text{AG} \times L)^*$ and $\varphi \in L$; σ is the prefix of φ in that case, and we say that φ is prefixed by σ. We note that we separate the elements of σ with a dot. Furthermore, in the tableau prefixes, we write $\alpha\langle\psi\rangle$ to mean the pair $(\alpha, \psi) \in \text{AG} \times L$. Therefore, for example, $(\alpha, \phi)(\beta, \chi)(\alpha, \psi)$ is written $\alpha\langle\phi\rangle.\beta\langle\chi\rangle.\alpha\langle\psi\rangle$ as a tableau prefix. We say that prefix σ is α-flat when agent α has axiom 5 and $\sigma = \sigma'.\alpha\langle\psi\rangle$ for some ψ. Each prefix possibly represents a state in a model, and a prefixed formula $\sigma\ \varphi$ declares that φ is true in that state. The tableau rules appear in Table 2.

A tableau branch is propositionally closed when $\sigma\ \mathtt{ff}$ or both $\sigma\ p$ and $\sigma\ \neg p$ appear in the branch for some prefix σ. We define the dependence relation $\xrightarrow{X}$ on prefixed formulas in a tableau branch as $\chi_1 \xrightarrow{X} \chi_2$, if χ_2 was introduced to the branch by a tableau rule with χ_1 as its premise, and χ_1 is not of the form $\sigma\ Y$, where $X < Y$. If in a branch there is a $\xrightarrow{X}$-sequence

$$\frac{\sigma\ \pi X.\varphi}{\sigma\ \varphi}\,(\mathsf{fix}) \quad \frac{\sigma\ X}{\sigma\ \mathsf{fx}(X)}\,(\mathsf{X}) \quad \frac{\sigma\ [\alpha]\varphi}{\sigma.\alpha\langle\psi\rangle\ \varphi}\,(\mathsf{B}) \quad \frac{\sigma\ \langle\alpha\rangle\varphi}{\sigma.\alpha\langle\varphi\rangle\ \varphi}\,(\mathsf{D}) \quad \frac{\sigma\ [\alpha]\varphi}{\sigma.\alpha\langle\varphi\rangle\ \varphi}\,(\mathsf{d})$$

$$\frac{\sigma\ [\alpha]\varphi}{\sigma.\alpha\langle\psi\rangle\ [\alpha]\varphi}\,(\mathsf{4}) \quad \frac{\sigma.\alpha\langle\psi\rangle\ [\alpha]\varphi}{\sigma\ \varphi}\,(\mathsf{b}) \quad \frac{\sigma\ [\alpha]\varphi}{\sigma\ \varphi}\,(\mathsf{t}) \quad \frac{\sigma.\alpha\langle\psi\rangle\ [\alpha]\varphi}{\sigma\ [\alpha]\varphi}\,(\mathsf{b4})$$

where, for rules (B) and (4), $\sigma.\alpha\langle\psi\rangle$ already appears in the branch; for (D), σ is not α-flat.

$$\frac{\sigma.\alpha\langle\psi\rangle\ [\alpha]\varphi}{\sigma\ [\alpha]\varphi}\,(\mathsf{B5}) \quad \frac{\sigma.\alpha\langle\psi\rangle\ \langle\alpha\rangle\varphi}{\sigma.\alpha\langle\psi\rangle.\alpha\langle\varphi\rangle\ \varphi}\,(\mathsf{D5}) \quad \frac{\sigma.\alpha\langle\psi\rangle\ [\alpha]\varphi}{\sigma.\alpha\langle\psi'\rangle\ [\alpha]\varphi}\,(\mathsf{B55}) \quad \frac{\sigma.\alpha\langle\psi\rangle.\alpha\langle\psi'\rangle\ \langle\alpha\rangle\varphi}{\sigma.\alpha\langle\psi\rangle.\alpha\langle\varphi\rangle\ \varphi}\,(\mathsf{D55})$$

where, for rule (B55), $\sigma.\alpha\langle\psi'\rangle$ already appears in the branch; for rule (D5), σ is not α-flat, and $\sigma\ \langle\alpha\rangle\varphi$ does not appear in the branch; for rule (D55), $\sigma\ \langle\alpha\rangle\varphi$ does not appear in the branch.

Table 2: The tableau rules for $\mathbf{L} = \mathbf{L}_n^\mu$. The propositional cases are omitted.

where X is a least fixed-point and appears infinitely often, then the branch is called fixed-point-closed. A branch is closed when it is either fixed-point-closed or propositionally closed; if it is not closed, then it is called open.

Theorem 2 (Soundness and Completeness of $\mathbf{L}_k^\mu$-Tableaux, [1])**.** *For every formula φ and logic $\mathbf{L}$, φ has a maximal $\mathbf{L}$-tableau with an open branch if and only if φ is $\mathbf{L}$-satisfiable.*

Although for many of our logics, we can give sound, complete, and terminating tableaux, in general it is possible for a tableau to be non-terminating. Moreover, some of our logics do not have a finite model property. To give a general upper bound for the satisfiability problem for our family of modal logics with recursion, we devised the following strategy. From the above, we have, possibly non-terminating, sound and complete tableaux for all our logics. Moreover, we know that μ-calculus satisfiability can be decided in exponential time. Therefore, our idea is to encode the tableaux themselves as μ-calculus formulas, interpreted over arbitrary frames. These formulas assert that a satisfying model encodes an open branch. The total overhead of this construction was that the initial tableaux constructed from a formula is exponential larger from it, and the final satisfiability checking for the formula that represented this tableaux costs exponential time. Therefore the resulting complexity upper bound is a double exponential.

We avoid presenting the full extensive construction of the formula that describes the tableau branch. We present the idea in Figure 1. The reader can read [2] for more details. A summary of our results from [2] can be seen in Table 3. The highlighted result is the ones that occurred due to the final idea of translating tableaux into formulas.

Theorem 3. *$\mathbf{L}$-satisfiability is in* 2EXP*.*

5 Conclusion and future work

We presented a simple translation method to prove the known PSPACE upper bound for the complexity of satisfiability for all logics without recursion. For the ones with recursion, we presented sound and complete tableaux, which in turn we encoded as μ-calculus formulas, thus proving that satisfiability is in 2EXP for all our logics.

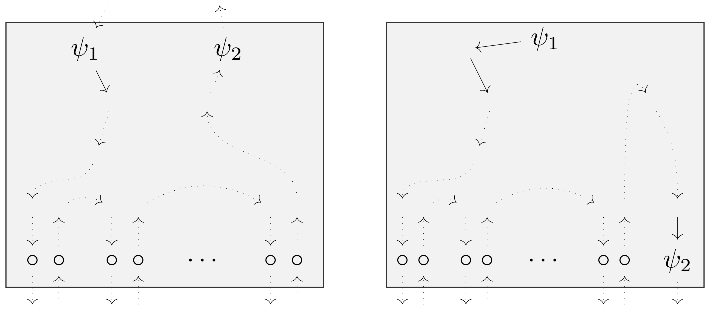

Figure 1: A finite $\xrightarrow{X}$-path from $\sigma\ \psi_1$ to $\sigma\ \psi_2$ may visit other tableau prefixes, and an infinite $\xrightarrow{X}$-path from $\sigma\ \psi_1$ may include finite segments that visit other prefixes, before continuing with an infinite path from a formula ψ_2. Each square area represents a tableau prefix and the part of an infinite $\xrightarrow{X}$-path that visits this prefix. We can describe this local behavior with exponentially many propositional variables, and the infinite path with a μ-calculus formula.

# agents	Restrictions on syntax/frames	Upper Bound	Lower Bound
≥ 2	frames with B or 5	2EXP	EXP-hard
	not B , 5	EXP	EXP-hard
	not 5, not μ. X	EXP	EXP-hard
1	with 5 (or B4)	NP	NP-hard
	with 4	PSPACE	PSPACE-hard
	Any other restrictions	EXP	EXP-hard

Table 3: The updated summary of the complexity of satisfiability checking for various modal logics with recursion. Our additional contributions from [2] are highlighted.

As Table 3 indicates, we currently do not have a tight complexity bound for the case of the multi-agent μ-calculus over symmetric or euclidean frames. The complexity of the model checking problem for the μ-calculus is an important open problem, known to have a quasi-polynomial time solution, but not known whether it is in P [7, 11, 16, 19, 20]. The problem does not depend on the frame restrictions of the particular logic, though one may wonder whether additional frame restrictions would help solve the problem more efficiently. Currently we are not aware of a way to use our translations to obtain such an improvement.

As, to the best of our knowledge, most of the logics described in this chapter have not been explicitly defined before, with notable exceptions such as [3, 8, 9], they also lack any axiomatizations and completeness theorems. We do expect the classical methods from [13, 17, 18] and others to work in these cases as well. However, it would be interesting and desirable to flesh out the details and see if there are any unexpected situations that arise.

Finally, given the importance of common knowledge for epistemic logic and the fact that it has been known that common knowledge can be thought of as a greatest fixed point already from [4, 15], we consider the logics we presented to be natural extensions of modal logic. We are interested in exploring what other natural concepts we can define with this enlarged language.

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The complexity of deciding characteristic formulae *

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1 Introduction

In concurrency theory, *characteristic formulae* serve as a bridge between model checking and preorder or equivalence checking. At an intuitive level, a characteristic formula provides a complete logical characterization of the behaviour of a process with respect to some notion of behavioural equivalence or preorder. For example, consider the widely used bisimulation equivalence relation [1]; Hennessy and Milner have shown in [2] that, under a mild finiteness condition, two processes are bisimilar if and only if they satisfy the same Hennessy-Milner logic (**HML**) formulae. Apart from its intrinsic theoretical interest, this seminal logical characterization of bisimilarity means that, when two processes are *not* bisimilar, there is always an **HML** formula that distinguishes between them. However, using the Hennessy-Milner theorem to show that two processes are bisimilar would involve verifying that they satisfy the same **HML** formulae and there are infinitely many of those. This is where characteristic formulae come into play. An **HML** formula φ is characteristic for process p, if every process q satisfies φ iff p and q are bisimilar. As a consequence, one can decide bisimulation equivalence between p and q by finding the characteristic formula $\chi(p)$ for p and checking whether $q \models \chi(p)$, that is a model-checking problem. Thus characteristic formulae allow one to reduce bisimilarity checking to model checking.

Conversely, Boudol and Larsen studied in [3] the problem of characterizing the collection of modal formulae for which model checking can be reduced to equivalence checking. See [4, 5, 6] for other contributions in that line of research. The aforementioned articles showed that characteristic formulae coincide with those that are consistent and prime. (A formula is prime if whenever it entails a disjunction $\varphi_1 \vee \varphi_2$, then it must entail φ_1 or φ_2.) Moreover, characteristic formulae with respect to the bisimulation relation coincide with the formulae that are consistent and complete, where a modal formula φ is complete, when for every modal formula ψ on the same propositional variables as φ, we can derive from φ either ψ or its negation. Note that in the case of bisimulation, a formula is prime iff it is complete. When one wants to reduce model checking to equivalence checking, the study of the complexity of identifying characteristic formulae modulo bisimilarity within (extensions of) **HML** is of relevance and has been addressed in [7, 8]. Typically, checking whether a formula is characteristic modulo bisimilarity has the same complexity as validity.

We described characteristic formulae using the example of bisimilarity, as it is the relation between processes that underlies the seminal Hennessy-Milner theorem and was used in much of the above-mentioned work. However there are a plethora of other preorder and equivalence relations that classify processes according to other possible behaviours; these and their logical characterizations have been extensively studied in concurrency theory—see e.g. [9, 10]. In this

*This work has been funded by the projects "Open Problems in the Equational Logic of Processes (OPEL)" (grant no 196050), "Mode(l)s of Verification and Monitorability" (MoVeMent) (grant no 217987), and "Learning and Applying Probabilistic Systems" (grant no. 206574-051) of the Icelandic Research Fund.of the Icelandic Research Fund.

work, we address the complexity of deciding and finding characteristic formulae with respect to four different preorders in van Glabbeek's branching-time spectrum, namely *simulation* ($\lesssim_S$), *complete simulation* ($\lesssim_{CS}$), *ready simulation* ($\lesssim_{RS}$), and *trace simulation* ($\lesssim_{TS}$) [9].

Our goal in this work is to study the complexity of determining whether a formula $\varphi \in \mathcal{L}_X$ is characteristic for some process p_φ modulo $\lesssim_X$, where $X \in \{S, CS, RS, TS\}$, or equivalently whether it is consistent and prime. For example, note that all consistent formulae in $\mathcal{L}_S$ that do not contain disjunctions are also prime. Thus, in this case deciding characteristic formulae reduces to deciding consistent formulae. However, when disjunctions are added to the language, the situation gets more complicated. For instance, formula $\langle a\rangle\mathbf{tt} \vee \langle b\rangle\mathbf{tt}$ is not prime, since $\langle a\rangle\mathbf{tt} \vee \langle b\rangle\mathbf{tt} \not\models \langle a\rangle\mathbf{tt}$ and $\langle a\rangle\mathbf{tt} \vee \langle b\rangle\mathbf{tt} \not\models \langle b\rangle\mathbf{tt}$, whereas formula $(\langle a\rangle\mathbf{tt} \vee \langle b\rangle\mathbf{tt}) \wedge \langle b\rangle\mathbf{tt}$ is prime.

In the sequel, we first give the necessary definitions and then we mention known complexity results on deciding preorders $\lesssim_S$, $\lesssim_{CS}$, and $\lesssim_{RS}$ respectively. We present our results on the complexity of deciding $\lesssim_{TS}$ and then, we state propositions and theorems establishing the complexity of identifying and finding characteristic formulae for the aforementioned preorders.

2 Definitions

Our semantic model is that of *labelled transition systems* (LTS) $\mathcal{S} = (P, A, \longrightarrow)$, where P is a set of states (or processes), A is a set of actions and $\longrightarrow \subseteq P \times A \times P$ is a transition relation on processes. We write $p \xrightarrow{a} q$ instead of $(p, a, q) \in \longrightarrow$. We say that a state p is deadlocked iff it has no outgoing transition. In this work, we consider finite LTSs.

For $X \in \{S, CS, RS, TS\}$, the preorder $\lesssim_X$ is the largest relation over the set of processes satisfying the following conditions for every p, q.

1. **Simulation (S):** $p \lesssim_S q \Leftrightarrow$ for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \lesssim_S q'$.
2. **Complete simulation (CS):** $p \lesssim_{CS} q \Leftrightarrow$
   (a) for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \lesssim_{CS} q'$, and
   (b) p is deadlocked iff q is deadlocked.
3. **Ready simulation (RS):** $p \lesssim_{RS} q \Leftrightarrow$
   (a) for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \lesssim_{RS} q'$, and
   (b) the initial sets of actions of p and q coincide. (The set of initial actions of a state is the collection of actions that label its outgoing transitions.)
4. **Trace simulation (TS):** $p \lesssim_{TS} q \Leftrightarrow$
   (a) for all $p \xrightarrow{a} p'$ there exists some $q \xrightarrow{a} q'$ such that $p' \lesssim_{TS} q'$, and
   (b) the sets of traces of p and q coincide. (The set of traces of p is the set of all possible sequences of actions that can be observed by executing p.)

It is well-known that $\lesssim_{TS} \subsetneq \lesssim_{RS} \subsetneq \lesssim_{CS} \subsetneq \lesssim_S$. We denote by $\mathcal{L}_S$, $\mathcal{L}_{CS}$, $\mathcal{L}_{RS}$, and $\mathcal{L}_{TS}$ respectively, the fragments of **HML** that characterize these four preorders [9, 6]. For $X \in \{S, CS, RS, TS\}$, $\mathcal{L}_X$ is defined to be the set of formulae given by the corresponding grammar as follows:

1. $\mathcal{L}_S$: $\varphi_S ::= \mathbf{tt} \mid \mathbf{ff} \mid \varphi_S \wedge \varphi_S \mid \varphi_S \vee \varphi_S \mid \langle a\rangle\varphi_S$.
2. $\mathcal{L}_{CS}$: $\varphi_{CS} ::= \mathbf{tt} \mid \mathbf{ff} \mid \varphi_{CS} \wedge \varphi_{CS} \mid \varphi_{CS} \vee \varphi_{CS} \mid \langle a\rangle\varphi_{CS} \mid \mathbf{0}$, where $\mathbf{0} = \bigwedge_{a \in A}[a]\mathbf{ff}$.

3. $\mathcal{L}_{RS}$: $\varphi_{RS} ::= \ \mathbf{tt} \ | \ \mathbf{ff} \ | \ \varphi_{RS} \wedge \varphi_{RS} \ | \ \varphi_{RS} \vee \varphi_{RS} \ | \ \langle a \rangle \varphi_{RS} \ | \ [a]\mathbf{ff}$.
4. $\mathcal{L}_{TS}$: $\varphi_{TS} ::= \ \mathbf{tt} \ | \ \mathbf{ff} \ | \ \varphi_{TS} \wedge \varphi_{TS} \ | \ \varphi_{TS} \vee \varphi_{TS} \ | \ \langle a \rangle \varphi_{TS} \ | \ \psi_{TS}$,
   $\psi_{TS} ::= \mathbf{ff} \ | \ [a]\psi_{TS}$

Truth in an LTS $\mathcal{S} = (P, A, \longrightarrow)$ is defined through relation $\models$ in the standard way. In particular,

- $p \models \langle a \rangle \varphi$ iff there is some $p \xrightarrow{a} q$ such that $q \models \varphi$ and
- $p \models [a]\varphi$ iff for all $p \xrightarrow{a} q$ it is the case that $q \models \varphi$.

We say that φ is *true* or *satisfied* in p if $p \models \varphi$. An **HML** formula is *consistent* or *satisfiable* if it is satisfied in a process p.

$\mathcal{L}_X$ characterizes $\lesssim_X$, where $X \in \{S, CS, RS, TS\}$, in the following sense: for all p, q, $p \lesssim_X q$ iff for every $\varphi \in \mathcal{L}_X$, $p \models \varphi \implies q \models \varphi$.

3 Deciding preorders

Let $\lesssim \in \{\lesssim_S, \lesssim_{CS}, \lesssim_{RS}\}$. Given two finite processes p and q, deciding whether $p \lesssim q$ can be done in polynomial time [9]. To the best of our knowledge, the complexity of deciding the trace simulation preorder has not been examined yet. The following propositions state that deciding trace simulation is hard.

Proposition 1. *Deciding $\lesssim_{TS}$ on finite processes is* PSPACE*-complete under polynomial-time Turing reductions.*

Proposition 2. *Deciding $\lesssim_{TS}$ on finite loop-free processes is* coNP*-complete under polynomial-time Turing reductions.*

Note that we use polynomial-time oracle reductions instead of the more standard Karp reductions between decision problems. This means that deciding $\lesssim_{TS}$ on finite loop-free processes is also NP-hard under polynomial-time Turing reductions. Moreover, Proposition 2 implies that if $p \lesssim_{TS} q$ can be solved in polynomial time for some finite loop-free p, q, then P = NP.

In Propositions 1 and 2, hardness is established by showing that the trace equivalence of two processes can be decided by making two oracle calls to the problem of deciding the trace simulation preorder. Since deciding trace equivalence is PSPACE- and coNP-hard under Karp reductions on finite and finite loop-free processes respectively [11, 12], we obtain our hardness results. Membership in PSPACE can be easily proven for Proposition 1, whereas membership in coNP for Proposition 2 is based on an NP algorithm for deciding $\not\lesssim_{TS}$ on finite loop-free processes.

4 Deciding characteristic formulae modulo some preorder

Recall that a formula is characteristic iff it is consistent and prime. We determine the complexity of deciding whether a formula is characteristic modulo one of the preorders $\lesssim_S$, $\lesssim_{CS}$, and $\lesssim_{RS}$, by providing results about the satisfiability and primality problems for the respective logics.

Theorem 3. *Let $\mathbf{\Lambda}$ be one of the modal logics $\mathcal{L}_S$ and $\mathcal{L}_{CS}$. Given $\varphi \in \mathbf{\Lambda}$, deciding whether φ is satisfiable and prime is in* P.

Theorem 4.

(a) Let $|Act| = k$, where k is a constant. Given $\varphi \in \mathcal{L}_{RS}$, deciding whether φ is satisfiable and prime is in P.

(b) Let $|Act|$ be unbounded. Satisfiability in $\mathcal{L}_{RS}$ is NP*-complete, whereas primality in $\mathcal{L}_{RS}$ is* coNP*-complete.*

Polynomial-time complexity of the satisfiability problem in Theorems 3 and 4(a) is proven by a uniform algorithm that can be appropriately adjusted in each case. For primality in $\mathcal{L}_S$, there are rules that allow us to check whether a given formula φ is prime by checking the relationship between polynomially many subformulae of φ. In conclusion, the problem can be reduced to the reachability problem in an alternating graph, the nodes of which represent tuples of φ's subformulae. This algorithm can be extended to solve primality in $\mathcal{L}_{CS}$ and $\mathcal{L}_{RS}$ with a bounded action set. We also obtain the following corollary.

Corollary 5. *Let $\mathbf{\Lambda}$ be either $\mathcal{L}_S$, $\mathcal{L}_{CS}$, or $\mathcal{L}_{RS}$ with a bounded action set.*

(a) Given a characteristic formula $\varphi \in \mathbf{\Lambda}$, there is a polynomial-time algorithm that outputs a process p, for which φ is characteristic within $\mathbf{\Lambda}$.

(b) Given $\varphi \in \mathbf{\Lambda}$ and process p, verifying whether φ is characteristic within $\mathbf{\Lambda}$ for p is in P.

5 Finding characteristic formulae modulo some preorder

Given a process p, the problem of constructing the characteristic formula for p has been studied for a variety of preorders and equivalences [13, 4, 14, 15]. To resolve the complexity of the problem we consider two different ways of representing formulae and measuring their size. Given a formula φ, the first approach is to write φ explicitly and define its size to be equal to the number of symbols that appear in φ as above; the second one involves representing φ using recursive equations called declarations, and defining its declaration-size as the number of required declarations. We denote the former by $|\varphi|$ and the latter by $\mathrm{decl}(\varphi)$. For example, formula $\varphi_2 = \langle a\rangle(\langle a\rangle\mathbf{tt} \wedge \langle b\rangle\mathbf{tt}) \wedge \langle b\rangle(\langle a\rangle\mathbf{tt} \wedge \langle b\rangle\mathbf{tt})$ has size $|\varphi_2| = 13$ and declaration-size $\mathrm{decl}(\varphi_2) = 3$, as it can be represented by the equations $\varphi_2 = \langle a\rangle\varphi_1 \wedge \langle b\rangle\varphi_1$, $\varphi_1 = \langle a\rangle\varphi_0 \wedge \langle b\rangle\varphi_0$, and $\varphi_0 = \mathbf{tt}$. The following propositions hold.

Proposition 6. *Let $\mathbf{\Lambda}$ be one of the modal logics $\mathcal{L}_S$, $\mathcal{L}_{CS}$, $\mathcal{L}_{RS}$ with a bounded action set. Given a finite loop-free process p, finding the characteristic formula $\chi(p)$ for p within $\mathbf{\Lambda}$ is* NP*-hard under polynomial-time Turing reductions, if $\chi(p)$ is explicitly written.*

Proposition 7. *Let $\mathbf{\Lambda}$ be one of the modal logics $\mathcal{L}_S$, $\mathcal{L}_{CS}$, $\mathcal{L}_{RS}$. Given a finite loop-free process p, finding the characteristic formula $\chi(p)$ for p within $\mathbf{\Lambda}$ is in* P*, if $\chi(p)$ is given as a set of declarations.*

For example, consider process p_2 of Figure 1. Formula $\varphi_2 = \langle a\rangle(\langle a\rangle\mathbf{tt} \wedge \langle b\rangle\mathbf{tt}) \wedge \langle b\rangle(\langle a\rangle\mathbf{tt} \wedge \langle b\rangle\mathbf{tt})$ is characteristic for p_2 within $\mathcal{L}_S$. As we already mentioned, φ_2 can be given much more efficiently in declarative form than in explicit form. In general, the characteristic formula (within $\mathcal{L}_S$) φ_n for process p_n, where p_n has the form of p_2 and length n, is of exponential size in $|p_n|$, when φ_n is given in explicit form.

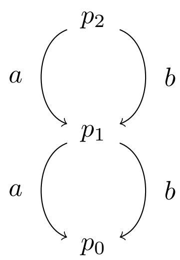

Figure 1: Process p_2 for which φ_2 is characteristic within $\mathcal{L}_S$.

Proposition 8. *Assume that for every finite loop-free process p, there is a characteristic formula within $\mathcal{L}_{TS}$ for p, denoted by φ_p, such that* $\mathrm{decl}(\varphi_p)$ *is polynomial in $|p|$ and every declaration is of polynomial size in $|p|$. Given a finite loop-free process p, if φ_p can be computed in polynomial time, then* $\mathsf{P} = \mathsf{NP}$.

Proposition 9. *Assume that the following two conditions are true:*

1. *For every finite loop-free process p, there is a characteristic formula within $\mathcal{L}_{TS}$ for p, denoted by φ_p, such that* $\mathrm{decl}(\varphi_p)$ *and every declaration are of polynomial size in $|p|$.*
2. *Given a finite loop-free process p and a formula φ in declarative form, deciding whether φ is characteristic within $\mathcal{L}_{TS}$ for p is in* NP.

Then $\mathsf{NP} = \mathsf{coNP}$.

Thus, when $\chi(p)$ is given as a set of declarations, we isolate a sharp difference between the complexity of finding $\chi(p)$ within any $\mathbf{\Lambda} \in \{\mathcal{L}_S, \mathcal{L}_{CS}, \mathcal{L}_{RS}\}$, and finding $\chi(p)$ within $\mathcal{L}_{TS}$.

6 Conclusions

Finally, we mention some problems that still remain open and whose solutions we are currently pursuing. First, we conjecture that for the trace simulation, deciding whether a formula is satisfiable is NP-complete, deciding primality of formulae is coNP-complete, whereas if we assume that $|A| = 1$, deciding both satisfiability and primality is in P. Yet another relevant problem is the complexity of deciding whether an **HML** formula φ is logically equivalent to a formula φ' in $\mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is one of $\mathcal{L}_S$, $\mathcal{L}_{CS}$, $\mathcal{L}_{RS}$, and $\mathcal{L}_{TS}$. Moreover, we want to address all the aforementioned problems for other relations in van Glabbeek's spectrum and over finite processes with loops.

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On the expressiveness of hyperlogics*

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Abstract

We compare the expressiveness between hyperlogics, i.e., logics interpreted over sets of traces, defined as extensions of LTL, FO, and the μ-calculus.

1 Introduction

Hyperlogics are a family of logics that started emerging 15 years ago. They were first suggested as a formalism rich enough to capture information flow security properties [5]. At their core, hyperproperties are extensions of properties of traces to properties of *sets of traces* (denoted T). Having properties of sets of traces captures situations from computer science, where a *set* of users (or executions) might exhibit some bad behavior, or might *together* assert some guarantee. The most popular hyperlogic is HyperLTL [4], the extension of LTL, which uses trace quantifiers and trace variables to refer to multiple traces. For example, the formula:

$$\forall\pi\forall\pi' G(a_\pi \equiv a_{\pi'}) \tag{1}$$

expresses that all traces must either satisfy a, or all traces must satisfy $\neg a$, at each spot. This property is trivially satisfied by any trace but can be violated when interpreted over sets.

An important question about logics of this type is whether they maintain (or somehow lift) language-theoretic, complexity, or expressive equivalence results from their non-hyper counterparts. For example, we know that every satisfiable LTL formula has a model that is an ultimately periodic trace [12]. On an even more fundamental level, Kamp's seminal theorem [9] (in the formulation due to Gabbay et al. [8]) states that LTL is expressively equivalent to first-order logic FO[<] over the natural numbers with order.

FO[$<, \mathbf{E}$], i.e. FO[<], equipped with the "equal level" arrity 2 predicate $\mathbf{E}$, was proposed by [7] to capture the expressive power of HyperLTL. This logic is essentially interpreted over multiple copies of the natural numbers with order, and thus the models of its sentences are sets of traces, just like with hyperlogics. Variables in FO[$<, \mathbf{E}$] are mapped to "places", i.e., pairs of a trace and an index in that trace, as opposed to a simple index in the case of FO[<]. $\mathbf{E}(x, y)$ holds only when the two quantified variables x, y are mapped to the same position of possibly different traces. For example, Property 1 is formulated as:

$$\forall x.\forall y.\mathbf{E}(x, y) \rightarrow (\mathbf{P}_a(x) \equiv \mathbf{P}_a(y)) \tag{2}$$

where $\mathbf{P}_a(x)$ is a unary predicate that encodes the occurrence of symbol a at position x. It turns out that this logic is strictly more expressive than HyperLTL [7]. Although the authors of that work do propose a logic (called HyperFO) that is expressively equivalent to HyperLTL by restricting FO[$<, \mathbf{E}$], there is still no temporal counterpart to the full FO[$<, \mathbf{E}$] logic.

*The work reported in this paper is supported by the project 'Mode(l)s of Verification and Monitorability' (MoVeMent) (grant no 217987) of the Icelandic Research Fund.

A property that is expressible in FO[$<$, **E**] but not HyperLTL, is: "there exists an $n \in \mathbb{N}$, such that $t(n) = a$, for all $t \in T$". This property is a *consensus property* (and very relevant to the context of hyperlogics), and it is also not expressible in HyperCTL* [3]. Thus, to produce a temporal equivalent of FO[$<$, **E**], one would have to look at more expressive logics. Such a logic could be the extension of μHML on hypertraces, which was recently studied by the authors and collaborators [1]. In this work, we discuss the spectrum of expressiveness between these three logics and prove that 1) the gap of expressivity between LTL and μHML is preserved in their hyper extensions, and that 2) FO[$<$, **E**] does not cover the full Hyper-μHML in expressiveness.

2 Preliminaries

Let AP denote the set of all atomic propositions. An atomic proposition a, where $a \in \mathsf{AP}$, expresses some fact about states. Thus, all the propositional information for a state is described by an *action* $\alpha \in \text{ACT} = 2^{\mathsf{AP}}$. TR stands for ACT^ω, the set of all traces. A *hypertrace* T is a subset of TR, and we denote with $\mathsf{HTrc} = 2^{\mathsf{TR}}$ the set of hypertraces. Let $t \in \mathsf{TR}$ be a trace. We use $t[i]$ to denote the element i of t, where $i \in \mathbb{N}$. Hence, $t[0]$ is the first element of t. We write $t[0, i]$ to denote the prefix of t up to and including element i, and $t[i, \infty]$ to denote the infinite suffix of t beginning with element i. We can also lift the suffix notation to hypertraces $T \in \mathsf{HTrc}$: $T[i, \infty] == \{t[i, \infty] \in \mathsf{TR} \mid t \in T\}$. In what follows, we consider formulas with trace variables and trace quantifiers. We will call a formula *closed* if a trace quantifier binds every occurrence of a trace variable.

HyperLTL We introduce here the logic HyperLTL as it was described originally in [4].

$$\begin{array}{rcllll} \psi & ::= & \exists \pi.\, \psi & \mid\ \forall \pi.\, \psi & \mid\ \varphi & \\ \varphi & ::= & a_\pi & \mid\ \neg\varphi & \mid\ \varphi \vee \varphi & \mid\ X\varphi \ \mid\ \varphi U \varphi \end{array}$$

True and false, written `tt` and `ff`, are respectively defined as $a_\pi \vee \neg a_\pi$ and $\neg\mathtt{tt}$. The satisfaction judgment for HyperLTL formulas is written $\Pi \models_T \psi$, where T is a set of traces, and $\Pi : \mathcal{V} \to \mathsf{TR}$ is a *trace assignment* (i.e., a *valuation*), which is a partial function mapping trace variables to traces in T. Let $\Pi[\pi \mapsto t]$ denote the same function as Π, except that π is mapped to t. One can think of these semantics in two layers: one for ψ as it is, and one for φ that only depends on Π. Satisfaction is defined as follows:

$$\begin{array}{lll} \Pi \models_T \exists\pi.\, \psi & \text{iff} & \text{there exists } t \in T : \Pi[\pi \mapsto t] \models_T \psi \\ \Pi \models_T \forall\pi.\, \psi & \text{iff} & \text{for all } t \in T : \Pi[\pi \mapsto t] \models_T \psi \\ \Pi \models_T a_\pi & \text{iff} & a \in \Pi(\pi)[0] \\ \Pi \models_T \neg\varphi & \text{iff} & \Pi \not\models_T \varphi \\ \Pi \models_T \varphi_1 \vee \varphi_2 & \text{iff} & \Pi \models_T \varphi_1 \text{ or } \Pi \models_T \varphi_2 \\ \Pi \models_T X\varphi & \text{iff} & \Pi[1,\infty] \models_T \varphi \\ \Pi \models_T \varphi_1 U \varphi_2 & \text{iff} & \text{there exists } i \geq 0 : \Pi[i,\infty] \models_T \varphi_2 \\ & & \quad \text{and for all } 0 \leq j < i \text{ we have } \Pi[j,\infty] \models_T \varphi_1 \end{array}$$

If $\Pi_\emptyset \models_T \varphi$ holds for the empty assignment $\Pi_\emptyset$, then T *satisfies* φ.

Hyper-μHML We present Hyper-μHML as a logic to specify hyperproperties. Hyper-μHML extends the linear-time interpretation of μHML [10, 11, 13] by allowing quantification over traces. We assume two disjoint and countably infinite sets Π and V of trace variables and recursion variables, ranged over by π and x, respectively.

Definition 1. *Formulae of Hyper-μHML are constructed as follows:*

$$\begin{aligned}\varphi ::= \mathsf{tt} \;\mid\; \mathsf{ff} \;\mid\; \varphi \wedge \varphi \;\mid\; \varphi \vee \varphi \;\mid\; \max x.\varphi \;\mid\; \min x.\varphi \;\mid\; x \\ \mid\; \exists \pi.\varphi \;\mid\; \forall \pi.\varphi \;\mid\; \pi = \pi \;\mid\; \pi \neq \pi \;\mid\; [a_\pi]\varphi \;\mid\; \langle a_\pi \rangle \varphi\end{aligned}$$

To help us simplify the definition of the semantics, we consider hypertraces of a fixed size k, and we identify hypertraces with k-tuples $T = (T(0), T(1), \ldots, T(k-1)) \in \mathsf{HTrc}_k = \mathsf{TR}^k$. The semantics of a Hyper-μHML formula φ is defined for each such k by exploiting two partial functions: $\rho\colon V \rightharpoonup 2^{\mathsf{HTrc}_k}$, that assigns a set of hypertraces of size k to all free recursion variables of φ, and $\sigma\colon \Pi \rightharpoonup \{0, 1, \ldots, k-1\}$, that assigns a position in each tuple T to each free trace variable of φ. The semantics is given by:

$$\begin{aligned}
&[\![\mathsf{tt}]\!]^\rho_\sigma = \mathsf{HTrc}_k \qquad [\![\mathsf{ff}]\!]^\rho_\sigma = \emptyset && [\![x]\!]^\rho_\sigma = \rho(x)\\
&[\![\varphi \wedge \varphi']\!]^\rho_\sigma = [\![\varphi]\!]^\rho_\sigma \cap [\![\varphi']\!]^\rho_\sigma && [\![\varphi \vee \varphi']\!]^\rho_\sigma = [\![\varphi]\!]^\rho_\sigma \cup [\![\varphi']\!]^\rho_\sigma\\
&[\![\max x.\psi]\!]^\rho_\sigma = \bigcup\{S \mid S \subseteq [\![\psi]\!]^{\rho[x\mapsto S]}_\sigma\} && [\![\min x.\psi]\!]^\rho_\sigma = \bigcap\{S \mid S \supseteq [\![\psi]\!]^{\rho[x\mapsto S]}_\sigma\}\\
&[\![\exists \pi.\varphi]\!]^\rho_\sigma = \bigcup_{i=0}^{k-1} [\![\varphi]\!]^\rho_{\sigma[\pi\mapsto i]} && [\![\forall \pi.\varphi]\!]^\rho_\sigma = \bigcap_{i=0}^{k-1} [\![\varphi]\!]^\rho_{\sigma[\pi\mapsto i]}\\
&[\![\pi = \pi']\!]^\rho_\sigma = \{T \in \mathsf{HTrc}_k \mid T(\sigma(\pi) = T(\sigma(\pi'))\} && [\![\pi \neq \pi']\!]^\rho_\sigma = \mathsf{HTrc}_k \setminus [\![\pi = \pi']\!]^\rho_\sigma\\
&[\![[a_\pi]\varphi]\!]^\rho_\sigma = \{T \mid \sigma(\pi)[0] = a \text{ implies } T[1,\infty] \in [\![\varphi]\!]^\rho_\sigma\}\\
&[\![\langle a_\pi\rangle\varphi]\!]^\rho_\sigma = \{T \mid \sigma(\pi)[0] = a \;\wedge\; T[1,\infty] \in [\![\varphi]\!]^\rho_\sigma)\}
\end{aligned}$$

Whenever φ is *closed*, the semantics is given by $[\![\varphi]\!]^\emptyset_\emptyset$, where $\emptyset$ denotes the partial function with empty domain, and we simply write $[\![\varphi]\!]$ instead of $[\![\varphi]\!]^\emptyset_\emptyset$. We use the standard notation $T \models \varphi$ to denote that the set of traces T satisfies φ (and similarly for $T \not\models \varphi$). As an example, consider the alphabet $\{a, b\}$. The property

$$\forall\pi.\, \mathrm{max\,x.}(\langle \mathrm{b}_\pi\rangle \mathrm{x} \vee (\exists \pi'.(\pi' \neq \pi) \wedge \langle \mathrm{a}_{\pi'}\rangle \mathrm{x})) \tag{3}$$

means that, for every trace, whenever there is an a, there is another trace that also has a.

The logic FO$[<, \mathbf{E}]$

Definition 2 (From [7])**.** *FO$[<, \mathbf{E}]$ is defined over the signature $\{\mathbf{E}, <\} \cup \{\mathbf{P}_a \mid a \in \mathsf{AP}\}$, i.e., with atomic formulas $x = y$, $x < y$, $\mathbf{E}(x, y)$, and $\mathbf{P}_a(x)$ for $a \in \mathsf{AP}$, and disjunction, conjunction, negation, and existential and universal quantification over elements.*

The semantics of this logic is the standard semantics of *FO* and comes in accordance with the semantics of FO$[<]$. We interpret FO$[<, \mathbf{E}]$ formulas over a set of traces $T \subseteq \textsc{Act}^\omega$ and an interpretation $I : \mathcal{V} \to T \times \mathbb{N}$, which assigns a tuple (t, n) to each variable x, with $t \in T$, $n \in \mathbb{N}$. Given a set of traces T, the operations $<, \mathbf{E}$, and $\mathbf{P}_a, a \in \mathsf{AP}$ are interpreted as:

- $<^T := \{((t, n), (t, n')) \mid t \in T \text{ and } n < n' \in \mathbb{N}\}$,
- $\mathbf{E}^T := \{((t, n), (t', n)) \mid t, t' \in T \text{ and } n \in \mathbb{N}\}$, and
- $\mathbf{P}^T_a := \{(t, n) \mid t \in T \text{ and } n \in \mathbb{N} \text{ and } a \in t(n)\}$.

3 Expressiveness comparisons

We start the comparison of expressiveness from the single-trace setting. Hyper-μHML is an extension of the linear-time interpretation of μHML. The logic μHML is expressive enough to strictly include LTL, and even CTL* in its usual, branching-time interpretation [2]. Quantification over traces and trace comparisons are allowed in any part of the formula, which means our syntax subsumes the syntax of HyperLTL, using straightforward translations. We show that the strictness of the inclusion of LTL in μHML is preserved for their hyper-trace extensions.

Theorem 1. *Hyper-μHML is strictly more expressive than HyperLTL.*

Proof. The simple inclusion follows from the embedding of LTL in μHML and the more liberal ability to quantify over traces. To demonstrate the strictness of this inclusion, we bring forward two arguments. First, we reference the work of Wolper in [14], which describes formulas of μHML that require an event a to occur at least in all even positions of a trace. The following μHML formula describes exactly this (over the set of actions a, b):

$$\varphi_e := \max x.([a]\langle a\rangle x \wedge [b]\langle a\rangle x) \tag{4}$$

Let φ_{h_e} be the formula that occurs if one adds an existential trace quantifier $\exists\pi$ at the beginning of φ_e, and replaces all modalities with π-indexed ones:

$$\varphi_{h_e} := \exists\pi.\max x.([a_\pi]\langle a_\pi\rangle x \wedge [b_\pi]\langle a_\pi\rangle x), \tag{5}$$

whose evaluation over singleton hypertraces coincides with the evaluation of φ_e. Assume now that a formula φ_{h-LTL} is expressively equivalent to φ_{h_e} over hypertraces. We would like to use this to extract an LTL formula that is expressively equivalent to φ_e. We cannot trivially claim that φ_{h-LTL} only contains a single quantifier $\exists\pi$. Instead, though, we know that over singleton hypertraces, say for $T = \{t_0\}$, $T \models \varphi_{h-LTL}$ iff $T \models \varphi_{h_e}$. Since T contains only a single trace, we know that all the trace variables in φ_{h-LTL} must be mapped to t_0. Consequently, all propositional variables that occur in φ_{h-LTL} must be mapped to t_0. Therefore, for this variable mapping, we get an LTL formula that expresses exactly that a trace (t_0) satisfies Wolper's property. We then replace all propositional variables with non-trace quantified ones and, remove all quantifiers, which brings us to plain LTL, and arrive at a contradiction. □

Remark 1. *In the proof above, we demonstrate that the property "there exists a trace for which a holds on at least all even positions" is not expressible in HyperLTL but is expressible in Hyper-μHML. The same argument can be repeated for any period k.*

Furthermore, we demonstrate that Hyper-μHML is more expressive than FO[$<, \mathbf{E}$]. Intuitively, one factor that gives Hyper-μHML significant expressive power is its ability to use quantifiers at any part of the syntax. This is also allowed in other temporal logics, such as, for example, $HyperCTL^*$. A key difference is that Hyper-μHML can nest quantifiers within a fixed-point operator. For example, we see that the property from Example 3 will potentially spawn an unbounded number of quantifiers due to the recursion unfolding caused by encountering a events. We argue that due to the ability to nest quantifiers at any point of our syntax, Hyper-μHML is more expressive than HyperLTL, and it can express properties that $HyperCTL^*$ and FO[$<, \mathbf{E}$] cannot.

Theorem 2. *Hyper-μHML contains properties not expressible in $HyperCTL^*$ and $FO[<, \mathbf{E}]$.*

Proof. For the first part, we refer the reader to the work of Bozzelli, Maubert, and Pinchinat [3], who show that the property "there is an $n \geq 0$ such that $a \neq t(n)$ for every $t \in T$" is not expressible in $HyperCTL^*$. In Hyper-μHML, this property is expressible (over the set of actions $\{a, b\}$) with the formula:

$$\min x.((\forall \pi \langle b_\pi \rangle \mathsf{tt}) \vee (\forall \pi' ([a_{\pi'}]x \wedge [b_{\pi'}]x))) \ . \tag{6}$$

In this formula, either all traces have b, or all traces take a step. Since this happens within the scope of a minimal fix-point, we get that to satisfy the formula, this process needs to terminate, and thus, we get exactly the property we wanted.

For the second part, we use Wolper's property φ_{h_e} (Property 5). Due to the expressive equivalence of LTL and FO[<] (from [8]), we can use a similar proof as for Theorem 1. The key is after projecting an FO[<, **E**] formula over uniset hypertraces to replace all occurrences of $\mathbf{E}(x, y)$ with $x = y$, as the two predicates coincide over such models. This leads us again to a property in FO[<] which expresses φ_E (Property 4, and we get a contradiction from the expressive equivalence of FO[<] and LTL (from [8]). □

4 Conclusion and future work

We have shown that the expressive power of Hyper-μHML is above HyperLTL, and possibly above (or at least incomparable with) FO[<, **E**]. We would like to extend Theorem 1 to fully characterize whether FO[<, **E**] is contained in Hyper-μHML. In case they are incomparable, it would suffice to produce a property in FO[<, **E**] that is not expressible in Hyper-μHML. Any properties we tried to that end, however, were not able to distinguish the two logics. Thus, we are left with the conjecture that Hyper-μHML subsumes FO[<, **E**]. At this point, we have partially produced an embedding of FO[<, **E**] into Hyper-μHML, and we believe one does exist. Finishing such an embedding would also imply that to produce a temporal equivalent of FO[<, **E**], one would need to find a middle ground between the syntax of Hyper-μHML and HyperLTL. On the other hand, a non-temporal equivalent of Hyper-μHML in the style of FO[<, **E**] could be MSO over hypertraces (and possibly with the equality predicate **E**).

In the future, we aim to answer the following questions. The first is to fully produce such an encoding and prove its correctness. The second is to find a temporal equivalent of FO[<, **E**]. We believe this is not a trivial question at all. For instance, increasing the quantification power of HyperLTL to allow non-normalized formulae would not be enough since HyperCTL*, which allows this, cannot express the consensus property. Moreover, we are interested in finding a classical logic characterization of Hyper-μHML. As we discussed, HyperLTL is expressively equivalent to a fragment of FO[<, **E**], as proven in [7], and as we have shown Hyper-μHML is not the temporal counterpart of FO[<, **E**]. We would like to fill this expressiveness gap. A good candidate for this could be some version of MSO over sets of traces. Indeed, there is work already done in this direction (see [6]), although so far, there seems to be no logic that can capture the properties 5, or 6. Finally, just like it is known that μHML corresponds to ω-regular languages, it would be interesting to find language-theoretic counterparts of HyperLTL, Hyper-μHML, and FO[<, **E**].

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Decomposition horizons and tame graph classes

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Abstract

Low treedepth decompositions are central to the structural characterizations of bounded expansion classes and nowhere dense classes, and the core of main algorithmic properties of these classes, including fixed-parameter (quasi) linear-time algorithms checking whether a fixed graph F is an induced subgraph of the input graph G. These decompositions have been extended to structurally bounded expansion classes and structurally nowhere dense classes, where low treedepth decompositions are replaced by low shrubdepth decompositions. In the emerging framework of a structural graph theory for hereditary classes of structures based on tools from model theory, it is natural to ask how these decompositions behave with the fundamental model theoretical notions of dependence (alias NIP) and stability.

Our first main result proves that the model theoretical notions of NIP and stable classes are transported by decompositions. Precisely: Let $\mathscr{C}$ be a hereditary class of graphs. Assume that for every p there is a hereditary NIP class $\mathscr{D}_p$ with the property that the vertex set of every graph $G \in \mathscr{C}$ can be partitioned into $N_p = N_p(G)$ parts in such a way that the union of any p parts induce a subgraph in $\mathscr{D}_p$ and $\log N_p(G) \in o(\log |G|)$. We prove that then $\mathscr{C}$ is (monadically) NIP. Similarly, if every $\mathscr{D}_p$ is stable, then $\mathscr{C}$ is (monadically) stable. Results of this type lead to the definition of decomposition horizons as closure operators. We establish some of their basic properties and provide several further examples of decomposition horizons.

Our second main result establishes that every stable hereditary graph class can be decomposed in such a manner into the much simpler classes of bounded shrubdepth, generalizing the initial result concerning low treedepth decompositions of nowhere dense classes.

1 Introduction and Previous Work

In the late 90's, Baker [2] introduced the shifting strategy, allowing a linear time approximation scheme for independent sets on planar graphs. The idea is to start a breadth-first search at a vertex v of a planar graph, which partitions the vertex set of the graph into layers $L_1, \dots, L_h$ and to fix an integer D. Then, for given $s \in [D]$, by deleting all the layers L_i with $i \equiv s \bmod D$, one gets a graph with treewidth bounded by $3D$, on which a maximum independent set can be found in linear time. Considering all the possible values of s, we obtain a $(1 + 1/D)$-approximate solution of the problem. Note that grouping the layers L_i with i in a same class modulo D yields a partition of the vertex set into D parts $V_0, \dots, V_{D-1}$ such that the union of any $p < D$ of them induces a subgraph with treewidth at most $3p + 4$.

This approach was further developed by DeVos et al. [7], who proved in particular that for every proper minor closed class of graphs $\mathscr{C}$ and every integer p, there exists an integer N_p such that the vertex set of every graph $G \in \mathscr{C}$ can be partitioned into N_p parts, each p of them inducing a subgraph with treewidth at most $p-1$.

This result has been further extended by two of the authors of the present paper in a characterization of both bounded expansion classes and nowhere dense classes. Before stating these results, recall that the *treedepth* of a graph G is the minimum depth of a rooted forest F, such that G is a subgraph of the closure of F (the graph obtained from F by adding edges between each vertex and its ancestors). With this definition, the characterization theorems read as follows.

Theorem 1.1 ([15]). *A class $\mathscr{C}$ has bounded expansion if and only if, for every parameter p, there is an integer N_p such that the vertex set of each graph $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph with treedepth at most p.*

Theorem 1.2 (see [16, 17]). *A class $\mathscr{C}$ is nowhere dense if and only if, for every parameter p and for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, such that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph with treedepth at most p.*

The notions of classes with bounded expansion and of nowhere dense classes are central to the study of classes of sparse graphs [16]. Note that the treewidth of a graph is bounded from above by its treedepth and hence by the result of DeVos et al. [7] and Theorem 1.1 every proper minor closed class has bounded expansion. Surprisingly, it appeared that for monotone classes of graphs, the notion of nowhere dense class of graphs coincides with fundamental dividing lines introduced in modern model theory [21]:

Theorem 1.3 ([1]). *For a monotone class of graphs $\mathscr{C}$, the following are equivalent:*

(1) $\mathscr{C}$ is nowhere dense;
(2) $\mathscr{C}$ is stable;
(3) $\mathscr{C}$ is monadically stable;
(4) $\mathscr{C}$ is NIP*;*
(5) $\mathscr{C}$ is monadically NIP.

For general hereditary classes of graphs, we do not have the collapse of the notions of stability, monadic stability, NIP, and monadic NIP stated in Theorem 1.3 for monotone classes. However, we still have the following collapses:

Theorem 1.4 ([5]). *A hereditary class of graphs is monadically* NIP *if and only if it is* NIP. *A hereditary class of graphs is monadically stable if and only if it is stable.*

The study of monadic stability and monadic NIP and their relations with first-order transductions [3] opened the way to the study of *structurally sparse* classes of graphs, that is of classes of graphs that are first-order transductions of classes of sparse graphs [6, 9, 10, 18–20]. Intuitively, a (first-order) transduction is a way to construct a set of target graphs from the vertex-colorings of a source graph by fixed first-order formulas, and, by extension, a new class of graphs from a given class of graphs.

Extending Theorem 1.1, first-order transductions of bounded expansion classes have been characterized in terms of low shrubdepth colorings. Recall the following high level characterization of classes with bounded shrubdepth [11, 12]: A class $\mathscr{D}$ has *bounded shrubdepth* if it is a transduction of a class of bounded depth rooted forests.

Theorem 1.5 ([10]). *A class $\mathscr{C}$ is a first-order transduction of a class with bounded expansion if and only if, for every parameter p, there is an integer N_p and a class $\mathscr{D}_p$ with bounded shrubdepth, such that the vertex set of each graph $G \in \mathscr{C}$ can be partitioned into at most N_p parts, each p of them inducing a subgraph in $\mathscr{D}_p$.*

Theorem 1.5 can be seen as a generalization of Theorem 1.1 as shrubdepth is a dense analogue of treedepth. On the other hand, only one direction of Theorem 1.2 has been extended to transductions of nowhere dense classes.

Theorem 1.6 ([8]). *Let $\mathscr{C}$ be a first-order transduction of a nowhere dense class. Then, for every parameter p there is a class $\mathscr{D}_p$ with bounded shrubdepth, such that for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, with the property that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$.*

Similar decompositions, where p parts induce a subgraph with bounded rankwidth were introduced in [13], while classes having such decompositions where p parts induce a subgraph with bounded linear rankwidth were discussed in [20]. However, it was not known whether such classes are monadically NIP. This question, which appears for instance in [20, Figure 3] and again in [19], will get a positive answer as a direct consequence of Theorem 2.1, which is our first main result.

The theoretical significance of first-order transductions of nowhere dense classes is witnessed by the following conjecture.

Conjecture 1.7 ([9]). *A class of graphs is monadically stable if and only if it is a first-order transduction of a nowhere dense class of graphs.*

Conjecture 1.7 can be refined as follows.

Conjecture 1.8. *For a hereditary class of graphs $\mathscr{C}$, the following properties are equivalent:*
(1) $\mathscr{C}$ is a first-order transduction of a nowhere dense class;
(2) $\mathscr{C}$ admits low shrubdepth decompositions with $n^{o(1)}$ parts;
(3) $\mathscr{C}$ is monadically stable;
(4) $\mathscr{C}$ is stable.

By Theorem 1.6, property (1) implies property (2). That property (2) implies property (3) will follow from our main result (Theorem 2.1). By Theorem 1.4, properties (3) and (4) are equivalent. Closing the chain of implications corresponds to Conjecture 1.7, which we now can decompose into two weaker statements: that property (3) implies property (2), and that property (2) implies property (1). Our second main result (Theorem 2.2) is that (3) implies (2).

2 Statement of the results

Our first main result show that NIP and stability are fixed under taking decompositions as in Theorems 1.1, 1.2, 1.5 and 1.6.

Theorem 2.1. *Let $\mathscr{C}$ be a hereditary graph class. Suppose that for every parameter p there is an* NIP *(resp. stable) class $\mathscr{D}_p$ such that for every graph $G \in \mathscr{C}$ there is an integer $N_p(G) \in |G|^{o(1)}$, with the property that the vertex set of G can be partitioned into at most $N_p(G)$ parts, each p of them inducing a subgraph in $\mathscr{D}_p$. Then $\mathscr{C}$ is* NIP *(resp. stable).*

In particular, this proves that property (2) implies property (4) in Conjecture 1.8, and so it follows that Conjectures 1.7 and 1.8 are equivalent. As mentioned after Theorem 1.6, this also proves that classes admitting low (linear) rankwidth decompositions are monadically NIP.

To place this theorem in a broader context, we introduce the notion of decomposition horizons. These seem to be of significant independent interest, and we prove some general properties. Theorem 2.1 can then be stated as "NIP and stability are decomposition horizons".

We define a *hereditary class property* to be a downset Π of hereditary graph classes, that is, a set of hereditary classes such that if $\mathscr{C} \in \Pi$ and $\mathscr{D}$ is a hereditary class with $\mathscr{D} \subseteq \mathscr{C}$, then $\mathscr{D} \in \Pi$.

Definition 1. *Let* Π *be a hereditary class property, let* $f : \mathbb{N} \to \mathbb{N}$ *be a non-decreasing function and let* p *be a positive integer. We say that a class* $\mathscr{C}$ *has an* f*-bounded* Π-decomposition *with parameter* p *if there exists* $\mathscr{D}_p \in \Pi$ *such that, for every graph* $G \in \mathscr{C}$*, there exists an integer* $N \le f(|G|)$ *and a partition* $V_1, \dots, V_N$ *of the vertex set of* G *with* $G[V_{i_1} \cup \dots \cup V_{i_p}] \in \mathscr{D}_p$ *for all* $i_1, \dots, i_p \in [N]$.

When f is a constant function, we say that $\mathscr{C}$ has a *bounded-size* Π*-decomposition* with parameter p; when f is a function with $f(n) = n^{o(1)}$, we say that $\mathscr{C}$ has a *quasi-bounded-size* Π*-decomposition* with parameter p. If a class $\mathscr{C}$ has a bounded-size (resp. a quasi-bounded-size) Π-decomposition with parameter p for each positive integer p, we say that $\mathscr{C}$ has *bounded-size* Π*-decompositions* (resp. *quasi-bounded-size* Π*-decompositions*).

For instance, by Theorem 1.1 and Theorem 1.2, considering the hereditary class property "bounded treedepth", we have that a class $\mathscr{C}$ has bounded-size bounded treedepth decompositions if and only if it has bounded expansion, and it has quasi-bounded-size bounded treedepth decompositions if and only if it is nowhere dense. With these definition in hand, it is natural to consider the following constructions of graph class properties:

Definition 2. *For a hereditary class property* Π *we define the properties* Π^+ *(resp.* Π^**) as follows:*

- $\mathscr{C} \in \Pi^+$ *if* $\mathscr{C}$ *has bounded-size* Π*-decompositions;*
- $\mathscr{C} \in \Pi^*$ *if* $\mathscr{C}$ *has quasi-bounded-size* Π*-decompositions.*

For every hereditary class property Π, we show that $(\Pi^+)^+ = \Pi^+$ and $(\Pi^*)^+ = \Pi^*$ (but we are not aware of any hereditary (NIP) class property Π, such that $\Pi^* \neq (\Pi^*)^*$). Also, for every two hereditary class properties Π_1 and Π_2, we show that $(\Pi_1 \cap \Pi_2)^+ = \Pi_1^+ \cap \Pi_2^+$ and $(\Pi_1 \cap \Pi_2)^* = \Pi_1^* \cap \Pi_2^*$, which suggests that, for every hereditary class property Π, there might exist an inclusion-minimum class Λ with $\Lambda^+ = \Pi^+$. On the other hand, if $(\Pi_i)_{i \in I}$ is a family of hereditary class properties indexed by a set I, then $(\bigcup_{i \in I} \Pi_i)^+ = \bigcup_{i \in I} \Pi_i^+$ and $(\bigcup_{i \in I} \Pi_i)^* = \bigcup_{i \in I} \Pi_i^*$. In particular, the inclusion order of decomposition horizons is a distributive lattice.

Definition 3. *We say that a hereditary class property* Π *is a* decomposition horizon *if* $\Pi^* = \Pi$*. If* Λ *is a hereditary class property, the* decomposition horizon *of* Λ *is the smallest decomposition horizon including* Λ*.*

For example, the hereditary class property of all hereditary classes excluding a fixed graph H is a decomposition horizon. We show that several hereditary class properties are decomposition horizons, including

- the class properties "bounded maximum degree after deletion of at most k vertices",
- the class property "transduction of a class with bounded maximum degree" (this property is equivalent to the model-theoretic property "mutually algebraic" [6], hence to the model-theoretic property "monadic NFCP" [14]),

- the class property "weakly sparse" (i.e. "biclique-free") of classes excluding a fixed biclique as a subgraph,
- the class property "nowhere dense".

Our examples include an infinite countable chain of decomposition horizons (the class properties"bounded maximum degree after deletion of at most k vertices"), witnessing some richness of the inclusion order on decomposition horizons.

Our second main result confirms (3) implies (2) from Conjecture 1.8.

Theorem 2.2. *Monadic stability is the decomposition horizon of the class property "bounded shrubdepth".*

From this, we obtain some combinatorial consequences for monadically stable graph classes. For example, we get the following very strong version of the Erdös-Hajnal property.

Corollary 1. *Every graph G in a hereditary stable class $\mathscr{C}$ has a clique or an independent set of size $\Omega_{\mathscr{C},\epsilon}(|G|^{1/2-\epsilon})$ for every $\epsilon > 0$. Furthermore, this cannot be improved to $\Omega_{\mathscr{C}}(|G|^{1/2})$.*

While Theorem 2.2 provides an analogue of Theorem 1.2 for monadically stable classes, monadically NIP hereditary classes seem to be more elusive. It was proved in [4] that for hereditary classes of ordered graphs, being NIP is equivalent to having bounded twin-width. On the other hand, classes with quasi-bounded-size bounded twin-width decompositions are NIP (as classes with bounded twin-width are NIP) and include transductions of nowhere dense classes (thus, conjecturally, all stable hereditary classes). Hence, it is a natural question whether every NIP hereditary class has quasi-bounded-size bounded twin-width decompositions.

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Intersecting sets in probability spaces and Shelah's classification

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Abstract

For $n \in \mathbb{N}$ and $\varepsilon > 0$, given a sufficiently long sequence of events in a probability space all of measure at least ε, some n of them will have a common intersection. A more subtle pattern: for any $0 < p < q < 1$, we cannot find events A_i and B_i so that $\mu(A_i \cap B_j) \leq p$ and $\mu(A_j \cap B_i) \geq q$ for all $1 < i < j < n$, assuming n is sufficiently large. This is closely connected to model-theoretic stability of probability algebras. We survey some results from our recent work in [7] on more complicated patterns that arise when our events are indexed by multiple indices. In particular, how such results are connected to higher arity generalizations of de Finetti's theorem in probability, structural Ramsey theory, hypergraph regularity in combinatorics, and model theory.

1 Intersections in a sequence of sets of positive measure

The following is a basic fact on intersections of sets of positive measure in probability spaces (there exist more precise infinitary/density versions, e.g. Bergelson's lemma in dynamics [5]):

Fact 1. *For every $\varepsilon \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ satisfying the following. If $(X, \mathcal{B}, \mu)$ is a probability space (i.e. $\mathcal{B}$ is a σ-algebra of subsets of X and μ is a countably additive probability measure on $\mathcal{B}$) and $A_i \in \mathcal{B}$ are measurable sets with $\mu(A_i) \geq \varepsilon$ for $1 \leq i \leq N$, then $\mu\left(\bigcap_{i \in I} A_i\right) > 0$ for some $I \subseteq [N] = \{1, \ldots, N\}$ with $|I| = n$.*

We sketch an overcomplicated proof of this fact in the remainder of Section 1, as a warm up for what comes later. If the random variables $\mathbf{1}_{A_i} : X \to \{0, 1\}$ in Fact 1 were independent, then of course $\mu\left(\bigcap_{i \in [n]} A_i\right) = \prod_{i \in [n]} \mu(A_i) \geq \varepsilon^n > 0$. We will reduce to this case. Assume from now on that for some fixed $\varepsilon > 0$ and n, no $N \in \mathbb{N}$ satisfies the claim.

1.1 Homogenizing the sequence

Using Ramsey's theorem, we can homogenize our sequence arbitrarily well. E.g., we could assume that for any fixed $\delta > 0$ and k, $\mu(A_{i_1} \cap \ldots \cap A_{i_k}) \approx^{\delta} \mu(A_{j_1} \cap \ldots \cap A_{j_k})$ for any $i_1 < \ldots < i_k$, $j_1 < \ldots < j_k$, and similarly for arbitrary Boolean combinations of the A_i's.

Using a compactness argument (e.g. taking Loeb measure on an ultraproduct of counterexamples), we can thus find some large probability space $(X, \mathcal{B}, \mu)$ and sets $A_i \in \mathcal{B}, \mu(A_i) \geq \varepsilon$ for $i \in \mathbb{N}$, still intersection of any n of them has measure 0, so that the sequence of random variables $(\mathbf{1}_{A_i} : i \in \mathbb{N})$ is *spreadable.*

*Chernikov was partially supported by the NSF CAREER grant DMS-1651321 and by the NSF Research Grant DMS-2246598.
†Towsner was partially supported by the NSF grant DMS-2054379.

1.2 de Finetti's theorem

Definition 1. *A sequence of* $[0,1]$*-valued random variables* $(\xi_i : i \in \mathbb{N})$ *is* spreadable *if for every* $n \in \mathbb{N}$ *and* $i_1 < \ldots < i_n, j_1 < \ldots < j_n$ *we have* $(\xi_{i_1}, \ldots, \xi_{i_n}) =^{dist} (\xi_{j_1}, \ldots, \xi_{j_n})$.

For example, every i.i.d. (independent, identically distributed) sequence of random variables is spreadable. The converse holds "up to mixing":

Fact 2 (de Finetti's theorem)**.** *If an infinite sequence of random variables* $(\xi_i : i \in \mathbb{N})$ *on* $(X, \mathcal{B}, \mu)$ *is spreadable then there exists a probability space* $(X', \mathcal{B}', \mu')$*, a Borel function* $f : [0,1]^2 \to [0,1]$ *and a collection of Uniform*$[0,1]$ i.i.d. *random variables* $\{\zeta_i : i \in \mathbb{N}\} \cup \{\zeta_\emptyset\}$ *on* X' *so that* $(\xi_i : i \in \mathbb{N}) =^{dist} (f(\zeta_i, \zeta_\emptyset) : i \in \mathbb{N})$.

This gives us an i.i.d. counterexample to Fact 1, and we can conclude.

1.3 Exchangeable versus spreadable sequences

More precisely, de Finetti obtained this conclusion under a stronger assumption that the sequence $(\xi_i : i \in \mathbb{N})$ is *exchangeable*, that is: for any $n \in \mathbb{N}$, any permutation $\sigma \in \mathrm{Sym}(n)$ and $i_1 < \ldots < i_n$ we have $(\xi_{i_1}, \ldots, \xi_{i_n}) =^{\mathrm{dist}} (\xi_{i_{\sigma(1)}}, \ldots, \xi_{j_{\sigma(n)}})$. And then Ryll-Nardzewski [20] proved that exchangeability is equivalent to spreadability. Curiously, Ryll-Nardzewski has a well-known theorem in model theory, but here he worked as a probabilist. It turns out that this result is closely connected to *stability* — a central notion in modern model theory.

2 Model theoretic stability of probability algebras

Modern model theory begins with *Morley's Categoricity Theorem*: for a countable theory T, if it has only one model of some uncountable cardinality (up to isomorphism), then it has only one model of *every* uncountable cardinality. Morley conjectured [18] a generalization: for a countable theory T, the number of its models of size κ is non-decreasing on uncountable κ.

In his solution of Morley's conjecture [21], Shelah isolated the importance of *stable theories* and developed a lot of machinery to analyze models of stable theories. Stability was rediscovered many times in various contexts, e.g. by Grothendieck in his work on Banach spaces, in dynamics as WAP systems (Weakly Almost Periodic), in machine learning as Littlestone dimension, etc.

In particular, probability algebras are stable, viewed as structures in continuous logic. This is implicit in Ryll-Nardzewski's theorem ("every indiscernible sequence is totally indiscernible"), later in Krivine and Maurey [17], explicit in Ben Yaacov [24]. A more general version was given by Hrushovski (proved using array de Finetti, discussed in Section 3.3), and Tao gave a short elementary proof [23]:

Fact 3. *For any* $0 \leq p < q \leq 1$ *there is* N *satisfying: if* $(X, \mathcal{B}, \mu)$ *is a probability space, and* $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{B}$ *satisfy* $\mu(A_i \cap B_j) \geq q$ *and* $\mu(A_j \cap B_i) \leq p$ *for all* $1 \leq i < j \leq n$*, then* $n \leq N$.

This result has many applications: Hrushovski's work on approximate subgroups [10], Tao's algebraic regularity lemma [22], work in topological dynamics by Tsankov, Ibarlucia [11], etc.

3 Intersecting multi-parametric families of events

We obtain a *higher arity* generalization of Fact 1:

Theorem 1 (Chernikov, Towsner [7]). *For every finite bipartite graph $H = (V_0, W_0, E_0)$ and $\varepsilon \in (0, 1]$ there exists a finite bipartite graph $G = (V, W, E)$ and $\delta > 0$ (depending only on H and ε) satisfying the following. Assume that $(X, \mathcal{B}, \mu)$ is a probability space, and for every $(v, w) \in V \times W$ a measurable set $A_{v,w} \in \mathcal{B}$ so that: for any $(v, w) \in E, (v', w') \notin E$ we have $\mu(A_{v,w}) - \mu(A_{v',w'}) \geq \varepsilon$. Then there exists an induced subgraph $H' = (V', W', E')$ of G (i.e. $V' \subseteq V, W' \subseteq W$ and $E' = E \cap (V' \times W')$) isomorphic to H so that:*

$$\mu\left(\left(\bigcap_{(v,w)\in E'} A_{v,w}\right) \cap \left(\bigcap_{(v,w)\in(V'\times W')\setminus E'} X \setminus A_{v,w}\right)\right) \geq \delta.$$

More precisely, Theorem 1 follows from [7, Lemma 10.13] and compactness. With high probability, a sufficiently large G taken at random will work. More generally, we prove it there for partite hypergraphs of any arity instead of just graphs. The question is motivated by *Keisler randomizations* of first-order structures [16] and whether they preserve NIP (Ben Yaacov, related to work of Talagrand on VC-dimension for functions [4]) and its higher arity generalization *n-dependence* (where Ben Yaacov's analytic proof for $n = 1$ does not seem to generalize).

In what follows we outline a proof of Theorem 1. The overall strategy is similar to the proof above for sequences of events, but each of the steps becomes harder.

3.1 Structural Ramsey theory, and infinite limits of Ramsey classes

Let $\mathcal{K}$ be a class of finite $\mathcal{L}_0$-structures, where $\mathcal{L}_0$ is a relational language (for example, finite graphs). For $A, B \in K$, let $\binom{B}{A}$ be the set of all $A' \subseteq B$ s.t. $A' \cong A$ (we work with substructures instead of embeddings for simplicity).

Definition 2. $\mathcal{K}$ *is* Ramsey *if for any $A, B \in K$ and $k \in \omega$ there is some $C \in K$ s.t. for any coloring $f : \binom{C}{A} \to k$, there is some $B' \in \binom{C}{B}$ s.t. $f \restriction \binom{B'}{A}$ is constant.*

The usual Ramsey theorem means: the class of finite linear orders is Ramsey. The subject of structural Ramsey theory started with the following fundamental result of Nesétril, Rödl [19] and Abramson, Harrington [1]:

Fact 4. *For any $k \in \mathbb{N}_{\geq 1}$, the class of all finite ordered k-hypergraphs is Ramsey.*

Fact 5. *Given a Ramsey class $\mathcal{K}$ of finite structures, there exists a unique (up to isomorphism) countable structure $\widetilde{K}$ (called the* Fraïssé limit *of $\mathcal{K}$) so that the class of its finite substructures is precisely $\mathcal{K}$ and $\widetilde{K}$ is* homogeneous*, i.e. if K_0 and K_1 are finite substructures of $\widetilde{K}$ and $f : K_0 \to K_1$ is an isomorphism, then f extends to an automorphism of the whole structure $\widetilde{K}$.*

E.g., if $\mathcal{K}$ is the class of all graphs, its limit $\widetilde{K}$ is the countable Rado's random graph; and if $\mathcal{K}$ is the class of finite linear orders, then its limit is $(\mathbb{Q}, <)$.

Understanding which structures are Ramsey is an active subject, with connections to model theory and topological dynamics (Ramsey property of $\mathcal{K}$ is equivalent to the extreme amenability of the group $\mathrm{Aut}(\widetilde{K})$ — via Kechris-Pestov-Todorcevic correspondence [15]).

3.2 Finding an "exchangeable" counterexample

For any $k \in \mathbb{N}_{\geq 1}$, using that the class of all finite ordered partite k-hypergraphs is Ramsey (viewed as structures in the language $E, P_1, \ldots, P_k, <$ with P_i a partition of vertices, $E \subseteq$

$P_1 \times \ldots \times P_k$ and $P_i < P_j$ for $i < j$, e.g. [6, Appendix A]), we let $\mathcal{OH}_k$ denote its *Fraïssé limit.* And we let $\mathcal{H}_k$ be its reduct forgetting the ordering.

Assuming that Theorem 1 fails, by Ramsey property and compactness (model theoretic jargon: extracting a generalized indiscernible) we can find some large probability space $(X, \mathcal{B}, \mu)$, $0 < r < s < 1$ and sets $A_{v,w} \in \mathcal{B}$ for all v, w vertices of $\mathcal{OH}_2 = (E; V, W)$ so that:

- $(v, w) \in E \implies \mu(A_{v,w}) \geq s$ and $(v, w) \notin E \implies \mu(A_{v,w}) \leq r$;
- for any two isomorphic (as ordered bipartite graphs) substructures H_1, H_2 of $\mathcal{OH}_2$,

$$(\mathbf{1}_{A_{v,w}} : v, w \in H_1) =^{\text{dist}} (\mathbf{1}_{A_{v,w}} : v, w \in H_2).$$

3.3 (Relatively) Exchangeable random structures

This indiscernibility guarantees certain "exchangeability" in the probabilistic sense. Exchangeable sequences (de Finetti, Section 1.2) and arrays (Aldous-Hoover-Kallenberg, see [14]) of random variables can be presented "up to mixing" using i.i.d. random variables, and we need a certain generalization to relational structures which were studied recently by a number of authors [8, 2, 12].

Definition 3. *(1) Let $\mathcal{L}' = \{R'_1, \ldots, R'_{k'}\}$, R'_i a relation symbol of arity r'_i. By a* random $\mathcal{L}'$-structure *we mean a (countable) collection of random variables $\left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i}\right)$ on some probability space $(\Omega, \mathcal{F}, \mu)$ with $\xi^i_{\bar{n}} : \Omega \to \{0, 1\}$.*

(2) Let now $\mathcal{L} = \{R_1, \ldots, R_k\}$ be another relational language, with R_i a relation symbol of arity r_i, and let $\mathcal{M} = (\mathbb{N}, \ldots)$ be a countable $\mathcal{L}$-structure with domain $\mathbb{N}$. We say that a random $\mathcal{L}'$-structure $\left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i}\right)$ is $\mathcal{M}$-exchangeable *if for any two finite subsets $A = \{a_1, \ldots, a_\ell\}, A' = \{a'_1, \ldots, a'_\ell\} \subseteq \mathbb{N}$*

$$\operatorname{qftp}_{\mathcal{L}}(a_1, \ldots, a_\ell) = \operatorname{qftp}_{\mathcal{L}}(a'_1, \ldots, a'_\ell) \implies$$
$$\left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in A^{r'_i}\right) =^{dist} \left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in (A')^{r'_i}\right).$$

3.4 A higher amalgamation condition on the indexing structure

Let $\mathcal{K}$ be a collection of finite structures in a relational language $\mathcal{L}$ closed under isomorphism.

Definition 4. *For $n \in \mathbb{N}_{\geq 1}$, we say that $\mathcal{K}$ satisfies the* n-disjoint amalgamation property *(n-DAP) if for every collection of $\mathcal{L}$-structures $(\mathcal{M}_i = (M_i, \ldots) : i \in [n])$ with $\mathcal{M}_i \in \mathcal{K}$, $M_i = [n] \setminus \{i\}$ and $\mathcal{M}_i|_{[n]\setminus\{i,j\}} = \mathcal{M}_j|_{[n]\setminus\{i,j\}}$ for all $i \neq j \in [n]$, there exists an $\mathcal{L}$-structure $\mathcal{M} = (M, \ldots) \in \mathcal{K}$ such that $M = [n]$ and $\mathcal{M}|_{[n]\setminus\{i\}} = \mathcal{M}_i$ for every $1 \leq i \leq n$.*

We say that an $\mathcal{L}$-structure $\mathcal{M}$ satisfies n-DAP if the collection of its finite substructures does. E.g., the generic k-hypergraph $\mathcal{H}_k$ satisfies n-DAP for all n [7, Proposition 9.6], but $(\mathbb{Q}, <)$ fails 3-DAP.

3.5 Presentation for random relational structures

Fact 6 (Crane, Towsner [8]; generalizing Aldous-Hoover-Kallenberg [3, 9, 13]). *Let $\mathcal{L}' = \{R'_i : i \in [k']\}, \mathcal{L} = \{R_i : i \in [k]\}$ be finite relational languages with all R'_i of arity at most r', and $\mathcal{M} = (\mathbb{N}, \ldots)$ a countable homogeneous $\mathcal{L}$-structure that has n-DAP for all $n \geq 1$. Suppose that*

$\left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i}\right)$ *is a random* $\mathcal{L}'$*-structure that is* $\mathcal{M}$*-exchangeable, such that the relations* R'_i *are symmetric with probability* 1.

Then there exists a probability space $(\Omega', \mathcal{F}', \mu')$, $\{0,1\}$*-valued Borel functions* $f_1, \ldots, f_{r'}$ *and a collection of* Uniform$[0,1]$ i.i.d. *random variables* $(\zeta_s : s \subseteq \mathbb{N}, |s| \leq r')$ *on* Ω' *so that*

$$\left(\xi^i_{\bar{n}} : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i}\right) =^{dist} \left(f_i\left(\mathcal{M}|_{\operatorname{rng}\bar{n}}, (\zeta_s)_{s \subseteq \operatorname{rng}\bar{n}}\right) : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i}\right),$$

where $\operatorname{rng}\bar{n}$ *is the set of its distinct elements, and* $\subseteq$ *denotes "subsequence".*

3.6 Getting rid of the ordering

Our counterexample from Section 3.2 is only guaranteed to be $\mathcal{OH}_n$-exchangeable (and the ordering is unavoidable in the Ramsey theorem for hypergraphs) — but the presentation theorem in Fact 6 requires n-DAP (and linear orders fail 3-DAP). However, using Fact 3 inductively, we can show that $\mathcal{OH}_n$-exchangeability already implies $\mathcal{H}_n$-exchangeability (i.e., with respect to the reduct forgetting the ordering), using that the theory of probability algebras is *stable*! (See [7, Lemma 10.15] for the details.)

Applying the exchangeable presentation to the counterexample, we finally reduce (modulo some mixing) to working with *independent* random variables in the proof of Theorem 1, and can conclude the proof.

4 Open questions and future directions

Question 1. Our proof of Theorem 1 is non-constructive and relies on a compactness argument. It would be interesting to obtain explicit bounds on $|G|$ and δ in terms of H and ε. Do there exist infinitary/density versions of this result (similarly to Fact 1)?

Question 2. Apart from k*-partite* k-hypergraphs, which other classes of structures satisfy an analog of Theorem 1? E.g., there is a growing list of Ramsey classes of finite structures, for which also an appropriate analog of Fact 6 holds. The following example illustrates that these two properties alone are not sufficient:

Example (Tim Austin) Theorem 1 does not hold for graphs (as opposed to *bipartite* graphs). Indeed, let H be the triangle K_3, and $\varepsilon = 1/2$. Consider any graph $G = (V, E)$. On some probability space (Ω, Σ, μ), let $(\pi_v : v \in V)$ be a process of independent uniform $\{0,1\}$-valued random variables, and consider the events $A_{v,w}$ defined by: $A_{v,w} := (\pi_v \neq \pi_w)$ if $(v,w) \in E$, and $A_{v,w} := \emptyset$ if $(v,w) \notin E$. Then $\mu(A_{v,w})$ is equal to $1/2$ if $(v,w) \in E$, but equal to 0 if $(v,w) \notin E$. However, for any induced triangle in G, say with vertices u, v, w, we have $\mu(A_{u,v} \cap A_{v,w} \cap A_{w,u}) = \mu(\pi_u \neq \pi_v \neq \pi_w \neq \pi_u) = \mu(\emptyset) = 0$.

As mentioned above, Theorem 1 is the main ingredient in our proof that Keisler randomization of first-order theories preserves n-dependence, for all $n \in \mathbb{N}_{\geq 1}$ ([7, Corollary 11.3]).

Question 3. Apart from n-dependence, what are the other higher arity tameness notions from model theory preserved under Keisler randomization? E.g., is FOP_n preserved?

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Formal power series in Second-Order Arithmetic

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1 Introduction

If $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers, then a well-known algebraic fact says that, for any field F[1], the (finitely generated polynomial) ring $F[\vec{X}_N] = F[X_1, X_2, \ldots, X_n]$ is Noetherian (i.e. satisfies the ascending chain condition on its ideals) for any natural number n. Classically, this result is known as the Hilbert Basis Theorem (HBT), and was established by Hilbert [8] via nonconstructive methods. Later on, Buchberger's Algorithm [6, Theorem 15.9] for computing Gröbner Bases in $F[\vec{X}_N]$ yielded a constructive (computable) proof of the Hilbert Basis Theorem for polynomial rings. After Buchberger's results, Simpson [12] showed that, in the context of Reverse Mathematics, HBT for $F[\vec{X}_N]$ is logically equivalent to the First-Order statement asserting the well-ordering of the ordinal number $\mathbb{N}^{\mathbb{N}}$ that corresponds (i.e. is isomorphic) to finite $\mathbb{N}-$sequences with the length-lexicographic ordering. This article is a precursor to a follow-up article that seeks to examine and classify the computability-theoretic properties of HBT for polynomial rings and its consequences such as the Artin-Rees Lemma, Krull Intersection Theorem, and related results concerning rings of formal power series.

In particular, we aim to exhibit the central role that the standard proof of HBT for the ring $R[\vec{X}_N]$ of polynomials plays in establishing similar results in the context of rings of formal power series. More specifically, Theorem 3.1 below formalizes [10, Theorem 3.3] in the context of RCA_0, and in so doing essentially establishes an effective reduction between the Hilbert Basis Theorem in the contexts of rings of polynomials and formal power series, and is the basis of all of our main results. Afterwards, Section 4 applies the basic module of Theorem 3.1 to show that, in the context of Reverse Mathematics, all known implications concerning HBT for polynonmial rings also hold for HBT in the context of formal power series.

2 Preliminaries

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote a possibly nonstandard set of natural numbers, and for any $N \in \mathbb{N}$, define

$$\mathbb{N}^N = \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}}_{N}.$$

For any $N \in \mathbb{N}$,

$$\vec{X}_N = \{X_0, X_1, \ldots, X_N\}$$

is a set of indeterminate variables, and we can speak of $\vec{X}_N-$monomials that are finite $\vec{X}_N-$products of the form

$$\prod_{i=0}^{N} X_i^{\alpha_i},\ \alpha_i \in \mathbb{N},$$

[1] Recall that a field is essentially any "number system" with commutative addition and multiplication operations such that any nonzero element has a multiplicative inverse.

so that each $\vec{X}_N-$monomial m is uniquely determined by its exponents $m \sim \langle \alpha_i : 0 \leq i \leq N \rangle \in \mathbb{N}^{\mathbb{N}+1}$. Now, if we define the *degree* of m to be $\sum_{i=0}^{N} \alpha_i \in \mathbb{N}$, then for each $n \in \mathbb{N}$, there are only finitely many monomials of degree n, and it follows that if we denote the set of $\vec{X}_N-$monomials by $\mathcal{M}$, then there is an $\mathcal{M}-$enumeration of nondecreasing degree. Moreover, we say that a monomial m_0 *divides* a monomial m_1 whenever the m_1-exponent of the indeterminate factor X_i is at least as large as that of m_0, for each $i = 0, 1, \ldots, N$. Also recall that, while polynomials consisting of finitely many summand terms always have a leading term of *maximal degree*, for formal power series consisting of infinite sums containing unbounded exponents the leading term is taken to be the $\mathcal{M}-$least one having *minimal degree*.

We assume a familiarity with basic Commutative Ring Theory, as found in [4, 1, 6, 10]. For us, R will always refer to a countable commutative ring with identity element $1 = 1_R \in R$. Recall that an *ideal* of R ($R-$ideal) is a subset of R closed under addition, subtraction, and multiplication by all $R-$elements. For any finite sequence $a_0, a_1, \ldots, a_n \in R$, $n \in \mathbb{N}$, define

$$\langle a_0, a_1, a_2, \ldots, a_n \rangle_R = \left\{ \sum_{i=0}^{n} r_i \cdot a_i : r_i \in R \right\};[2]$$

this is the smallest $R-$ideal containing $a_0, a_1, \ldots, a_N$. Recall that R is *Noetherian* if it satisfies the ascending chain condition (ACC) on its ideals. This is equivalent to saying that for any given infinite sequence $\{a_i\}_{i \in \mathbb{N}} \subseteq R$ there exists $N_0 \in \mathbb{N}$ such that the first N_0-many elements of A, $A_0 = \{a_0, a_1, \ldots, a_{N_0}\} \subseteq A$, generates A, i.e. each a_i, $i \in \mathbb{N}$, can be written as an $R-$linear combination of the elements of A_0. If R is a ring, then its *generalized division algorithm* is the relation

$$x \in \langle a_0, a_1, \ldots, a_N \rangle_R, \; N \in \mathbb{N}, \; x, a_0, a_1, \ldots, a_N \in R.$$

Finally, recall that the Hilbert Basis Theorem (HBT) says that, for each ring R and $n \in \mathbb{N}$, the polynomial ring

$$R[\vec{X}_N] = R[X_0, X_1, \ldots, X_n]$$

is Noetherian whenever R is Noetherian.

We will be examining HBT in the context of Reverse Mathematics for rings of formal power series over various coefficient rings R and sets of indeterminate variables $\vec{X}_N = \{X_0, X_1, \ldots, X_N\}$, $N \in \mathbb{N}$. Formal power series are infinitary objects, and so we will formally represent them in the context of Reverse Mathematics and RCA_0 numerically via their Turing (Gödel) codes. More specifically, a formal power series ring is a set $X \subseteq \mathbb{N}$ such that every $x \in X$ is the code of a formal power series, and X is closed under addition, subtraction, and multiplication of power series (codes). Other algebraic definitions, such as ideals and generating sets, are also defined via codes. The reader should keep in mind that, for us, specifying a formal power series amounts to giving an algorithm for computing its infinitely many coefficients, one coefficient for each monomial summand.

2.0.1 Reverse Mathematics, RCA_0, and induction

We assume familiarity with the arithmetical hierarchy consisting of the Σ_n and Π_n arithmetic formulas; more information on this topic can be found in either [14, Chapter 4] or [5, Section 5.2]. Throughout this article we work in the context of Reverse Mathematics and Subsystems of Second-Order Arithmetic[3] that always assumes a hypothesis denoted RCA_0 which, generally

[2] Note the subscript R on the lefthand side; for us, it distinguishes *ideals* from *sequences*.

[3] The program of Reverse Mathematics was first introduced by H. Friedman in the 1970s. More information on this modern branch of Mathematical Logic, including an introduction and historical remarks, can be found in [13, 5].

speaking, validates computable mathematical constructions via a Δ^0_1−comprehension axiom, along with a restricted induction scheme called $\mathsf{I}\Sigma_1$ that grants induction for arithmetic formulas of complexity Σ_1 consisting of a Δ^0_1−predicate preceded by a single existential quantifier. For more information on the formalism of Reverse Mathematics and RCA_0, we refer the reader to either [13, Chapter II] or [5, Chapter 5]. Induction schemes are arithmetical axioms that only pertain to the first-order theory of any subsystem of Second-Order Arithmetic. Throughout this article we will only work with arithmetical subsystems of Second-Order Arithmetic over RCA_0 that follow from Σ_2−induction ($\mathsf{I}\Sigma_2$); the next subsection describes these specific axioms in more detail.

2.1 Preliminary Combinatorics: the Infinite Pigeonhole Principle, the Well-Ordering of $\mathbb{N}^{\mathbb{N}}$, and the existence of monomial division chains

2.1.1 The Infinite Pigeonhole Principle

Recall the Infinite Pigeonhole Principle says that if $f : A \to B$ is a function with infinite domain A and finite range B, then for some $b \in B$ the fiber

$$f^{-1}(b) = \{a \in A : f(a) = b\}$$

is infinite. In the context of Reverse Mathematics (i.e. over RCA_0) a theorem of Hirst [9] says that the Infinite Pigeonhole Principle is equivalent to a bounding principle for Σ_2−formulas that produces uniform bounds for finite sets of existential witnesses to Σ_2−formulas, and so over RCA_0 we denote the Infinite Pigeonhole Principle by $\mathsf{B}\Sigma_2$.

2.1.2 The well-ordering of $\mathbb{N}^{\mathbb{N}}$

There is an arithmetical principle that follows from $\mathsf{I}\Sigma_2$ and says that the ordinal number $\mathbb{N}^{\mathbb{N}}$ is well-ordered. This is equivalent to saying that the length-lexicographic ordering on finite sequences of natural numbers is a well-order. We denote this principle by $\mathsf{WO}(\mathbb{N}^{\mathbb{N}})$. Simpson [12] has shown that $\mathsf{WO}(\mathbb{N}^{\mathbb{N}})$ is equivalent to saying that the finitely generated polynomial ring $F[\vec{X}_N] = F[X_0, X_1, \ldots, X_N]$, $N \in \mathbb{N}$, with coefficients in a field F is Noetherian. Along the way Simpson also shows the equivalence between $\mathsf{WO}(\mathbb{N}^{\mathbb{N}})$ and the Noetherian criterion for monomials that says if $M = \{m_i\}_{i\in\mathbb{N}} \subseteq F[\vec{X}_N]$ is an infinite sequence of $\vec{X}_N$−monomials (i.e. finite products of indeterminates in $\vec{X}_N$) then there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have that m_n is divisible by some element of $M_0 = \{m_i\}_{i=0}^{n_0}$, i.e. M_0 generates M.

2.1.3 The existence of monomial division chains

Recently in [3] the author has studied a combinatorial principle that plays a key role in the proof of the Hilbert Basis Theorem, called MDC, that says if $M = \{m_i\}_{i=0}^{\infty}$ is an infinite sequence of $\vec{X}_N = \{X_0, X_1, \ldots, X_N\}$−monomials, $N \in \mathbb{N}$, then there exists an infinite subsequence $\{i_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ such that for each $k \in \mathbb{N}$ we have that m_{i_k} divides $m_{i_{k+1}}$. Moreover, building on results of Simpson [12] and Chong, Slaman and Yang [2], the author has shown MDC to be equivalent to $\mathsf{B}\Sigma_2 + \mathsf{WO}(\mathbb{N}^{\mathbb{N}})$ over RCA_0, while Simpson [11] has shown that $\mathsf{B}\Sigma_2 + \mathsf{WO}(\mathbb{N}^{\mathbb{N}})$ is strictly stronger than either $\mathsf{B}\Sigma_2$ or $\mathsf{WO}(\mathbb{N}^{\mathbb{N}})$, and that $\mathsf{B}\Sigma_2 + \mathsf{WO}(\mathbb{N}^{\mathbb{N}})$ is strictly weaker than $\mathsf{I}\Sigma_2$.

3 Transfering the Division Algorithm from $R[\vec{X}_N]$ to $R[[\vec{X}_N]]$

The following theorem is the essential key to all of our results. Its proof is essentially a formalization of [10, Theorem 3.3] in $\mathsf{RCA_0}$.

Theorem 3.1 (The Division Algorithm for power series rings with Noetherian coefficients, $\mathsf{RCA_0}$). *Suppose that R is a ring, $n \in \mathbb{N}$, and let*

- $\vec{X}_N = \{X_0, X_1, \dots, X_n\}$ *be a set of $n-$many indeterminates corresponding to rings $R[\vec{X}_N]$ and $R[[\vec{X}_N]]$, and such that*
  - $\mathcal{M} = \{m_i\}_{i=0}^{\infty}$ *is an enumeration of the $\vec{X}_N-$monomials in nondecreasing order of $\mathbb{N}-$degree,*
- $F = \{f_k\}_{k\in\mathbb{N}} \subseteq R[[X]]$ *be an enumeration of an $R[[\vec{X}_N]]-$ideal with*

$$f_k = \sum_{i=0}^{\infty} a_{k,i} m_i, \ a_{k,i} \in R,$$

and such that $\ell_k \in \mathbb{N}$ is ($\mathbb{N}-$)least such that $a_{k,\ell_k} \neq_R 0$. In this case we have that $s_k = a_{k,\ell_k} m_{\ell_k}$ denotes the leading summand of f_k.

Now, suppose that there exists some $N_0 \in \mathbb{N}$ that witnesses the Noetherian property that says:

$$\langle s_k : k \in \mathbb{N}\rangle_{R[\vec{X}_N]} = \langle s_0, s_1, \dots, s_{N_0}\rangle_{R[\vec{X}_N]}, \tag{1}$$

<u>then</u> we also have that

$$F = \langle f_k : k \in \mathbb{N}\rangle_{R[[\vec{X}_N]]} = \langle f_k : 0 \le k \le N_0\rangle_{R[[\vec{X}_N]]}.$$

Proof. Let

$$S_0 = \{s_0, s_1, \dots, s_{N_0}\}, \quad F_0 = \{f_0, f_1, \dots, f_{N_0}\},$$

and $k = k_0 \in \mathbb{N}$. By hypothesis we have that

$$f_{k_0} = a_{k_0,\ell_{k_0}} m_{\ell_{k_0}} + \sum_{\ell > \ell_{k_0}} a_{k_0,\ell} m_\ell = s_{k_0} + \sum_{\ell > \ell_{k_0}} a_{k_0,\ell} m_\ell,$$

and moreover we can write the leading summand $s_{k_0} = a_{k_0,\ell_{k_0}} m_{\ell_{k_0}} \in R[\vec{X}]$ of f_{k_0} as an $R[\vec{X}_N]-$linear combination of $\{s_0, s_1, \dots, s_{N_0}\}$. Therefore, if we have that

$$s_{k_0} = \sum_{i=0}^{N_0} c_{k_0,i} s_i, \ c_{k_0,i} \in R[\vec{X}_N],$$

then it follows that

$$f_k - \sum_{i=0}^{N_0} c_{k,i} f_i = f_{k_1} \in F$$

is such that $\ell_{k_1} > \ell_{k_0}$. Furthermore, we can repeat the argument, in infinitely many stages indexed by $i \in \mathbb{N}$, to obtain an infinite sequence of numbers $\{k_i\}_{i\in\mathbb{N}}$ corresponding to power series $\{f_{k_i}\}_{i\in\mathbb{N}} \subseteq F$ such that for every $i \in \mathbb{N}$ we have that

$$\ell_{k_{i+1}} > \ell_{k_i};$$

in other words, the $\mathcal{M}-$index of the leading summand of $f_{k_{i+1}}$ is strictly greater than that of f_{k_i}. Now, the degrees of the monomials in any enumeration of $\mathcal{M}$ always grow uniformly, and thus we have that $\lim_i \deg(m_i) = \infty$. Also, because our sets S_0 and F_0 are fixed thorughout the construction, at each stage $i \in \mathbb{N}$, in order to obtain the cancellation required for $\ell_{i+1} > \ell_i$, we must have that

$$\lim_j \deg(c_{k_j,i}) = \infty,$$

uniformly in $i = 0, 1, \ldots, N_0$. Finally, by our construction it follows that if we set

$$c_i = \sum_{j=0}^{\infty} c_{k_j,i}, \ i = 0, 1, \ldots, N_0,$$

then $c_i \in R[[\vec{X}_N]]$ and

$$f_k = \sum_{i=0}^{N_0} c_i f_i.$$

□

Remark 3.2. *The key assumption in the previous theorem is the existence of $N_0 \in \mathbb{N}$, which essentially assumes a division algorithm for $R[\vec{X}_N]$, $N \in \mathbb{N}$. It would benefit the reader to keep in mind that the hypotheses in the theorems that follow, all of which utilize Theorem 3.1, are chosen so as to guarantee the existence of the number N_0 in the previous proof, and that the necessary hypotheses for producing N_0 depend upon the properties of R and N.*

4 Transfering the Noetherian property from R to $R[[\vec{X}_N]]$ (via $R[\vec{X}_N]$)

Let F be a field and R be a ring with a generalized division algorithm. The goal of this section is to apply Theorem 3.1 to successively more general power series rings of the form $R[[X]]$, $F[\vec{X}_N]$, and finally $R[[\vec{X}_N]]$. Each application corresponds to a different subsystem of Second-Order Arithmetic.

In the proofs of each of the theorems below $F = \{f_k\}_{k\in\mathbb{N}}$ will always denote the ideal of $R[[\vec{X}_N]]$, $\vec{X}_N = \{X_0, X_1, \ldots, X_n\}$, $n \in \mathbb{N}$, for which we produce a finite set of generators via Theorem 3.1 above. Also, as in Theorem 3.1, recall that $\mathcal{M} = \{m_i\}_{i\in\mathbb{N}}$ denotes an enumeration of $\vec{X}_N-$monomials of nondecreasing $\mathbb{N}-$degree, and for each $k \in \mathbb{N}$, $\ell_k \in \mathbb{N}$ is least such that the leading summand of f_k is of the form $a_{\ell_k} \cdot m_{\ell_k}$ for some $0 \neq_R a_k$. With all of this notation and definitions in mind and out of the way, the main focus of our proofs will be the construction of the number N_0 mentioned in the hypothesis of Theorem 3.1 above.

Theorem 4.1 (RCA_0)**.** *$R[[X]]$ is Noetherian whenever R is a Noetherian ring possessing a generalized division algorithm.*

Proof. To construct N_0 in the current context, there are two cases to consider. The first case says that

$$a_{k+1} \notin \langle a_0, a_1, \ldots, a_k\rangle_R$$

for infinitely many $k \in \mathbb{N}$. In this case it follows that R is not Noetherian, which is a contradiction. So we are in the second case which says that there exists $N_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq N_0$, we have that

$$a_k \in \langle a_0, a_1, \ldots, a_{N_0}\rangle_R.$$

Arbitrary Frege Arithmetic

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Abstract

This paper focuses on a less-known version of Abstractionism, that we'll call Arbitrary Logicism. It is obtained by means of a double revision of Frege's Logicist program: on the one side, weakening the Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary, I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such a non-canonical interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms.

Keywords: Abstraction principles, Logicism, Arbitrariness, Frege, Arithmetic

Abstractionist theories in philosophy of mathematics are systems composed by a logical theory augmented with an abstraction principle (AP), of form: $\forall X \forall Y (@X = @Y) \leftrightarrow E(X, Y)$[1] – that introduces, namely rules and implicitly defines, a term-forming operator @ by means of an equivalence relation E. As is well-known, the seminal abstractionist program, Frege's Logicism, failed[2]: Russell's Paradox proved its inconsistency and, *a fortiori*, its non-logicality. In the last century, both the issue of consistency and the issue of logicality have been resumed in the abstractionist debate (cf. [13], [7], [1], [4], [3]). More precisely, on the one side, different revisions of Frege's original system have been proposed in order to avoid Russell's Paradox and to obtain a consistent system that is strong enough to derive (at least, a relevant portion of) Peano Arithmetic. On the other side, given a semantical definition of logicality as permutation invariance, some abstraction principles have been proved to be logical ([1], [4]).

Nevertheless, many concerns are still open. Particularly, regarding the preliminary condition of consistency, the ways out of Russell's Paradox proposed so far do not precisely mirror a corresponding explanation of the origin of the contradiction and often imply a weakening of the hoped strength of the theory (cf. [11], [14], [6])[3]; regarding the issue of logicality, an undesired dilemma overshadows the abovementioned results: precisely in case of logical (i.e. permutation invariant) abstraction principles, their implicit *definienda* turn out to be non logical ([1]) – so preventing a full achievement of the Logicist goal.

My preliminary aim consists in arguing that these – apparently unrelated –problems have a common source in some unquestioned assumptions of Frege's project (inherited also by the following abstractionist programs). I argue that such assumptions are part of what we can call the Traditional view of abstraction, that includes the choice of classical logic as the base theory, with the related semantical consequence of full referentiality of the vocabulary, and the

[1] In the rest of the paper, I'll adopt this axiomatic version of AP. Given full Comprehension Axiom Schema (that will be assumed in the systems that we'll investigate), it is provably equivalent to the schematic form: $@x.\alpha(x) = @x.\beta(x) \leftrightarrow E(\alpha(x), \beta(x))$. Cf. [12]

[2] It was proposed with the foundational purpose to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms.

[3] In [5] and [2], second-order Peano Axioms are recovered but by appealing to stronger logical resources – i.e. double-sorted variables

choice of a so-called Canonical interpretation function for all the (both primitive and defined) expressions of the language.

In the rest of the talk, I show that by renouncing one or both of these problematic assumptions we can recover consistency and/or logicality. More precisely, I propose a double revision of Frege's Logicist program: on the one side, weakening the Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary (cf. [3]), I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such an arbitrary interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms. This means that this system – that we'll call Arbitrary Logicism, precisely renouncing to the Traditional assumptions mentioned above, is able to recover both Frege's goals of consistency and logicality.

The logical part of the language of Arbitrary Logicism, L_F, includes denumerably many first-order variables $(x, y, z, ...)$, denumerably many second-order variables $(X, Y, Z, ...)$, logical connectives $(\neg, \rightarrow)$ and a first-order existential quantifier $(\exists)$[4]. We can also usefully define a predicative monadic constant (E!), whose extension is equal to the range of identity: E!a $=_{def}$ $\exists x(x = a)$. The only non-logical primitive symbol is the term-forming operator ϵ which applies to monadic second-order variables to produce complex singular terms $(\epsilon(X))$[5].

The theory involves, as its logical part, the axioms and inference rules of non-inclusive negative free logic with identity (NF$^=$):

NF1) $\forall v\alpha \rightarrow (E!t \rightarrow \alpha(t/v))$;

NF2) $\exists v E!v$;

NF3) $s = t \rightarrow (\alpha \rightarrow \alpha(t//s))$[6];

NF4) $\forall v(v = v)$;

NF5) $P\tau_1, ..., \tau_n \rightarrow E!\tau_i$ (with $1 \leq i \leq n$);

$\forall$I): $E!a...\phi(a/x) \vdash \forall x\phi$;

$\exists$E): $\phi(a/x), E!a...\psi, \exists x\phi \vdash \psi$, where a is a new individual constant which does not occur in ϕ and ψ.

[4]We can also define the other connectives and the universal quantifier $\forall xAx =_{def} \neg\exists x\neg Ax$.

[5]Let D be the full first-order domain (then, the second-order domain is constituted by its power-set $\wp(D)$). The satisfaction clauses for the formulas of L_F are defined in terms of an evaluation function V and an assignment function I that ascribes elements of D to the first-order terms and elements of $\wp(D)$ to the second-order terms:

- $V(Pt_1, ..., t_n) = 1 \leftrightarrow I(t_1), ..., I(t_n) \in D \wedge < I(t_1), ..., I(t_n) > \in I(P)$; 0 otherwise;
- $V((s) = (t)) = 1 \leftrightarrow I(s), I(t) \in D \wedge I(s) = I(t)$; 0 otherwise;
- $V(E!t) = 1 \leftrightarrow I(t) \in D$; 0 otherwise;
- $V(\neg\alpha) = 1 \leftrightarrow V(\alpha) = 0$; 0 otherwise;
- $V(\alpha \wedge \beta) = 1 \leftrightarrow \alpha = 1 \wedge \beta = 1$; 0 otherwise;
- $V(\alpha \vee \beta) = 1 \leftrightarrow \alpha = 1 \vee \beta = 1$; 0 otherwise;
- $V(\forall v\alpha) = 1 \leftrightarrow \forall s \in D, V_{(t,s)}(\alpha(t/v)) = 1$ – where t is not in α and $V_{(t,s)}$ is the valuation function on the model $< D, I^* >$ such that $I^* = I$, except that $I^*(t) = s$.
- $V(\forall V\alpha) = 1 \leftrightarrow \forall S \subseteq D, V_{(T,S)}(\alpha(T/V)) = 1$ – where T is not in α and $V_{(T,S)}$ is the valuation function on the model $< D, I^* >$ such that $I^* = I$, except that $I^*(T) = S$.

[6]Where $\alpha(t//s))$ is the result of replacing one or more occurrences of s in A by t.

Additionally, the theory involves an axiom-schema of universal instantiation for second-order variables ($\forall X\phi(X) \to \phi(Y)$), a rule of universal generalisation (GEN), a second-order comprehension axiom schema (CA: $\exists X\forall x(Xx \leftrightarrow \alpha)$) and *modus ponens* (MP)[7].

The abstraction principle that characterizes this theory is obtained by weakening the right-to-left conditional of Basic Law V (BLV: $\forall F\forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx))$, i.e. BLVa (arbitrarily interpreted), by means of the condition of Permutation Invariance (cf. [1], [3]).

W-BLV: $\forall F\forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx) \wedge \epsilon(\pi(F)) = \pi(\epsilon F))$[8]

As well known, the ϵ operator (as defined by standard BLV), also arbitrarily interpreted, is not Permutation Invariant – because, roughly speaking, by being inconsistent it is unable to define or rule any function. We can emphasize that, given an arbitrary interpretation, Permutation Invariance fails precisely for the argument that determines its inconsistency. In other words, as can be pointed out for other consistent revisions of BLV, in any case in which it is safely restricted, ϵ turns out to satisfy Permutation Invariance, namely it is such that $\pi(\epsilon) = \epsilon$, i.e. $\forall X\forall y(\epsilon X = y \leftrightarrow \epsilon(\pi(X)) = \pi(y))$. Then, the second conjunct of the right-hand side of W-BLV requires that – no matter which object y is identical to ϵF – ϵ satisfies Permutation Invariance for the considered arguments[9].

Accordingly, W-BLV, as a bi-conditional, turns out to be satisfied by any concept instantiating the universal quantifier. On the one side, given an arbitrary interpretation of the abstraction operator, for any concept different from Russellian concept (R), $\pi(\epsilon) = \epsilon$. On the other side, we can consider Russell's Paradox as a *reductio ad absurdum* of the alleged truth of both the sides of the bi-conditional for the concept R: the contradiction proves that ϵR – as legitimately admitted on a free logical background – does not exist, namely it is a term devoid of denotation; accordingly, it is not identical to itself (so, falsifying the left-hand side of W-BLV) and, even if R, as any other concept, is co-extensional with itself, it falsifies Permutation Invariance of the operator[10]. Accordingly, also the right-hand side of W-BLV is false and also the instance of the bi-conditional for the concept R is verified.

Such a restricted version of W-BLV allows us to derive a corresponding restricted version of Hume's Principle. Nevertheless, the same restriction, on HP, is trivially satisfied by any instantiation, so it actually does not represent a weakening of the principle itself and allow us to derive the main arithmetical results, including Frege's Theorem.

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[7] From these axioms we can also derive the following theorems: T1) $\forall x E!x$; T2) $t = t \leftrightarrow E!t$; T3) $(\neg E!s \wedge \neg E!t) \to (\alpha \to \alpha(t//s))$.

[8] This abstraction principle is clearly circular because the extensional operator occurs on both the sides of the biconditional. The idea that circularity defeats the definitional role of such principle (or, in general, of implicit definitions) is controversial. Anyway, in this framework, what we need is a principle that rules the behavior of a new symbol of the language and W-BLV carries out this task.

[9] This revision of BLV (particularly of BLVa) is featured by a restriction that, with respect to many other (syntactical ones), is expressible into the language. Indeed, the permutation π of the operator or of the concepts mentioned in the right-hand side of the bi-conditional can be defined as abbreviation of the effects of any first-order bi-jective function f: $D_1 \to D_1$ on the entities (sets, relations or functions) further up in the type hierarchy.

[10] This last claim follows from the definition of π and the result of non-existence of ϵR: on the one side, $\epsilon(\pi(R)) = \epsilon(\{\pi(x)|x \in R\}) = \epsilon(X)$ – where X is any other concept (based on π); on the other side, $\pi(\epsilon R)$, given that ϵR is not denoting, is another well-formed term without denotation; then, the identity between ϵX (for any X that is obtained by means of a permutation of R) and the empty term $\pi(\epsilon R)$ is false.

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Preservation theorems on sparse classes revisited

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Abstract

We revisit the work studying homomorphism preservation in sparse classes of structures initiated in [Atserias et al., JACM 2006] and [Dawar, JCSS 2010]. These established that first-order logic has the homomorphism preservation property in any sparse class that is monotone and addable. It turns out that the assumption of addability is not strong enough for the proofs given. We demonstrate this by constructing classes of graphs of bounded treewidth which are monotone and addable but fail to have homomorphism preservation. We also show that homomorphism preservation fails on the class of planar graphs. On the other hand, the proofs can be recovered by replacing addability by a stronger condition of amalgamation over bottlenecks. This is analogous to a similar condition formulated for extension preservation in [Atserias et al., SiCOMP 2008].

1 Introduction

Preservation theorems have played an important role in the development of finite model theory. They provide a correspondence between the syntactic structure of first-order sentences and their semantic behaviour. In the early development of finite model theory it was noted that many classical preservation theorems fail when we limit ourselves to finite structures. An important case in point is the Łoś-Tarski or *extension* preservation theorem, which asserts that a first-order formula is preserved by embeddings between all structures if, and only if, it is equivalent to an existential formula. Interestingly, this was shown to fail on finite structures [9] much before the question attracted interest in finite model theory [6]. On the other hand, the *homomorphism* preservation theorem, asserting that formulas preserved by homomorphisms are precisely those equivalent to existential-positive ones, was remarkably shown to hold on finite structures by Rossman [8], spurring applications in constraint satisfaction and database theory.

However, even before Rossman's result, these preservation properties were investigated on subclasses of the class of finite structures. In this context, restricting to a subclass weakens both the hypothesis and the conclusion, therefore leading to an entirely new question. Thus, while the class of all finite structures is combinatorially wild, it contains *tame* classes which are both algorithmically and model-theoretically better behaved [4]. A study of preservation properties for such restricted classes of finite structures was initiated in [3] and [2], which looked at homomorphism preservation and extension preservation respectively. The focus was on tame classes defined by *wideness* conditions, allowing for methods based on the *locality* of first-order logic.

The main result asserted in [3] is that homomorphism preservation holds in any class $\mathcal{C}$ which is *almost wide* and is *monotone* and *addable*. From this, it is concluded that homomorphism preservation holds for any class $\mathcal{C}$ whose Gaifman graphs exclude some graph G as a minor, as long as $\mathcal{C}$ is monotone and addable. The result was extended from almost wide to *quasi-wide* classes in [5], from which homomorphism preservation was deduced for classes that locally

*Supported by a George and Marrie Vergotis Scholarship awarded by Cambridge Trust, an Onassis Foundation Scholarship, and a Robert Sansom Studentship

exclude minors and classes that have bounded expansion, again subject to the proviso that they are monotone and addable. Quasi-wide classes were later identified with *nowhere dense* classes, which are now central in structural and algorithmic graph theory [7].

The main technical construction in [3] is concerned with showing that classes of graphs which exclude a minor are indeed almost wide. The fact that homomorphism preservation holds in monotone and addable almost wide classes is deduced from a construction of Ajtai and Gurevich [1] which shows the "density" of minimal models of a first-order sentence preserved under homomorphisms, and the fact that in an almost wide class a collection of such dense models must necessarily be finite. While the Ajtai and Gurevich construction is carried out within the class of all finite structures, it is argued in [3] that it can be carried out in any monotone and addable class because of "the fact that disjoint union and taking a substructure are the only constructions used in the proof" [3, p. 216].

The starting point of the present paper is that this argument is flawed. The construction requires us to take not just disjoint unions, but unions that identify certain elements: in other words *amalgamations* over sets of points. On the other hand, we can relax the requirement of monotonicity to just hereditariness. The conclusion is that homomorphism preservation holds in any class $\mathcal{C}$ that is quasi-wide, hereditary and closed under amalgamation over bottleneck points. The precise statement is given in Theorem 4.1 below. We also show that the requirements formulated in [3] are insufficient by constructing a class that is almost wide (indeed, has bounded treewidth), is monotone and addable, but fails to have the homomorphism preservation property. The class of planar graphs is an interesting case as it is used in [2] as an example of a hereditary, addable class with excluded minors in which extension preservation fails. We show that homomorphism preservation also fails in this class, strengthening the result of [2].

2 Preliminaries

We fix a finite relational vocabulary τ; by a structure we implicitly mean a τ-structure. Given two structures A, B, a homomorphism $f : A \to B$ is a map such that for all relation symbols R and tuples $\bar{a}$ from A we have $\bar{a} \in R^A \implies f(\bar{a}) \in R^B$. If moreover $f(\bar{a}) \in R^B \implies \bar{a} \in R^A$ then f is said to be *strong*. An injective strong homomorphism is called an *embedding*.

A structure B is said to be a *weak substructure* of a structure A if $B \subseteq A$ and the inclusion map $\iota : B \hookrightarrow A$ is a homomorphism. Likewise, B is an *induced substructure* of A if the inclusion map is an embedding. An induced substructure B of A is said to be *free in A* if there is some structure C such that A is the disjoint union $B + C$. Finally, a substructure B of A is said to be *proper* if the inclusion map is not full. We say that a class of structures is *monotone* if it is closed under weak substructures, and it is *hereditary* if it is closed under induced substructures. Moreover a class is called *addable* if it is closed under taking disjoint unions.

Given structures A, B, S and embeddings $f : S \to A$ and $g : S \to B$, we write $A \oplus_{S,f,g} B$ for the quotient of the disjoint union $A + B$ by the equivalence relation generated by $\{(f(s), g(s)) : s \in S\}$. Whenever $S \subseteq A \cap B$, we write $A \oplus_S B$ for $A \oplus_{S,\iota_A,\iota_B} B$ where ι_A, ι_B are the corresponding inclusion maps, and call this the *free amalgam of A and B over S*.

Fixing a graph H, we say that a graph G is H-free and H-minor-free if it does not contain H as an induced subgraph and minor respectively. By Wagner's Theorem, a graph is planar if and only if it is K_5-minor-free and $K_{3,3}$-minor-free. Finally, a class of graphs $\mathcal{C}$ is said to be *quasi-wide* if for every $r \in \mathbb{N}$ there exist $s_r \in \mathbb{N}$ and $f_r : \mathbb{N} \to \mathbb{N}$ such that for every $m \in \mathbb{N}$ and every $G \in \mathcal{C}$ there exist disjoint sets $A, S \subseteq V(G)$ such that A is r-independent in $G \setminus S$.

We say that a formula ϕ is preserved by homomorphisms (respectively extensions) over a class of structures $\mathcal{C}$ if for all $A, B \in \mathcal{C}$ such that there is a homomorphism (respectively embedding) from A to B, $A \models \phi$ implies that $B \models \phi$. We say that a class of structures $\mathcal{C}$ has the *homomorphism preservation property* (HPP) (respectively *extension preservation property*, EPP) if for every formula ϕ preserved by homomorphisms (respectively extensions) over $\mathcal{C}$ there is an existential-positive (respectively existential) formula ψ such that $M \models \phi \iff M \models \psi$ for all $M \in \mathcal{C}$.

Given a formula ϕ and a class of structures $\mathcal{C}$, we say that $M \in \mathcal{C}$ is a *minimal induced model* of ϕ in $\mathcal{C}$ if $M \models \phi$ and for any proper induced substructure N of M with $N \in \mathcal{C}$ we have $N \not\models \phi$. The relationship between minimal models and preservation is highlighted by the following theorem.

Theorem 2.1. *Let $\mathcal{C}$ be a hereditary class of finite structures. The $\mathcal{C}$ has the HPP (respectively EPP) if and only if every formula preserved by homomorphisms (respectively extensions) over $\mathcal{C}$ has finitely many minimal induced models in $\mathcal{C}$. So, if $\mathcal{C}$ has the EPP then it has the HPP.*

3 Preservation can fail on classes of small treewidth

Theorem 4.4 of [3] can be paraphrased in the language of this paper as saying that *homomorphism preservation holds over any monotone and addable class of bounded treewidth.* Here, we provide a simple counterexample to this, exhibiting a monotone and addable class of graphs of treewidth 3 where homomorphism preservation fails.

Definition 3.1. Fix $k \in \mathbb{N}$ and $n_i \geq 3$ for every $i \in [k]$. We define the *bouquet of cycles of type* $(n_1, \dots, n_k)$, denoted by $W_{n_1,\dots,n_k}$, as the graph obtained by taking the disjoint union of k cycles of length $n_1, \dots, n_k$ respectively, and adding an apex vertex, i.e. a vertex adjacent to every vertex in these cycles. Whenever $k = 1$, we refer to the graph W_n as the *wheel of order n*.

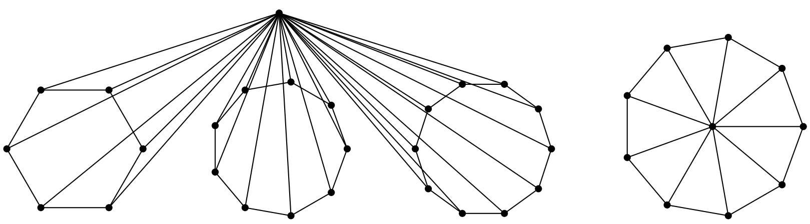

Figure 1: The bouquet of cycles of type $(6, 9, 10)$ and the wheel of order 9 respectively.

First, observe that each bouquet has treewidth 3. Indeed, taking a tree decomposition of each cycle of width 2, and adding the apex to every bag in the decomposition gives the required tree decomposition. The advantage of working with bouquets of cycles is that, unlike single cycles, there is a formula that defines their existence as free induced subgraphs. To see this, we let

$$\psi(x, z) := \exists u \exists v [u \neq v \wedge u \neq x \wedge v \neq x \wedge E(z, u) \wedge E(z, v) \wedge \forall w (E(w, z) \rightarrow w = u \vee w = v \vee w = x)],$$

$$\text{and } \phi := \exists x \exists y [E(x, y) \wedge \forall z (z \neq x \wedge \text{dist}(x, z) \leq 2 \rightarrow E(x, z) \wedge \psi(x, z)].$$

Intuitively, ϕ asserts the following: "there is a vertex x of degree at least one such that every other vertex reachable from x by a path of length two is adjacent to x and has exactly two distinct neighbours which are not x".

Lemma 3.2. *Let G be an arbitrary finite graph. Then $G \vDash \phi$ if, and only if, it contains a bouquet of cycles as a free induced subgraph.*

It is evident that ϕ is not preserved by homomorphisms over the class of all undirected graphs. However, when restricting to subgraphs of disjoint unions of wheels we no longer have non-free-occurring bouquets of cycles in the class. This is precisely the core of the following theorem.

Theorem 3.3. *The monotone and addable closure of $\{W_{2n+1} : n \in \mathbb{N}\}$ does not have the HPP.*

4 Preservation under bottleneck amalgamation

The main result of this section is the corrected version of Theorem 4.4 in [3] and its generalisation, Theorem 9 in [5]. More precisely, we establish homomorphism preservation on hereditary quasi-wide classes which are closed under certain free amalgams. While the existence of arbitrary amalgams certainly suffices, it prohibits any sort of sparsity in the class. Indeed, any hereditary class of undirected graphs with the free amalgamation property contains arbitrarily large 1-subdivided cliques, and hence, cannot be quasi-wide.

The proof proceeds by obtaining a concrete bound on the size of minimal models of ϕ in $\mathcal{C}$, and concluding by Theorem 2.1. The existence of this bound is guaranteed by quasi-wideness, as any large enough structure contains a large scattered set after removing a small number of bottleneck points. To isolate the bottleneck points $\bar{p}$ of M we consider a structure $\bar{p}M$ in an expanded language which is bi-interpretable with M, and work with the corresponding interpretation ϕ^k of ϕ; in particular $\bar{p}M$ contains a large scattered set itself and it models ϕ^k. Then, by removing a carefully chosen point from the scattered set of $\bar{p}M$, we obtain a proper induced substructure $\bar{p}N$ of $\bar{p}M$ such that $N \in \mathcal{C}$ by hereditariness. To argue that this still models ϕ^k, we use a relativisation of the locality argument of Ajtai and Gurevich from [1]. While in its original version the argument only considers disjoint copies of M, working with the interpretation $\bar{p}M$ of M corresponds to taking free amalgams of M over the set of bottleneck points; this is precisely the subtlety that was missed in [3] and [5].

Theorem 4.1. *Let $\mathcal{C}$ be a hereditary class such that for every $r \in \mathbb{N}$ there exist $k_r \in \mathbb{N}$ and $f_r : \mathbb{N} \to \mathbb{N}$ satisfying that for every $m \in \mathbb{N}$ and $M \in \mathcal{C}$ of size at least $f_r(m)$ there exist disjoint sets $A, S \subseteq M$ such that $|A| \geq m$, $|S| \leq k_r$, A is r-independent in $M \setminus S$, and $\oplus_S^n M \in \mathcal{C}$ for every $n \in \mathbb{N}$. Then homomorphism preservation holds over $\mathcal{C}$.*

Obtaining homomorphism preservation for quasi-wide classes therefore amounts to verifying closure under amalgams over bottleneck points. This is precisely the case for K_4-minor-free and outerplanar graphs. Another class with this property is already known to exist by [2], that is, the class $\mathcal{T}_k$ of all graphs of treewidth bounded by k, for any $k \in \mathbb{N}$.

Theorem 4.2. *The classes of K_4-minor-free graphs and outerplanar graphs have the HPP.*

5 Preservation fails on planar graphs

In this section we witness that homomorphism preservation fails on the class of planar graphs. Previously, it was established [2] that the extension preservation property fails on planar graphs. Since extension preservation implies homomorphism preservation on hereditary classes by Theorem 2.1, our result strengthens the above. Our construction will in fact also reveal that homomorphism preservation fails on the class of K_5-minor-free graphs.

Definition 5.1. Fix $n \in \mathbb{N}$. Define D_n as the undirected graph on vertex set

$$V(D_n) = \{v_1, v_2\} \cup \{a_i : i \in [n]\} \cup \{b_i : i \in [n]\}, \text{ and edge set}$$

$$E(D_n) = \{(v_1, a_i) : i \in [n]\} \cup \{(v_2, b_i) : i \in [n]\} \cup \{(a_i, b_i) : i \in [n]\} \cup \{(a_i, a_{i+1}) : i \in [n-1]\}$$
$$\cup \{(b_i, b_{i+1}) : i \in [n-1]\} \cup \{(a_{i+1}, b_i) : i \in [n-1]\} \cup \{(a_1, a_n), (b_1, b_n), (a_1, b_n)\}.$$

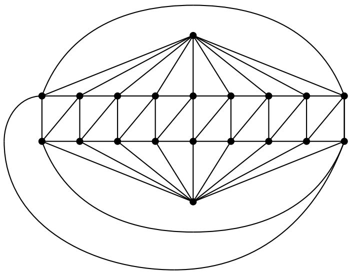

Figure 2: A planar embedding of D_9.

We proceed to characterise the K_5-minor-free homomorphic images of D_n.

Theorem 5.2. *Fix $n \geq 4$. Then any K_4-free and K_5-minor-free homomorphic image of D_n contains an induced copy of D_m for some $m \geq 4$ such that $m \mid n$.*

We then show that the existence of the graphs D_n as induced subgraphs is definable among K_4-free K_5-minor-free graphs by a simple first-order formula. Indeed, consider the formula

$$\chi(x_1, x_2, y_1, z_1, y_2, z_2) = E(x_1, y_2) \wedge E(y_1, y_2) \wedge E(z_1, y_2) \wedge E(z_1, z_2) \wedge E(y_2, z_2) \wedge E(z_2, x_2),$$

$$\text{and } \phi = \exists x_1, x_2, y, z[E(x_1, y) \wedge E(y, z) \wedge E(z, x_2) \wedge \forall a, b(E(x_1, a) \wedge E(a, b) \wedge E(b, x_2))$$
$$\to \exists c, d\ \chi(x_1, x_2, a, b, c, d))]$$

Proposition 5.3. *Let H be a finite K_4-free and K_5-minor free graph. Then $H \vDash \phi$ if and only if, there is some $n \geq 4$ such that H contains D_n as an induced subgraph.*

Putting the above together, we deduce the main theorem of this section.

Theorem 5.4. *The class of planar graphs does not have the HPP.*

Proof. Let $\hat{\phi}$ be the disjunction of ϕ with the formula that induces a copy of K_4, i.e. $\hat{\phi} := \phi \vee \exists x_1, x_2, x_3, x_4 \bigwedge_{i \neq j} E(x_i, x_j)$. We argue that $\hat{\phi}$ is preserved by homomorphisms over the class of planar graphs. Indeed, let $f : G \to H$ be a homomorphism with G, H planar such that $G \vDash \hat{\phi}$. Clearly, if H contains a copy of K_4 then $H \vDash \hat{\phi}$. Without loss of generality we may assume that $G \vDash \phi$ and G, H are K_4-free. It follows by Proposition 5.3 that there exists some $n \geq 4$ such that G contains D_n as a subgraph. Theorem 5.2 thus implies that H that there is some $m \geq 4$ such that H contains D_m as a subgraph. Proposition 5.3 then implies that that $H \vDash \phi$, and thus $H \vDash \hat{\phi}$ as required. To conclude, observe that the minimal models of $\hat{\phi}$ over the class of planar graphs are K_4 and the graphs D_n for $n \geq 4$; since these are infinitely many Theorem 2.1 implies that $\hat{\phi}$ is not equivalent to an existential-positive formula over the class of planar graphs. ☐

Since we only use exclusion of K_5-minors, the same proof relativises to the following theorem.

Theorem 5.5. *The class of all K_5-minor-free graphs does not have the HPP.*

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Undecidability of expansions of Laurent series fields by cyclic discrete subgroups

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Abstract

In 1987, Pheidas showed that the field of Laurent series $\mathbb{F}_q((t))$ with a constant for the indeterminate t and a predicate for the natural powers $\{t^n \mid n > 0\}$ of t is existentially undecidable. We show that the same result holds true if t is replaced by any element α of positive t-adic valuation.

Introduction. Hilbert's Tenth Problem asks for an *algorithm* that, given a polynomial $f(X_1, \ldots, X_n)$ with integer coefficients, will determine whether or not it has a root in integers $\mathbb{Z}$, see [10, 11]. Building on previous work by Robinson, Davis, and Putnam, Matiyasevich famously showed that no such algorithm exists [16]. Hilbert's Tenth Problem can be equivalently phrased as asking whether or not the positive existential theory $\mathrm{Th}_{\exists^+}(\mathbb{Z})$ in the first-order language of rings $\mathcal{L}_{\mathrm{ring}} = \{0, 1, +, \cdot\}$ is decidable [13, 1.1] (in what follows, we will omit the symbols of $\mathcal{L}_{\mathrm{ring}}$ when speaking of ring structures). In the context of model theory, it is both natural to consider other structures $\mathcal{M}$ that may differ from $\mathbb{Z}$ and to extend the family of sentences that we look at (e.g. the existential theory $\mathrm{Th}_\exists(\mathcal{M})$ or the entire theory $\mathrm{Th}(\mathcal{M})$). Many classical results in logic and model theory subsume answers to decidability questions.

Before Matiyasevich's negative solution to Hilbert's Tenth Problem, it was already known by Gödel's work on his Incompleteness Theorems [8] that the full first-order $\mathcal{L}_{\mathrm{ring}}$-theory $\mathrm{Th}(\mathbb{Z})$ is undecidable. In the 1930s and 1950s, Tarski [18, 19] determined the $\mathcal{L}_{\mathrm{ring}}$-theories of the real and complex fields $\mathbb{R}$, $\mathbb{C}$ (the archimedean local fields) and consequently showed that both are decidable. Ax and Kochen [4] studied the model theory of non-archimedean local fields, i.e., p-adic fields K (finite field extensions of the p-adic numbers $\mathbb{Q}_p$) and Laurent series fields

$$\mathbb{F}_q((t)) = \left\{ \sum_{i=-k}^{\infty} a_i t^i \,\middle|\, a_i \in \mathbb{F}_q,\ k \in \mathbb{Z} \right\}$$

over finite fields $\mathbb{F}_q$ with $q = p^n$ elements, p a prime number. It follows from their work that the theory $\mathrm{Th}(K)$ of any p-adic field K is decidable. Whether or not the Laurent series fields are decidable, is a major open question in the model theory of valued fields. In 2016, Anscombe and Fehm [2] made substantial progress towards this question by proving the decidability of the existential theory $\mathrm{Th}_\exists(\mathbb{F}_q((t)))$ of Laurent series fields. For other recent results in this direction, we refer to Anscombe, Dittmann, and Fehm [1, 7].

It is natural to consider the structures mentioned above in expansions of the language of rings. Van den Dries [20] considered the real ordered field with a new predicate for $2^{\mathbb{Z}}$, the cyclic multiplicative subgroup generated by 2. He proves the surprising result that $(\mathbb{R}, 2^{\mathbb{Z}})$ is decidable by showing quantifier elimination in a natural expansion of $(\mathbb{R}, 2^{\mathbb{Z}})$. This still holds if 2 is replaced by a recursive real number $\alpha > 1$. In the same paper, van den Dries asks if his results can be generalised to the structure $(\mathbb{R}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}})$. In 2010, Hieronymi [9] gave a negative answer: for two real numbers $\alpha, \beta > 1$ satisfying $\alpha^{\mathbb{Z}} \cap \beta^{\mathbb{Z}} = \{1\}$, the theory $\mathrm{Th}(\mathbb{R}, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ is undecidable. Expansions of $\mathbb{Q}_p$ by discrete cyclic (multiplicative) subgroups have been studied by Mariaule

[14, 15]. He proves that for $\alpha \in \mathbb{Q}_p$ of positive p-adic valuation $v_p(\alpha) > 0$, the theory $\mathrm{Th}(\mathbb{Q}_p, \alpha^{\mathbb{Z}})$ is decidable, whereas $\mathrm{Th}(\mathbb{Q}_p, \alpha^{\mathbb{Z}}, \beta^{\mathbb{Z}})$ is undecidable whenever $v_p(\beta) > 0$ and $\alpha^{\mathbb{Z}} \cap \beta^{\mathbb{Z}} = \{1\}$. Ax already knew (unpublished) that $\mathrm{Th}(\mathbb{F}_q((t)), t^{\mathbb{Z}})$ is undecidable. An elementary proof was given by Becker, Denef, and Lipshitz [5]. Later, a considerable strengthening was obtained by Pheidas [17]. This is particularly interesting, as not much is known about these fields from the point of view of (un)decidability. He shows:

Theorem (Pheidas)**.** *Let $P = \{t^n \mid n > 0\}$ be the set of powers of the indeterminate t. Then $\mathrm{Th}_\exists(\mathbb{F}_q((t)), t, P)$ is undecidable.*

Note that by virtue of Anscombe and Koenigsmann [3], who show that the valuation ring $\mathbb{F}_q[[t]]$ in $\mathbb{F}_q((t))$ is existentially $\mathcal{L}_{\mathrm{ring}}$-definable without parameters, it follows moreover that $\mathrm{Th}_\exists(\mathbb{F}_q((t)), t, t^{\mathbb{Z}})$ is undecidable (observe that $t^{\mathbb{Z}} \cap \mathbb{F}_q[[t]] = P \cup \{1\}$). We generalise this theorem to arbitrary cyclic discrete subgroups of $\mathbb{F}_q((t))$, i.e., subgroups generated by an element α of positive t-adic valuation $v_t(\alpha)$.

Theorem. *Let $\alpha \in \mathbb{F}_q((t))$ be an element with $v_t(\alpha) > 0$. Then the existential theory of the structure $(\mathbb{F}_q((t)), \alpha, \alpha^{\mathbb{Z}})$ is undecidable.*

See Remark 10 for a more general formulation. Another way of viewing $\alpha^{\mathbb{Z}}$ is to think of it as the image of a homomorphism from the value group $\mathbb{Z}$ into the multiplicative group $\mathbb{F}_q((t))^\times$. When $v_t(\alpha) = 1$, such a homomorphism is called a cross-section.

Pheidas' work. Pheidas proves his theorem in two steps. His key tool is the following (somewhat unusual) relation on natural numbers that goes back to Denef [6] and is sometimes called *p-divisibility*. We write

$$n \mid_p m \text{ if and only if } \exists k \in \mathbb{N}\ m = n \cdot p^k.$$

His proof now proceeds as follows.

(I) Prove that $\mathrm{Th}_\exists(\mathbb{N}, 0, 1, +, \mid_p)$ is undecidable by giving an existential definition of multiplication in this structure and invoking the Matiyasevich/MRDP theorem.

(II) Show that the relation $n \mid_p m$ can be effectively coded in $\mathbb{F}_q((t))$ by an existential formula via $P = \{t^n \mid n > 0\}$.

To generalise from t to arbitrary α, we precisely follow Pheidas' strategy. The main content of this note is to explain how Pheidas' coding needs to be modified in this more general context.

Essential to the coding is the unique arithmetic of $\mathbb{F}_q((t))$.

Remark 1. *In characteristic p, both the Frobenius map $x \longmapsto x^p$ and the Artin-Schreier map $x \longmapsto x^p - x$ are additive. Moreover, the Frobenius map is an automorphism on the finite field $\mathbb{F}_q$ and a non-surjective monomorphism on $\mathbb{F}_q((t))$ with image*

$$\mathbb{F}_q((t^p)) = \left\{ \sum_{i=-k}^{\infty} a_{pi} t^{pi} \,\middle|\, a_{pi} \in \mathbb{F}_q,\ k \in \mathbb{Z} \right\}.$$

This is the field of p^{th} powers in $\mathbb{F}_q((t))$.

Lemma 2. *Fix an element $\alpha \in \mathbb{F}_q((t))$ with $v_t(\alpha) > 0$ not divisible by p. We can characterise the relation $n \mid_p m$ for natural $m, n > 0$ as follows:*

$$n \mid_p m \quad \textit{iff} \quad m \geq n \wedge \exists a \in \mathbb{F}_q((t))\ \alpha^{-m} - \alpha^{-n} = a^p - a. \tag{1}$$

Proof. Pheidas' proof [17, Lem. 1] for $\alpha = t$ goes through in this case. We will use this opportunity to show his beautiful argument.

Assume $n \mid_p m$ holds such that $m = n \cdot p^k$ for some $k \in \mathbb{N}$. In that case, the element

$$a = \alpha^{-np^{k-1}} + \alpha^{-np^{k-2}} + \ldots + \alpha^{-n}$$

witnesses the right-hand side of (1). Conversely, assume that for positive integers $m \geq n$, there exists $a \in \mathbb{F}_q((t))$ satisfying $\alpha^{-m} - \alpha^{-n} = a^p - a$. Write $m = m_0 p^{v_p(m)}$ and $n = n_0 p^{v_p(n)}$, where both $m_0, n_0 > 0$ are not divisible by p. By the first part of the proof, we can find $b, c \in \mathbb{F}_q((t))$ with

$$\alpha^{-m} - \alpha^{-m_0} = b^p - b$$
$$\alpha^{-n} - \alpha^{-n_0} = c^p - c.$$

Setting $d = a - b + c$, we can combine these three equations to $\alpha^{-m_0} - \alpha^{-n_0} = d^p - d$. If $m_0 = n_0$, we are done since $m \geq n$. Otherwise, we may assume $m_0 \neq n_0$, in which case

$$v_t(d^p - d) = v_t(\alpha^{-m_0} - \alpha^{-n_0}) = -v_t(\alpha) \max\{m_0, n_0\}.$$

We know $v_t(d) < 0$ implies that $v_t(d^p - d)$ is divisible by p, which is in contradiction to our assumptions that $v_t(\alpha)$, m_0, n_0 are not divisible by p. □

Remark 3. *Note that (1) still holds in the case when we can write $\alpha = \beta^{p^k}$, $k \in \mathbb{N}$, where $v_t(\beta)$ is not divisible by p. Indeed, for $m \geq n$, we have*

$$\exists a \in \mathbb{F}_q((t)) \; \alpha^{-m} - \alpha^{-n} = \beta^{-mp^k} - \beta^{-np^k} = a^p - a$$

iff $np^k \mid_p mp^k$ iff $n \mid_p m$.

The general case. This characterisation of $\mid_p$ given by (1) will not work for all possible values of α, as we can see by the following counterexample.

Example 4. *Consider $p = q = 3$, i.e., the local field $\mathbb{F}_3((t))$ and the element*

$$\alpha = (t^{-3} + 1 + t + t^2)^{-1}$$

with $v_t(\alpha) = 3$ divisible by $p = 3$. Then $\alpha^{-2} - \alpha^{-1} = a^3 - a$ has a solution in $\mathbb{F}_3((t))$,

$$a = t^{-2} + t^{-1} - t + t^2 + \sum_{i \geq 0} (-1)^i (-t^{4 \cdot 3^i} + t^{6 \cdot 3^i}),$$

but the relation $1 \mid_3 2$ does not hold.

Hence a new observation is needed. For this purpose, we define the following unusual function, which we call the "p^{th}-powers-omitting t-adic valuation" for lack of a better name.*

Definition 5. *Given $x \in \mathbb{F}_q((t))$, written as a Laurent series*

$$x = \sum_{i=-k}^{\infty} a_i t^i,$$

*Note that, strictly speaking, $\hat{v}_t$ is not a valuation on $\mathbb{F}_q((t))$: it does not satisfy $x = 0 \iff \hat{v}_t(x) = \infty$ and it is also not a group homomorphism.

define $\hat{v}_t(x)$ *to be the integer*

$$\hat{v}_t(x) = \min\{i \mid a_i \neq 0 \wedge p \nmid i\},$$

and $\hat{v}_t(x) = \infty$ *if this minimum does not exist, i.e., if* $x \in \mathbb{F}_q((t^p))$.

Curiously, it captures exactly the kind of algebraic-combinatorial behaviour of $\mathbb{F}_q((t))$ that becomes invisible to v_t.

Lemma 6. *Assume that* $\alpha \in \mathbb{F}_q((t))$ *is not a* p^{th} *power, but* $p \mid v_t(\alpha) > 0$. *Let* $N \in \mathbb{N}$ *be not divisible by* p. *Then*

$$\hat{v}_t(\alpha^N) = (N-1)v_t(\alpha) + \hat{v}_t(\alpha).$$

Proof. Decompose α as $\alpha = \beta + \gamma$, where $\beta \neq 0$ contains all monomials with exponent divisible by p and $\gamma \neq 0$ contains all monomials with exponent not divisible by p. By our assumptions,

$$v_t(\beta) = v_t(\alpha) < \hat{v}_t(\alpha) = \hat{v}_t(\gamma).$$

Considering the binomial theorem for $(\beta + \gamma)^N$, we observe that

$$\binom{N}{N-1}\beta^{N-1}\gamma$$

must contain the monomial with smallest exponent not divisible by p. Thus

$$\hat{v}_t(\alpha^N) = \hat{v}_t(N\beta^{N-1}\gamma) = (N-1)v_t(\beta) + \hat{v}_t(\gamma) = (N-1)v_t(\alpha) + \hat{v}_t(\alpha). \qquad \square$$

Lemma 7. *Fix an element* $\alpha \in \mathbb{F}_q((t))$ *with valuation* $v_t(\alpha) = C > 0$ *divisible by* p. *Assume additionally that* α *is not a* p^{th} *power, so that* $\hat{v}_t(\alpha^{-1}) = D \in \mathbb{Z}$. *Then for any choice of* $N > 0$ *satisfying*

$$N > \frac{D}{C} + 1 \quad \text{and} \quad p \nmid N,$$

we have

$$n \mid_p m \quad \text{iff} \quad m \geq n \wedge \exists a \in \mathbb{F}_q((t))\ \alpha^{-mN} - \alpha^{-nN} = a^p - a$$

for all $m, n > 0$.

Proof. If $n \mid_p m$ holds, we essentially take the same witness $a \in \mathbb{F}_q((t))$ as in Lemma 2. As for the converse, let us consider positive integers $m \geq n$ such that there exists $a \in \mathbb{F}_q((t))$ with

$$\alpha^{-mN} - \alpha^{-nN} = a^p - a.$$

By repeating the same steps as in the proof of Lemma 2, we can write $m = m_0 p^{v_p(m)}$, $n = n_0 p^{v_p(n)}$ and find $d \in \mathbb{F}_q((t))$ such that

$$\alpha^{-m_0 N} - \alpha^{-n_0 N} = d^p - d. \tag{2}$$

We are done if $m_0 = n_0$. So assume without loss of generality that $m_0 > n_0 \geq 1$. Instead of considering the t-adic valuation on both sides of (2), we look at the p^{th}-powers-omitting t-adic valuation instead. By Lemma 6 and $p \nmid m_0 N$, we observe

$$\hat{v}_t(\alpha^{-m_0 N} - \alpha^{-n_0 N}) = -(m_0 N - 1)C + D. \tag{3}$$

If we evaluate the right-hand side of (2), we get

$$\hat{v}_t(d^p - d) = \hat{v}_t(d) \geq v_t(d). \tag{4}$$

Since $v_t(d) < 0$, we can use

$$pv_t(d) = v_t(d^p - d) = v_t(\alpha^{-m_0 N} - \alpha^{-n_0 N}) = -m_0 NC,$$

together with (2), (3), and (4), to deduce the inequality

$$-(m_0 N - 1)C + D \geq \frac{-m_0 NC}{p}.$$

After rearranging, we have

$$N \leq \frac{Cp + Dp}{m_0 C(p-1)} = \frac{C+D}{C} \frac{p}{m_0(p-1)} \leq \frac{D}{C} + 1,$$

contradicting our choice of N. Hence $m_0 = n_0$. □

In Example 4, it would suffice to take $N = 2$.

By combining Lemma 2 and Lemma 7, we can complete our coding of $|_p$ inside $\mathbb{F}_q((t))$.

Proposition 8. *Fix an element $\alpha \in \mathbb{F}_q((t))$ with valuation $v_t(\alpha) > 0$. Then there exists a parameter $N > 0$, depending on α, such that*

$$n \mid_p m \quad \textit{iff} \quad m \geq n \wedge \exists a \in \mathbb{F}_q((t))\ \alpha^{-mN} - \alpha^{-nN} = a^p - a$$

holds for all $m, n > 0$.

Proof. Write $\alpha = \beta^{p^k}$, $k \in \mathbb{N}$, such that β is not a p^{th} power in $\mathbb{F}_q((t))$. We consider two cases:

Case 1. p does not divide $v_t(\beta)$. By Lemma 2 and Remark 3, we can choose $N = 1$.

Case 2. p divides $v_t(\beta)$. By Lemma 7 and Remark 3, we can choose N to be the smallest natural number not divisible by p bigger than $\hat{v}_t(\beta^{-1})/v_t(\beta) + 1$. □

From this, we conclude our main theorem.

Theorem 9. *Let $\alpha \in \mathbb{F}_q((t))$ be an element with $v_t(\alpha) > 0$. Then the existential theory of the structure $(\mathbb{F}_q((t)), \alpha, \alpha^{\mathbb{Z}})$ is undecidable.*

Proof. First, we identify $\{\alpha^n \mid n > 0\}$ in this structure. This set is given by $\alpha^{\mathbb{Z}} \cap \mathbb{F}_q[[t]] \setminus \{1\}$. In [3], Anscombe and Koenigsmann show that $\mathbb{F}_q[[t]]$ is existentially $\mathcal{L}_{\text{ring}}$-definable in $\mathbb{F}_q((t))$ without parameters, so the same is true of $\{\alpha^n \mid n > 0\}$ inside $(\mathbb{F}_q((t)), \alpha, \alpha^{\mathbb{Z}})$. By Proposition 8, we can interpret $(\mathbb{N}, 0, 1, +, |_p)$ in $(\mathbb{F}_q((t)), 0, 1, +, \cdot, \alpha, \alpha^{\mathbb{Z}})$ using existential formulas. By (I), $\text{Th}_\exists(\mathbb{N}, 0, 1, +, |_p)$ is undecidable, so $\text{Th}_\exists(\mathbb{F}_q((t)), \alpha, \alpha^{\mathbb{Z}})$ must also be undecidable. □

Remark 10. *Pheidas formulates his theorem in slightly more general terms: for any integral domain F of characteristic p, quotient field K of F, and intermediate ring $F[t] \subseteq R \subseteq K((t))$, the existential theory $\text{Th}_\exists(R, t, P)$ is undecidable. The same is true of our result: as long as $\alpha \in R$, we have that $\text{Th}_\exists(R, \alpha, \{\alpha^n \mid n > 0\})$ is undecidable (essentially by the same proof).*

More recently, an adaption of Pheidas' theorem via the so-called Krasner-Kazhdan-Deligne philosophy was obtained by Kartas [12], who shows that the asymptotic theory of all p-adic fields is undecidable in the language of rings with a cross-section. We hope to further adapt these types of results to infinitely ramified valued fields.

Acknowledgements. First and foremost, I would like to express my gratitude to Philipp Hieronymi who advised this work in 2022. Moreover, I would like to thank Philip Dittmann, Konstantinos Kartas, and Jochen Koenigsmann for valuable comments that helped improve the exposition of this note.

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Value-aligning legitimate robot judges using λ-calculus

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Abstract

In our presentation, we intend to showcase the central role that formal philosophy ought to play in the engineering of AI models that exercise judicial power *legitimately.* We begin by arguing that according to the value of legitimacy, such AI models ought to provide justifications for their outputs that have the same logical form as the justifications found in case-law. We further argue that the conceptual re-engineering methods of Carnapian explication & narrow reflective equilibrium can be combined to guide formal philosophers in the practice of descriptively formalising those logical forms. Finally, we exhibit how this can be done when the formal philosopher uses λ-calculus to perform such formalisations. To make our case, we refer to two state-of-the-art methodologies of logically reconstructing judicial reasoning, those of LogiKEy & Catala.

1 Introduction

Algocracies, i.e., political orders where political power is exercised inter alia *by* or *via* algorithms, are already a reality [9]. A prime example of such algorithmic political authorities are AI models that exercise judicial power. Such is the example of AI that partakes in the interpretation & application of the law, the so-called "*robot judges*"[22] (e.g., identifying irregularities in contracts between the public & private sector [10], predicting the probability of recidivating [23], deciding the outcome of a criminal law case [24]). Since such AI models exercise judicial power, they now constitute political authorities themselves, and hence, they should be checked & balanced so as to avoid any abuse of power. Such a check & balance is the alignment of those models towards the value of legitimacy: *is the exercise of power by the robot judge legitimate* (*cf.* [9])?

The practice of engineering AI that abides by specific values is known in the literature as *value-alignment* [23]. Despite its seemingly "*practical*" character, value-alignment lies at the core of a traditional problem in diverse fields of analytic philosophy (e.g., philosophy of logic, (meta-)ethics, philosophy of science), that of deciding the truth value of the so-called *evaluative judgements.* Specifically, an evaluative judgement is a judgement of whether a particular value (e.g., the value of legitimacy) is applicable to a particular case (e.g., the exercise of power by a judge) ([18]; *cf.*[20]). In analytic meta-semantics, an evaluative judgement is essentially construed as a question of whether an object is subsumed by a concept or whether a term is subsumed by a specific predicate or whether a particular is an instantiation of a universal [2, 15]. The challenge with evaluative judgements' truth value is that evaluative judgements are accused of not being objective [18]. For instance, since the value of legitimacy has a different meaning in the European Enlightenment-rooted political tradition than the Chinese Confucian-based one[11], the evaluative judgement of whether a judicial authority is legitimate has a different answer in each of the two traditions. A method to overcome this (seemingly) subjective nature of evaluative judgement is to reduce it to specific *factual* judgements, where the latter can be minimally construed as empirical judgments about states of the physical world. I.e., judgements about a *non*-subjective *ordo essendi* for which there is a strong intersubjective consensus about its characteristics. E.g., a judge's decision is legitimate only if the *factual* condition of the judge not being bribed is true. In practice, before an evaluative judgement is (partially) reduced to factual judgements, it is customary to reduce it to other evaluative judgements and then reduce those evaluative judgements to factual ones (*see e.g.* [1, 14]). Such reductions of evaluative judgement to factual ones are called *operationalisations* [7]. In our case, the evaluative judgment of AI's legitimacy-alignment should be operationalised minimally to AI's alignment towards the following two values: (i) the epistemic value of foreseeability. Specifically, the application of the law should be foreseeable.; (ii) the legal value of legality which can be minimally construed as the so-called supremacy of law: everyone, even those that exercise power, should abide by the law [8, 14].

The necessity of alignment towards the values of legality & foreseeability imposes to the robot judge the factual requirement of providing a justification for its output that has the *same* logical form with the justifications provided by the judicial authorities that have the authority by the law to judge the cases in question. In particular, for the application of the law to be foreseeable, AI should apply the *same* reasoning methods in *similar* circumstances. Otherwise, one can not know with certainty the circumstances under which they violate the law. However, different judicial authorities propose the application of different reaosning methods to similar cases raising the question of which one should the robot judge adopt. For instance, in order to establish a causal relation between an action of a defendant (e.g., shooting the victim) and the alleged harm induced by that action (e.g., the victim dying), Anlgo-American criminal law courts often employ the so-called *but-for test*: but-for the action of the defendant (legal cause), the harm (legal effect) would not have happened [17]. However, the European Court of Human Rights (ECtHR) has explicitly rejected the but-for test as a reasoning method of establishing legal causal relations between actions and harm (*E. and others v. UK*, no. 33218/96, 2002, ¶99; *cf.* [21]). Subsequently, different construals of causal reasoning can lead to different (conflicting) judgements making the application of the law less foreseeable.

The foregoing bring about the question of which should be the reasoning methods used in the application of the law. E.g., should we use the but-for test or not? The answer is given by the value of legality. Specifically, the supremacy of law dictates that the legitimate reasoning methods are those employed by the judicial authorities that are prescribed by law to judge the case in question (e.g., criminal courts judging criminal law cases). Any divergence from the reasoning methods found in the case law of those authorities undermines the value of legality.

Considering the above, we ought to impose specific logical constraints to AI that exercises judicial power so as to output justifications of the desired logical form (for an overview of such logical constraints to connectionist AI *see* [12]). Therefore, one needs to logically reconstruct the judicial reasoning used by the legitimate judicial authorities so as to determine which should those logical constraints be. Which then brings about the question of which *methodology* should one use to perform such descriptive logical reconstructions of judicial reasoning.

Two such methodologies of logical reconstruction are LogiKEy[1] and Catala,[2] which can be abstracted in the following four-steps high-level schema:[3] (i) parsing case & statutory law documents so as to mine in natural language the reasoning methods a judicial authority uses; (ii) choosing an *object logic* to formally model the mined reasoning methods. In Catala, they choose a prioritised default typed logic. The priority relation is used to model exceptions in the application of the law by prioritising conditions that override the default way that the law

[1] Introduced in [4]. The acronym stands for **Logic** and **K**nowledge **E**ngineering Framework and Methodolog**y**.
[2] Introduced in [16]. It is "*[n]amed after Pierre Catala [...] a pioneer of French legal informatics*"[13].
[3] Note that there can (and should) be a fluctuation among those steps, albeit the basic order remains as is.

is applied. In juxtaposition to the choice of a unique object logic, LogiKEy can be employed for (a combination of) different object logics like modal logics of preferences or dyadic deontic logics [3, 4].; (iii) *translating* the logical reconstructions of the object logic to λ-terms in a target λ-calculus. LogiKEy λ-translates formulae ϕ_i of the object logic while Catala λ-translates terms t_i.; (iv) translating those λ-translations to (functional) programming code. In Catala, this code is directly used to engineer AI, while in LogiKEy, it is used by automated theorem provers (ATPs) for verifying whether the object logic is a faithful reconstruction of judicial reasoning (more on *faithfulness* below). Catala can generate code in programming languages from diverse programming paradigms (e.g., OCaml, (Java)Script, Python), while LogiKEy can employ only programming languages used by ATPs that use HOL (e.g., Isabelle/HOL).

The existence of different methodologies of logically reconstructing judicial reasoning like Catala & LogiKEy necessitates the employment of a *methodology* that can evaluate which of the available logical reconstruction methodologies is more adequate. Since a *good* methodology for logically reconstructing judicial reasoning is a methodology that generates *good* models of that reasoning, the evaluation of that methodology's goodness is reduced to the evaluation of goodness of the models that it generates. To evaluate a model's goodness, we can identify a list of criteria that a good model should satisfy. For such a list, we have to look no further than formal philosophy's early pioneer Rudolf Carnap's *explication* criteria. Specifically, explication is a conceptual re-engineering method of identifying a particular concept in a particular discourse (e.g., the concept of causal reasoning in the discourse of the ECtHR), and then, re-engineering this concept in a (non-)formal form in a different discourse (e.g., the discourse of formal philosophy of law) such that the re-engineered concept being *similar* to the initial concept, more *exact*, more *fruitful*, and more *simple*. I.e., the re-engineered concept needs to corroborate the epistemic values of similarity, exactness, fruitfulness, and simplicity. The initial concept is called *explicandum* and its explicated form is called *explicatum*. In our case, the reasoning method used by judicial authorities (e.g., causal reasoning) is the *explicandum* and its model in the object logic is the *explicatum*. The λ-translations of the object logic and the subsequent (functional) programming codes are *not* explicata since they are formalisations of the object logic in the *same* discourse as the object logic.

Considering the above, one should prefer the logical reconstruction methodologies that corroborate explication's four epistemic values the strongest. Subsequently, the choice of a logical reconstruction methodology becomes a *value*-driven decision. Hence, we are faced once more with the problem of the objectivity of evaluative judgements. We can once more overcome it by operationalising the four evaluative judgements. Since we want to use those operationalisations to engineer legitimate models of judicial reasoning, they should be grounded on the operationalisation of the value of legitimacy (e.g., choose operational definitions of similarity that make the application of the law more foreseeable). In what follows, we briefly exhibit how such an operationalisation of similarity can be used to compare LogiKEy's & Catala's adequacy.[4] We focus on the role that λ-calculus plays in the two methodologies due to its key contribution to the corroboration of the value of *similarity*.

The first *similarity* operational requirement is imposed by the value of foreseeability: legitimacy-aligned logical reconstruction need to make explicit necessary and/or sufficient conditions under which a reasoning method should be used to interpret the law so as for that interpretation to be foreseeable. I.e., the *explicandum* & the *explicatum* need to be *intentionally* similar. Intentional similarity conditions are many times determined by specific theories of interpretation of the law. In LogiKEy, λ-translations can be used to model such interpretation theories in ATPs. Through those ATP embeddings, we can evaluate whether those theorems

[4] A more thorough operationalisation of the 4 values can be found in one of the authors' MSc Thesis: [14].

are true in the object logic. The use of HOL in the LogiKEy approach is necessary for the theorem verification task. Specifically, in order to evaluate the truth of theorems expressed in λ-terms, one has to provide a semantical interpretation of those terms that allows those theorems to be truth bearers. To deal with this challenge, LogiKEy's λ-translations in HOL have bodies with the same syntactic structure as formulae of classical higher-order logic (e.g., $\lambda w.\varphi(w) \wedge \psi(w)$). *Via* this syntactical similarity, λ-translations can be semantically interpreted using Henkin models. Thus, if one uses a Henkin-sound ATP, then a proof of a theorem entails its truth. The only thing left is to prove that there is a truth preservation between the HOL's & object logic's semantics, what in [4] call as *faithfulness of the embedding*. *Contra* to LogiKEy, Catala's λ-translations do not have a semantical interpretation, and hence, its current form does not suffice to verify the truth of interpretation theories.

However, both Catala and LogiKEy can be used to corroborate the similarity requirement of *coherence*: a formal model of a judicial reasoning should verify paradigmatic applications of that reasoning method in judicial judgements so as to corroborate the coherence among those judgements. The requirement for coherence is imposed by both the value of legality and value of foreseeability. Firstly, formal models of judicial reasoning should produce the same judgements as those of the legitimate human judicial authorities (legality). Secondly, paradigmatic cases of how a reasoning method is applied in past cases are used to determine future applications of the law (foreseeability). In philosophy of law, the construal of the epistemic value of coherence as the satisfaction of paradigmatic judgements is *on par* with another landmark conceptual re-engineering method, that of *narrow reflective equilibrium* (*NRE*) advocated by pioneers in philosophy of law like John Rawls & Richard Dworkin (*see e.g.* [19]). The combination of explication and NRE can be used to balance out their perils and maximise their efficiency[6].

In order to verify paradigmatic applications of a reasoning method, both LogiKEy and Catala make use of λ-calculi's translatability to functional programming code: one can give as input to the code the facts of past cases and then verify whether the output of the code coincides with the respective judgements. For that to be possible, both methods need to employ some kind of judgement-preservation theorems: judgements that can be derived in the object logic should be derivable in the target λ-calculi as well. In the case, of LogiKEy, this is secured once more through HOL's semantical interpretation *via* Henkin models. On the other hand, in order to ensure judgement-preservation without semantics, Catala adopts the GOFAI rule-based modelling of judicial reasoning. Specifically, certain λ-translations have the syntactic structure of IF-THEN *rules with exceptions*: the *heads* of the rules are potential judgements and the *bodies* are sufficient conditions for each of those judgements as well as possible exceptions to those conditions. By following the reduction rules of the target λ-calculus, those rule-like λ-terms can be reduced to either their heads (i.e., specific judgements) or to their exceptions. Using this rule-reduction schema, Catala ensures judgement-preservation though what they call *correctness theorem*: a rule-like λ-translation is reduced to the same λ-term that the λ-translation of the object logic's true judgement (or its true exception) is reduced to.

Summing up, it seems that LogiKEy corroborates the value of similarity better than Catala since it can be used to enhance both intentional similarity & coherence. Having said that, before concluding on which of the two methodologies is more adequate, one has to evaluate their performance in other similarity requirements (e.g., extensional & relational similarity) as well as to the rest three explication values [14, 6, 5]. Another conclusion from the above analysis is the ways that λ-calculus can be used corroborate the value of similarity setting a precedent for other logical reconstruction methodologies to follow. The foregoing remarks are a quick taste of how explication & NRE can guide a formal philosophers practice of modelling *legitimate* models of judicial reasoning that can be employed by robot judges.

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Generic groups and WAP

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Abstract

We consider the logic space of countable (enumerated) groups and show that closed subspaces corresponding to some standard classes of groups have (do not have) generic groups.

1. INTRODUCTION. The recent paper [1] considers countable groups as elements of a Polish space in the following way. An **enumerated group** is the set ω together with a multiplication function $\cdot : \omega \times \omega \to \omega$, an inversion function $^{-1} : \omega \to \omega$, and an identity element $e \in \omega$ defining a group. The space $\mathcal{G}$ of enumerated group is the closed subset of the space $\mathcal{X} = \omega^{\omega\times\omega} \times \omega^{\omega} \times \omega$ (under the product topology). If U is a universal extension of the theory of groups then the space of enumerated groups satisfying U, say $\mathcal{G}_U$, is a closed subspace of $\mathcal{G}$. Since all these spaces are Polish (separable and completely metrizable), Baire category methods can be applied.

In fact, instead of U any abstract property of groups, say P, can be considered. It naturally defines the invariant subset $\mathcal{G}_P$ of $\mathcal{G}$. Assuming that $\mathcal{G}_P$ is Polish it is studied in [1] which group-theoretic properties define comeagre subsets of $\mathcal{G}_P$. In particular, are there **generic groups** in $\mathcal{G}_P$ (i.e.groups forming a comeagre isomorphism class)? The main results of [1] state that there is no generic group in $\mathcal{G}$ and moreover, when P is the property of left orderability the space $\mathcal{G}_P$ does not have a generic group. The authors deduce this using involved arguments in the style of model-theoretic forcing.

A. Ivanov studied the problem of existence of generic objects in [2] in full generality. He introduced some general condition which is now called the weak amalgamation property (WAP) and showed that it together with the joint embedding property characterizes the existence of generics. (In [2] this condition was called the almost amalgamation property. It was considered later for automorphisms in the very influential paper [4] were the name WAP was used.)

The goal of our paper is to demonstrate that the results of [1] mentioned above can be deduced by an application of [2]. We also add some new important examples of this kind. Furthermore, the approach of [1] is extended to actions of groups on first-order structures, for example on $(\mathbb{Q}, <)$. We also show how [2] works in the contrary direction, i.e. for proving of existence of generics in $\mathcal{G}_U$ for some natural group varieties U.

In this text we omit proofs. They can be found in [3]. Theorems 4 and 7 are already available to the reader at this stage. These are our new results concerning Burnside varieties. Other results below describe our approach and technical details.

General preliminaries. Fix a countable ω-categorical structure M in a language L. Let T be an expansion of $Th(M)$ in some $L \cup \bar{r}$ where $\bar{r} = (r_1, \ldots, r_\iota)$ is a sequence of additional relational/functional symbols. We assume that T is axiomatizable by sentences of the following form:

$$(\forall \bar{x}) \left(\bigvee_i (\phi_i(\bar{x}) \wedge \psi_i(\bar{x})\right),$$

where $\phi_i(\bar{x})$, is a quantifier-free formula in the language $L \cup \bar{r}$, and $\psi_i(\bar{x})$ is a first order formula of the language L. Let $\mathcal{X}_M$ be the space of all $\bar{r}$-expansions of M to models of T. This is a topological space with respect to so called **logic topology**. In the following few paragraphs we introduce some notions from [2]. They will be helpful in a description of the logic topology too.

For a tuple $\bar{a} \subset M$ we define a diagram $\phi(\bar{a})$ of $\bar{r}$ on $\bar{a}$ as follows. To every functional symbol from $\bar{r}$ we associate a partial function from $\bar{a}$ to $\bar{a}$. When r_i is a relational symbol from $\bar{r}$ we choose a formula from every pair $\{r_i(\bar{a}'), \neg r_i(\bar{a}')\}$, where $\bar{a}'$ is a subtuple from $\bar{a}$ of the corresponding length.

Let $\mathcal{B}_T$ be the set of all theories $D(\bar{a})$, $\bar{a} \subset M$, such that each of them consists of $Th(M, \bar{a}) \cup T$ together with a diagram of $\bar{r}$ on $\bar{a}$ satisfied in some $(M, \bar{\mathbf{r}}) \models T$. Since M is atomic, each element of $\mathcal{B}_T$ is determined by a formula of the form $\phi(\bar{a}) \wedge \psi(\bar{a})$, where $\psi(\bar{x})$ is a complete formula for M realized by $\bar{a}$ and $\phi(\bar{a})$ is a quantifier-free formula in the language $L \cup \bar{r}$. The corresponding $\phi(\bar{x}) \wedge \psi(\bar{x})$ is called **basic**.

For every diagram $D(\bar{a}) \in \mathcal{B}_T$ the set

$$[D(\bar{a})] = \{(M, \bar{\mathbf{r}}) \in \mathcal{X}_M \mid (M, \bar{\mathbf{r}}) \text{ satisfies } D(\bar{a})\}$$

is clopen in the logic topology. In fact the family $\{[D(\bar{a})] \mid D(\bar{a}) \in \mathcal{B}_T\}$ is usually taken as a base of this topology. It is metrizable. Each $D(\bar{a}) \in \mathcal{B}_T$ can be viewed as an expansion of M by finite relations corresponding to $\bar{r}$. When $\mathbf{r}_i$ is a functional symbol the corresponding relation is $\mathsf{Graph}(\mathbf{r}_i)$ the graph of the corresponding partial function on $\bar{a}$. We will say that $\bar{a}$ is the domain of this diagram: $\bar{a} = \mathsf{Dom}(D(\bar{a}))$.

Definition 0. *An expansion* $(M, \bar{\mathbf{r}})$ *is called* **generic** *if it has a comeagre isomorphism class in* $\mathcal{X}_M$.

The set $\mathcal{B}_T$ is ordered by the relation of extension: $D(\bar{a}) \subseteq D'(\bar{b})$ if $\bar{a} \subseteq \bar{b}$ and $D'(\bar{b})$ implies $D(\bar{a})$ under T (in particular, the partial functions defined in $D'(\bar{b})$ extend the corresponding partial functions defined in $D(\bar{a})$).

In these terms we formulate the definitions of JEP, AP and WAP.

- The family $\mathcal{B}_T$ has the **joint embedding property** if for any two elements $D_1, D_2 \in \mathcal{B}_T$ there is D_3 from $\mathcal{B}_T$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.

- The family $\mathcal{B}_T$ has the **amalgamation property** if for any $D_0, D_1, D_2 \in \mathcal{B}_T$ with $D_0 \subseteq D_1$ and $D_0 \subseteq D_2$ there is $D_3 \in \mathcal{B}_T$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ fixing $\mathsf{Dom}(D_0)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.

- The family $\mathcal{B}_T$ has the **weak amalgamation property** if for every $D_0 \in \mathcal{B}_T$ there is an extension $D_0' \in \mathcal{B}_T$ such that for any $D_1, D_2 \in \mathcal{B}_T$ with $D_0' \subseteq D_1$ and $D_0' \subseteq D_2$ there is $D_3 \in \mathcal{B}'$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ fixing $\mathsf{Dom}(D_0)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.

By Theorem 1.2 from [2]:

$\mathcal{X}_M$ *has a generic expansion* $(M, \bar{\mathbf{r}})$ *if and only if the family* $\mathcal{B}_T$ *has JEP and WAP.*

I. Enumerated groups. The basic case. The structure M is just ω and L consists of one constant symbol 1 which is interpreted by number 1. Let $\bar{r}$ consist of the binary function of multiplication $\cdot$ and a unary function $^{-1}$. Let T be the universal theory of groups with the unit 1. [1]

[1] It is not necessary to fix 1 in the language of M. We do it just for convenience of notations in Scenario **II** below.

Each element of $\mathcal{B}_T$ consists of a tuple $\bar{c} \subset \omega$ containing 1 and partial functions for $\cdot$ and $^{-1}$ on $\bar{c}$. The space $\mathcal{X}_M$ corresponding to T is the logic space of all countable groups, where the unit is fixed. *We emphasize that this is exactly the space $\mathcal{G}$ of enumerated groups from [1].*

If instead of T above one takes a universal extension of the theory of groups, say $\bar{T}$, then the corresponding space of enumerated groups is the closed subspace $\mathcal{G}_{\bar{T}} \subseteq \mathcal{G}$.

In the paragraph below we will have several sorts in M. The sort just described will always occur. We will denote it by Gp. In particular if $(M, \bar{\mathbf{r}}) \in \mathcal{X}_M$ then we denote by $\mathsf{Gp}_{\bar{\mathbf{r}}}$ the group defined by $\bar{\mathbf{r}}$ on this sort.

II. The case of an action. Let M_0 be an atomic structure of some language L_0. Define M to be M_0 with an additional sort ω called Gp and the constant symbol 1 interpreted by 1. The symbols $\bar{r}$ include $\cdot$, $^{-1}$ and a new symbol ac for a function $M_0 \times \mathsf{Gp} \to M_0$. The theory T contains the universal axioms of groups on Gp with 1. We also add the axioms for an action:

$$\mathsf{ac}(x, z_1 \cdot z_2) = \mathsf{ac}(\mathsf{ac}(x, z_1), z_2) \ \ , \ \ \mathsf{ac}(x, 1) = x.$$

The space $\mathcal{X}_M$ is the logic space of all countable expansions of M where the group structure is defined on Gp, with a fixed unit 1 and an action ac .

It is natural to add the universal axioms that ac preserves the structure of M_0. In this way we obtain the space of actions on M_0 by automorphisms.

2. SEEKING GENERICS.

The following definition concerns cases **I,II**.

Definition 1. *Let $\bar{c}$ be a subtuple of $\bar{c}'$ and $D(\bar{c}') \in \mathcal{B}_T$. We say that $D(\bar{c}')$ is* **t-isolating** *(term-isolating) for $\bar{c}$ if for any two members of $[D(\bar{c}')]$ the groups $\langle \bar{c} \cap \mathsf{Gp} \rangle$ coincide (on the sets of group words of the alphabet $\bar{c} \cap \mathsf{Gp}$).*

In (basic) case **I** let G be a group which is defined on the sort Gp and $G = \langle \bar{c} \rangle$ for some $\bar{c} \subseteq \omega$ (it is not assumed that $G = \omega$). We say that G is *t-isolated* by $D(\bar{c}')$ if $D(\bar{c}')$ is t-isolating for $\bar{c}$ and G is the corresponding $\langle \bar{c} \rangle$.

Definition 2. (Case **I**) *We say that an abstract group G is* **t-isolated** *if it has a Gp-copy, say $\langle \bar{c} \rangle$, which is isolated by some $D(\bar{c}')$ for $\bar{c}$ where $\bar{c}' \subset \langle \bar{c} \rangle$.*

We say that t-isolated diagrams are *dense* in $\mathcal{B}_T$ if any $D_0(\bar{c}) \in \mathcal{B}_T$ extends to some $D(\bar{c}')$ which is t-isolating for $\bar{c}$ and $\bar{c}' \subset \langle \bar{c} \rangle$.

The following proposition is our tool for seeking generics.

Proposition 3. *Under the circumstances of case* **I** *assume that t-isolated diagrams are dense in $\mathcal{B}_T$. Then WAP for $\mathcal{B}_T$ is equivalent to the following property:*

> *any t-isolated G_0 can be extended to a t-isolated G_1 such that any two t-isolated extensions of G_1 can be amalgamated over G_0.*

Let us note that every finite group $F = \langle \bar{c} \rangle$ is t-isolated. Using this we can conclude that *if all groups satisfying T are residually finite then t-isolated diagrams are dense in $\mathcal{B}_T$. Furthermore, then t-isolated groups satisfying T are exactly finite ones.*

Under the scenario of case **I** abelian groups form a closed subspace of $\mathcal{X}_M$. Proposition 3 gives an easy proof that this subspace has a generic group, a result proved in [5]. The following situation is more complicated.

Theorem 4. *Let $c, p \in \omega \setminus \{0, 1\}$ and p be prime $> \mathsf{max}(2, c)$. The closed subspace $\mathcal{G}_{c_{nil}p} \subseteq \mathcal{G}$ of all* **nilpotent groups of degree** *c* **and of exponent** *p has a generic group.*

3. STRONG UNDECIDABILITY IMPLIES NON-WAP. We will assume below the setup of one of the cases **I - II**, where T is a universal theory including group axioms for the sort Gp.

The following proposition is a generalization of the theorem of Kuznetsov that a recursively presented simple group has decidable word problem.

Proposition 5. *Assume that $D(\bar{c}) \in \mathcal{B}_T$ and an extension $D'(\bar{c}') \in \mathcal{B}_T$ is t-isolating for $\bar{c}$. Let $(M, \bar{\mathbf{r}}) \in [D'(\bar{c}')]$. Assume that $D'(\bar{c}')$ is a computably enumerable set. Then in $(M, \bar{\mathbf{r}})$ the elements of $\bar{c}$ which belong to the group sort $\mathsf{Gp}_{\bar{\mathbf{r}}}$ generate a recursively presented group with decidable word problem.*

Our main tool for proving the absence of generics in $\mathcal{G}_T$ is as follows.

Theorem 6. *Assume that $D(\bar{c}) \in \mathcal{B}_T$ (where $\bar{c}$ includes all distinguished elements of M) satisfies the property that every extension $D'(\bar{c}')$ is computably enumerable and for every $(M, \bar{\mathbf{r}}) \in [D'(\bar{c}')]$ the group $\langle \bar{c} \cap \mathsf{Gp}_{\bar{\mathbf{r}}} \rangle$ (defined in $(M, \bar{\mathbf{r}})$) has undecidable word problem.*

Then $D(\bar{c})$ does not have an extension required by WAP for $\mathcal{B}_T$.

We now give one of our applications of Theorem 6 for concrete spaces of enumerated groups.

Theorem 7. *There is a constant C such that for every odd integer $n \geq C$, the space $\mathcal{G}_{exp.n}$ of all groups of exponent n does not have a generic group.*

Remark 8. A similar approach works for the logic spaces of semigroups and rings. We prove that there is neither generic semigroup nor generic associative ring.

Generics over rationals. Consider Scenario **II**, where $M_0 = (\mathbb{Q}, <)$. Define M to be M_0 with the additional sort Gp and the constant symbol 1 interpreted by $1 \in \omega$. The symbols $\bar{r}$ are $\cdot$, $^{-1}$ and ac for an action $M_0 \times \mathsf{Gp} \to M_0$ by automorphisms.

The theory T contains the universal axioms of groups on the sort Gp with the unit 1 and the axioms for an action by automorphisms. The space $\mathcal{X}_M$ is the logic space of all countable expansions of M where the group structure is defined on Gp, with a fixed unit 1 and an action ac . We denote it by $\mathcal{X}_{(\mathbb{Q},<)}$.

Theorem 9. *The space $\mathcal{X}_{(\mathbb{Q},<)}$ does not have a generic action.*

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On the Extension of Argumentation Logic

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Abstract

This paper shows how Argumentation Logic can be further extended to cover more fully paraconsistent forms of reasoning. The extension is based on the notion of non-acceptable self-defeating arguments as a generalization of the Reductio ad Absurdum principle.

1 Motivation and Background

Argumentative inference relies on the central normative condition of the acceptability of a (set of) argument(s). Informally, this condition states that "a (set of) argument(s) is acceptable iff it defends against all its counter-arguments". An acceptable argument thus forms a "case" that supports *satisfactorily* its claim and hence the claim is a possible or credulous conclusion under the argumentative reasoning.

When this is applied to formal logical reasoning where arguments are sets of logical formulae, e.g., propositional formulae, a case corresponds to a set of formulae which can be enveloped in a model of the theory and thus a credulous conclusion corresponds to a *satisfiable* formula. We can then show that such a form of Argumentation Logic (AL) is logically equivalent to classical Propositional Logic (PL). This equivalence though holds only when reasoning under a set of given premises that are classically consistent. When the premises are inconsistent, AL does not trivialize like PL but smoothly extends PL into a paraconsistent logic.

Technically, AL does this by encompassing the proof rule of Reduction ad absurdum through the notion of *non-acceptability* of arguments, namely the contrary notion of acceptability of arguments. Non-acceptable arguments are "self-defeating" arguments. Informally, such an argument is one that either forms a counter-argument to itself or that it is a counter-argument to an argument that it necessarily needs in order to defend against some counter-argument to it. In other words, a non-acceptable or self-defeating argument invalidates its possible case of support by rendering the set of arguments in the case incompatible with each other.

In this paper, we will explore further the notion of non-acceptable arguments and study how this can give in the AL reformulation of PL new acceptable sets of arguments (under inconsistent premises) that were not recognized as such before. Whereas in the previous work on AL in [3] this was carried out only for the limiting case of non-acceptability of a self-attacking counter-argument, in this paper we will show how more complex forms of self-defeating non-acceptable arguments can be identified and used to "neutralize" the effect of such arguments when they appear as counter-arguments to other arguments.

Section 2, reviews the acceptability semantics for general abstract argumentation frameworks under which the classical Propositional Logic is reformulated as an Argumentation logic. Section 3 discusses the non-acceptability of arguments. Section 4 defines the proposed extension of the acceptability semantics and applies this to the specific case of AL as a reformulation of PL. Section 5 concludes with a brief discussion of future work.

2 Acceptability semantics of Argumentation

Let us briefly review the area of Abstract Argumentation and its semantics [1, 2] as developed and used in the area of Artificial Intelligence. In abstract argumentation we are not interested in the internal structure of arguments but only in their relative properties.

Definition 1. *[Abstract Argumentation Framework]*
An abstract argumentation framework is a triple, $\langle Arg, Att, \mathcal{D}ef\rangle$, where

- *Arg is a set (of arguments)*
- *Att is a binary (partial) relation on Arg (attack relation)*
- *$\mathcal{D}ef$ is a binary (partial) relation on Arg (defence relation)*

Given $A, \Delta, D \subseteq Arg$, we say that A **attacks** Δ (written $A \rightsquigarrow \Delta$) iff there exists $a \in A$ and $b \in \Delta$ such that $(a, b) \in Att$ and that D **defends against** A (written $D \twoheadrightarrow A$) iff $(d, c) \in \mathcal{D}ef$ for some $d \in D$ and $c \in A$.

A typical realization of a triple argumentation framework in some language, $\mathcal{L}$, for constructing and comparing arguments is given by: (1) a is in conflict in $\mathcal{L}$ with b for $(a, b) \in Att$ to hold and (2) a is at least as strong in $\mathcal{L}$ as b for $(a, b) \in \mathcal{D}ef$ to hold. In such realizations, the attack relation is symmetric and the defence relation is a subset of the attack relation.

The semantics of an abstract argumentation framework is defined via subsets of arguments that satisfy an acceptability property, $\mathcal{A}cc(\Delta, \Delta_0)$, whose informal meaning is that the set of arguments Δ is acceptable in the context of a given set of arguments Δ_0, only when Δ can defend against all its counter-arguments.

Definition 2 (Acceptability property)**.**
Let $AF = \langle Arg, Att, \mathcal{D}ef\rangle$ be an abstract argumentation framework and $\Delta, \Delta_0 \subseteq Arg$. Then:

- *$\mathcal{A}cc(\Delta, \Delta_0)$ iff*
  - *$\Delta \subseteq \Delta_0$, or*
  - *for any $A \subseteq Arg$ such that $A \rightsquigarrow \Delta$:*
    - *$A \not\subseteq \Delta \cup \Delta_0$, and*
    - *there exists $\Delta' \subseteq Arg$ such that $\Delta' \twoheadrightarrow A$ and $\mathcal{A}cc(\Delta', \Delta \cup \Delta_0)$*

In other words, counter-arguments must be defended against by arguments that are themselves acceptable in the extended context of $\Delta \cup \Delta_0$, and hence Δ can contribute to its own defense. Formally, the acceptability property is defined through the least fixed point of an associated monotonic operator on the binary Cartesian product of sets of arguments $\mathcal{R} = 2^{Arg} \times 2^{Arg}$.

Definition 3 (Acceptability operator)**.** *Let $AF = \langle Arg, Att, \mathcal{D}ef\rangle$ be an abstract argumentation framework. The acceptability operator $\mathcal{A} : \mathcal{R} \to \mathcal{R}$ is defined as follows. Given $r \in \mathcal{R}$ and $\Delta, \Delta_0 \subseteq Arg$, $(\Delta, \Delta_0) \in \mathcal{A}(r)$ iff:*

- *$\Delta \subseteq \Delta_0$, or*
- *for any $A \subseteq Arg$ such that $A \rightsquigarrow \Delta$:*
  - *$A \not\subseteq \Delta \cup \Delta_0$, and*
  - *there exists $\Delta' \subseteq Arg$ such that $\Delta' \twoheadrightarrow A$ and $(\Delta', \Delta \cup \Delta_0) \in r$*

We denote by $\mathcal{A}^{fix}$ the least fixed point of this operator. Then the semantics of an argumentation framework is given through the subsets of arguments Δ that are acceptable with respect to the empty set of arguments, i.e. such that $(\Delta, \{\}) \in \mathcal{A}^{fix}$ holds. We say that such sets of arguments are *acceptable*.

Example 1. *Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be the abstract argumentation framework where*

- $Arg = \{a, b\}$
- $Att = \{(a, b), (b, a)\}$
- $\mathcal{D}ef = \{(b, a)\}$

In this framework its two arguments attack each other but only b is able to defend against its counter-argument of a, e.g., because b is stronger than a. We can then see that the set $\{b\}$ is acceptable whereas the set $\{a\}$ is not acceptable as it cannot defend against its counter-argument $A = \{b\}$. Instead, if the defense relation contained also (a, b), e.g., when the two arguments are of equal strength, then both $\{a\}$ and $\{b\}$ would be acceptable sets of arguments.

2.1 Propositional Logic as Argumentation Logic

We will review the reformulation [3] of classical Propositional Logic and its paraconsistent extension of Argumentation Logic as a realization of the abstract argumentation framework and its acceptability semantics.

Definition 4 (Argumentation Logic Framework)**.** *We denote by $\vdash_{MRA}$ the Natural Deduction direct derivation relation of propositional logic modulo Reduction ad Absudrum (MRA), i.e. without the proof rule of Reduction ad Absudrum.*

Let T be a propositional theory. The argumentation logic framework corresponding to T is the triple $AF^T = \langle Args, Att, Def \rangle$ with:

- $Arg = \{\Sigma \mid \Sigma$ *is a finite set of propositional sentences*$\}$
- *given* $\Delta, \Gamma \in Arg$, *with* $\Delta \neq \{\}$, $(\Gamma, \Delta) \in Att$ *iff* $T \cup \Gamma \cup \Delta \vdash_{MRA} \bot$
- *given* $\Delta \in Arg$, $(\{\overline{\phi}\}, \Delta) \in Def$, *where* $\overline{\phi}$ *is the complement of some sentence* $\phi \in \Delta$ *and* $(\{\}, \Delta) \in Def$ *whenever* $T \cup \Delta \vdash_{MRA} \bot$.

We see that the attack relation is symmetric, i.e. arguments are always counter-arguments of each other when together they are directly inconsistent in the context of the given premises T. The defense relation essentially expresses the fact that any argument can be defended against by *undermining* one of its premises. In logical terms the defense relation expresses the property that for any formula ϕ we are free to choose this or its complement. The second part of the defense relation expresses the fact that if an argument is self-inconsistent with respect to the given premises then this can be trivially defended against by the "safe" empty argument (which in turn can not be attacked). We will see below that when we extend the acceptability semantics, this second part of the defense relation will not need to be stated explicitly at this level but will be captured at the extended acceptability semantic level.

We will denote by $\mathcal{AL}^{fix}$ (or simply by $\mathcal{AL}$) the least fix point of the corresponding operator $\mathcal{A}$ in the general abstract argumentation frameworks as above in definition 3. We then have [3] a logical correspondence between propositional logic (for classically consistent premises T) and the argumentation acceptability semantics. For any formula ϕ: ϕ is acceptable, i.e.,

$(\{\phi\}, \{\}) \in \mathcal{AL}^{fix}$ if and only if there is a model of T in which ϕ is true. Furthermore, for classically inconsistent premises which are directly consistent, i.e. consistent under the restricted derivation of $\vdash_{MRA}$, the argumentation semantics does not trivialize but smoothly extends the propositional deductive semantics into such cases of inconsistent premises. The full technical details of these results can be found in [3]. For the purposes of this paper, it is important to point out that the results rest on the correspondence between proofs via Reductio ad Absurdum and the non-acceptability of formulae, i.e. formulae ϕ such that $(\{\phi\}, \{\}) \notin \mathcal{AL}^{fix}$ holds.

3 Non-acceptable Arguments

In this section, we will examine further the nature of non-acceptable arguments and the relative defeatedness of such arguments in the context of a given set of arguments.

Example 2 (Motivating Example 1)**.** *Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be the abstract argumentation framework where*

- $Arg = \{a, b\}$
- $Att = \{(a, b)\}$
- $\mathcal{D}ef = \{\}$

The argument set $\{b\}$ is attacked by the argument set $\{a\}$. Trivially then, $(\{b\}, \{a\}) \notin \mathcal{A}^{fix}$ i.e., $\{b\}$ is non-acceptable in the context of $\{a\}$, as $\{b\}$ is attacked by an argument that belongs to the context . We will also say that $\{b\}$ is ***defeated in the context of*** *$\{a\}$.*

Example 3 (Motivating Example 2)**.** *Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be the abstract argumentation framework where*

- $Arg = \{a, b\}$
- $Att = \{(a, a), (a, b)\}$
- $\mathcal{D}ef = \{\}$

The argument set $\{a\}$ is self-attacking and hence it is non-acceptable or defeated in its own context. We consider this argument as a ***self-defeating argument*** *exactly because it contains one of its attacks. This property of $\{a\}$ being self-defeating is not affected by the argument $\{b\}$.*

The above example shows a simple (and limiting) case of a non-acceptable self-defeating argument. More complex forms of such arguments exist as it is illustrated in the next example.

Example 4 (Motivating Example 3)**.** *Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be the abstract argumentation framework where*

- $Arg = \{a, b, a_1, d_1\}$
- $Att = \{(a, b), (a_1, a), (a, d_1), (d_1, a_1)\}$
- $\mathcal{D}ef = \{(d_1, a_1)\}$

Argument a is attacked by a_1 which can only be defended against by argument d_1. But a attacks this defense of d_1, i.e. d_1 is defeated in the context of a. Hence, as in the example above, a is non-acceptable and we can consider it as self-defeating, but now in an indirect way, because a renders its necessary defending argument(s) non-acceptable or defeated in its own context.

These more complex forms of self-defeated arguments arise from the recursive nature of non-acceptability given by negating the recursive definition of acceptability.

Proposition 1 (Non-acceptability).
Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be an abstract argumentation framework and $\Delta, \Delta_0 \subseteq Arg$. Let $non\_Acc(\Delta, \Delta_0)$ denote the statement $(\Delta, \Delta_0) \notin \mathcal{A}^{fix}$. Then the following holds[1]*:*

$non\_Acc(\Delta, \Delta_0)$ *iff* $\Delta \not\subseteq \Delta_0$ *and*

$\exists A \subseteq Arg$ *such that* $A \rightsquigarrow \Delta$ *and*

- $A \subseteq \Delta \cup \Delta_0$*, or*
- $\forall \Delta' \subseteq Arg$ *s.t.* $\Delta' \twoheadrightarrow A$*:* $non\_Acc(\Delta', \Delta \cup \Delta_0)$*.*

A non-acceptable argument A such that $(A, \{\}) \notin \mathcal{A}^{fix}$ holds, is one where when we collect recursively the defenses against one of its counter-arguments and recursively the defenses against attacks of the earlier defenses we end up with a collection of defenses that is self-attacking.

4 Extended Acceptability semantics

The extension of the notion of acceptability of arguments follows the simple idea that counter-arguments that are non-acceptable or self-defeating can be dealt with without the need to explicitly defend against them. It is sufficient to recognize that such attacks are self-defeating.

Definition 5 (Extended Acceptability).
Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be an abstract argumentation framework and $\Delta, \Delta_0 \subseteq Arg$. Then a set of arguments Δ **is acceptable in the context of** *Δ_0, denoted by $\mathcal{A}cc^+(\Delta, \Delta_0)$, when the following holds:*

$\mathcal{A}cc^+(\Delta, \Delta_0)$ *iff*

- $\Delta \subseteq \Delta_0$*, or*
- *for any* $A \subseteq Arg$ *such that* $A \rightsquigarrow \Delta$*:*
  - $A \not\subseteq \Delta \cup \Delta_0$*, and*
  - $(A, \{\}) \notin \mathcal{A}^{fix}$*, or* $\exists\ \Delta' \subseteq Arg$ *such that* $\Delta' \twoheadrightarrow A$ *and* $(\Delta', \Delta \cup \Delta_0) \in \mathcal{A}^{fix}$

Proposition 2. *Let $AF = \langle Arg, Att, \mathcal{D}ef \rangle$ be an abstract argumentation framework and $\Delta, \Delta_0 \subseteq Arg$. Then $(\Delta, \Delta_0) \in \mathcal{A}^{fix} \implies \mathcal{A}cc^+(\Delta, \Delta_0)$.*

Example 5 (Examples 1 and 4 cnt.). *In both of these examples $(\{b\}, \{\})$ does not belong to $\mathcal{A}^{fix}$, i.e. the argument set $\{b\}$ is not acceptable. However, $\mathcal{A}cc^+(\{b\}, \{\})$ holds because the only (minimal) attack against $\{b\}$, namely the set $\{a\}$, is self-defeating. Hence the argument set $\{b\}$ is acceptable in the extended semantics.*

[1] Proofs in this paper are omitted due to lack of space.

4.1 AL$^+$: Extended Argumentation Logic

We will now apply the extended acceptability semantics to obtain an extended form of Argumentation Logic (AL). We simply apply definition 5 to the case of AL. In effect, this will give us a generalized form of proof by contradiction under inconsistent premises.

Definition 6 (Extended Argumentation Logic)**.** *Let $AF^T = \langle Args, Att, Def \rangle$ be the argumentation logic framework corresponding to a (directly consistent) propositional theory T. Then the extended argumentation logic, $\mathcal{AL}^+$, is given by:*

$\mathcal{AL}^+(\Delta, \{\})$ holds iff for any $A \subseteq Arg$ such that $A \rightsquigarrow \Delta$:

- *$A \not\subseteq \Delta$, and*
- *$(A, \{\}) \notin \mathcal{AL}$, or there exists $\Delta' \subseteq Arg$ such that $\Delta' \twoheadrightarrow A$ and $(\Delta', \Delta) \in \mathcal{AL}$*

Hence a set of formulae is acceptable in $\mathcal{AL}^+$ either because its attacks could be defended acceptably, as before in the basic logic of $\mathcal{AL}$, or because its attacks are non-acceptable in $\mathcal{AL}$.

The following result shows that the extended argumentation logic, $\mathcal{AL}^+$, is a "proper" extension of $\mathcal{AL}$ when the given premises T are classically consistent.

Theorem 1. *Let T be a classically consistent theory and $AF^T = \langle Arg, Att, \mathcal{D}ef \rangle$ its corresponding argumentation logic framework. Let also ϕ be a propositional formula such that $(\{\phi\}, \{\}) \notin \mathcal{AL}$ holds. Then $\mathcal{AL}^+(\{\phi\}, \{\})$ does not hold.*

Thus the extension of the logic does not trivialize the original logic and specifically classical Propositional Logic for consistent premises. We also know from proposition 2 that $\mathcal{AL}^+$ contains the original logic of $\mathcal{AL}$. The following example, taken from [3], clarifies the link between the extended AL and the original AL and how the former gives genuinely new cases of acceptable formulae.

Example 6. *Consider the following two theories of propositional logic:*

- $T_1 = \{\neg(\beta \wedge \alpha), \neg\alpha\}$ $\qquad$ $T_2 = \{\neg(\beta \wedge \alpha), \neg(\alpha \wedge \gamma), \neg(\alpha \wedge \neg\gamma)\}$

It is easy to see that the argument $\{\beta\}$ is acceptable in $\mathcal{AL}$ relative to theory T_1. Its minimal attack $\{\alpha\}$ is directly self-inconsistent and hence self-attacking (i.e. $T_1 \cup \{\alpha\} \vdash_{MRA} \bot$) and so it can be defended by $\{\}$. The argument $\{\beta\}$ is also acceptable in $\mathcal{AL}$ relative to theory T_2, even though its attack α is not directly inconsistent. The defense against the attack of $\{\alpha\}$, namely $\{\neg\alpha\}$, is such that $(\{\neg\alpha\}, \{\beta\}) \in \mathcal{AL}$. Notice, however, that this attack of $\{\alpha\}$ is itself a non-acceptable self-defeating argument, as it cannot defend acceptably against its attack by $\{\gamma\}$: the only possible defense of $\{\neg\gamma\}$ in non-acceptable in the context of $\{\alpha\}$, because $\{\alpha\}$ attacks $\{\neg\gamma\}$. Therefore, recognizing the non-acceptability of the attack $\{\alpha\}$ is an alternative way to enforce the acceptability of $\{\beta\}$. The extended acceptability semantics of $\mathcal{AL}^+$ uses this alternative way. Importantly, it does so in the same way for both theories T_1 and T_2.

The extended acceptability semantics becomes relevant when the theory of premises is inconsistent, and attacks like $\{\alpha\}$ above cannot be defended acceptably by $\{\neg\alpha\}$.

Example 7 (Example 6 cnt.)**.** *Consider the following theory, obtained from T_2 by making also $\neg\alpha$ non acceptable: $T_3 = T_2 \cup \{\neg(\neg\alpha \wedge \delta), \neg(\neg\alpha \wedge \neg\delta)\}$. The attack $\{\alpha\}$ cannot be acceptably defended because the possible defense of $\{\neg\alpha\}$ is non-acceptable in a way similar to the non-acceptability of $\{\alpha\}$ shown above (replacing $\{\gamma\}$ with $\{\delta\}$). Nevertheless, as $\{\alpha\}$ is by itself non-acceptable, it is reasonable to accept $\{\beta\}$ as acceptable, as $\mathcal{AL}^+$ does.*

5 Conclusions

We have shown how to extend Argumentation Logic to capture the intuitive idea that for attacks which are by themselves self-defeating it is not necessary to defend against. This extension is based on definition 5. We can then consider applying this definition iteratively to give possible further extensions of acceptability and study the properties of such extensions.

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Identity types in predicate logic

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1 Introduction

Model constructions based on equivalence relations and partial equivalence relations (PERs) are instrumental in type theory, programming language theory and categorical logic. They provide well-behaved models of intensional type theory (e.g. [7]) and polymorphism (e.g. [6]). PERs are also a key feature in the tripos-to-topos construction [5]. This work draws on Martin-Löf's identity types [2, 4] to describe universal properties of certain forms of such constructions.

Universal properties of constructions based on equivalence relations have appeared in works by Lawvere and subsequent authors on exact completions (particularly the ex/reg completion [3, 8]) at the generality of categories, and in more recent works by Maietti and Rosolini on quotient completions [10, 11] and by Pasquali on the 'elementary completion' [12] at the generality of indexed posets. The present work builds upon Pasquali's completion result and the construction that underlies it, the latter of which was in essence originally described by Maietti and Rosolini for their 'effective quotient completion' [11] and we shall refer to as the *ER-descent construction* (Definition 3.3).

Theorem 1.1 (Pasquali [12], §3)**.** *The ER-descent construction gives a right 2-adjoint to the inclusion of the 2-category of elementary doctrines and strict-natural morphisms in the 2-category of primary doctrines (= indexed* $(\wedge, \top)$*-posets over finite-product categories) and strict-natural morphisms.*

This result will be reinterpreted in terms of identity types, in the following way. In view that indexed preorders are an interpretation of many-sorted predicate logic, and many-sorted predicate logic in turn is a (extremely) truncated version of dependent type theory, we formulate an adaptation of the inductive axioms of identity types to indexed preorders, calling the resulting notion *identity objects* (Definition 3.1). It turns out that a primary doctrine has identity objects if and only if it is an elementary doctrine (Theorem 3.2). Therefore, Theorem 1.1 is telling us that the ER-descent construction is a right 2-adjoint completion that adds identity objects.

We then describe a universal property analogous to this for the *PER-descent construction* (Definition 4.1), the partial equivalence relations version of the ER-descent construction. This construction can be seen as a step in the tripos-to-topos construction (cf. [9]). The universal property is obtained by considering a suitable weakening of identity objects, called *partial identity objects* (Definition 4.2). We show that the PER-descent construction is a right-biadjoint completion that adds partial identity objects (Theorem 4.6).

Partial identity objects can be promoted to identity objects by a comonadic construction we call *virtualisation* (Definition 5.2 and Remark 5.3). It serves as another step in the tripos-to-topos construction (cf. [9]), where it in particular turns the PERs emerged from the PER-descent construction into equivalence relations. Virtualisation in fact plays a role in the definition of partial identity objects (Remark 4.3). We establish universal properties of virtualisation (Theorem 5.6). It is an ambidextrously biadjoint completion with respect to oplax-natural morphisms of indexed preorders. With respect to pseudonatural morphisms of indexed preorders, it is merely a left-biadjoint completion.

2 2-categories of indexed preorders

Let Pre, $\mathrm{Pre}^{\wedge}$ and $\mathrm{Pre}^{\wedge,\top}$ denote the category of preorders, $\wedge$-preorders and $(\wedge,\top)$-preorders respectively.

Definition 2.1. A *(strictly) indexed preorder*, $\wedge$*-preorder* and $(\wedge,\top)$*-preorder* P consists of a category P^0 and a functor $P^1\colon (P^0)^{\mathrm{op}} \to \mathrm{Pre}$, $\mathrm{Pre}^{\wedge}$ and $\mathrm{Pre}^{\wedge,\top}$ respectively.

Definition 2.2. Let P and Q be indexed preorders. An *oplax-natural morphism* $F\colon P \to Q$ consists of a functor $F^0\colon P^0 \to Q^0$ and an oplax natural transformation

$$\begin{array}{ccc} (P^0)^{\mathrm{op}} & \xrightarrow{(F^0)^{\mathrm{op}}} & (Q^0)^{\mathrm{op}} \\ & \overset{F^1}{\Rightarrow} & \\ {\scriptstyle P^1} \searrow & & \swarrow {\scriptstyle Q^1} \\ & \mathrm{Pre}. & \end{array}$$

A *pseudonatural morphism* is an oplax-natural morphism F for which F^1 is a pseudonatural transformation. A *strict-natural morphism* is an oplax-natural morphism F for which F^1 is a strict natural transformation.

Terminology 2.3. Let P and Q be indexed preorders. An oplax-natural morphism $F\colon P \to Q$ *preserves binary products* if the functor F^0 preserves binary products, and *preserves meets* or *top* if each component of the oplax natural transformation F^1 does so respectively.

Definition 2.4. A *2-morphism* $\rho\colon F \to G\colon P \to Q$ between oplax morphisms of indexed preorders is a natural transformation $\rho\colon F^0 \to G^0\colon P^0 \to Q^0$ such that

$$\begin{array}{ccc} & P^1(X) & \\ {\scriptstyle F^1_X} \swarrow & \leq & \searrow {\scriptstyle G^1_X} \\ Q^1F^0(X) & \xleftarrow[\rho_X{}^* := Q^1(\rho_X)]{} & Q^1G^0(X) \end{array} \tag{1}$$

as homomorphisms of preorders for each object $X \in P^0$.

The following defines notations for some 2-categories of indexed preorders that we will need.

Proposition 2.5. *Indexed*

a. $\wedge$*-preorders over binary-product categories,*

b. $(\wedge,\top)$*-preorders,* *c.* $(\wedge,\top)$*-preorders over binary-product categories,*

their structure-preserving

1. oplax-natural *2. pseudonatural*

morphisms, and 2-morphisms form a 2-category, denoted

1a. $\mathrm{IdxPre}^{\times,\wedge}_{\mathrm{on}}$, *1b.* $\mathrm{IdxPre}^{\wedge,\top}_{\mathrm{on}}$, *1c.* $\mathrm{IdxPre}^{\times,\wedge,\top}_{\mathrm{on}}$

2a. $\mathrm{IdxPre}^{\times,\wedge}_{\mathrm{pn}}$, *2b.* $\mathrm{IdxPre}^{\wedge,\top}_{\mathrm{pn}}$, *2c.* $\mathrm{IdxPre}^{\times,\wedge,\top}_{\mathrm{pn}}$

respectively.

3 Identity objects and ER-descent construction

Let P be an indexed $(\wedge, \top)$-preorder over a binary-product category.

Definition 3.1. An *identity object* on an object $X \in P^0$ is

1. (*formation*) an element $\mathrm{Id}_X \in P^1(X \times X)$,

such that

2. (*introduction* or *reflexivity*) $\top \leq (X \xrightarrow{\delta} X \times X)^*(\mathrm{Id}_X)$, and
3. (*elimination*) for any object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(p) \leq (X \times Y \xrightarrow{\delta \times Y} X \times X \times Y)^*(q),$$

   then $(X \times X \times Y \xrightarrow{\pi_1,\pi_2} X \times X)^*(\mathrm{Id}_X) \wedge p \leq q$.

We say P *has identity objects* if each X has an identity object.

This Martin-Löf notion of equality is equivalent to Lawvere's hyperdoctrine equality [1] as extracted by Maietti and Rosolini [10] in the notion of elementary doctrine:

Theorem 3.2. *An indexed $(\wedge, \top)$-poset over a finite-product category has identity objects if and only if it is an elementary doctrine.*

This means Pasquali's 'elementary completion' result (Theorem 1.1) is telling us that the ER-descent construction is a right 2-adjoint completion adding identity objects. Let us review this construction.

Definition 3.3. An *equivalence relation* on an object $X \in P^0$ is an element $\sim \in P^1(X \times X)$ that is

1. (*reflexive*) $\top \leq (X \xrightarrow{\delta} X \times X)^*(\sim)$
2. (*symmetric*) $\sim \leq (X \times X \xrightarrow{\pi_2,\pi_1} X \times X)^*(\sim)$, and
3. (*transitive*) $(X \times X \times X \xrightarrow{\pi_1,\pi_2} X \times X)^*(\sim) \wedge (X \times X \times X \xrightarrow{\pi_2,\pi_3} X \times X)^*(\sim) \leq (X \times X \times X \xrightarrow{\pi_1,\pi_3} X \times X)^*(\sim)$.

The *ER-descent construction* on P is the indexed preorder $\mathrm{ER}(P)$ where an object in $\mathrm{ER}(P)^0$ is a pair $(X, \sim)$ with $X \in \mathrm{Ob}(P^0)$ and $\sim \in P^1(X \times X)$ an equivalence relation, an arrow $(X, \sim_X) \to (Y, \sim_Y)$ is an arrow $f \colon X \to Y$ in P^0 satisfying $\sim_X \leq P^1(f \times f)(\sim_Y)$, and $\mathrm{ER}(P)^1(X, \sim)$ the full subpreorder of $P^1(X)$ on those elements p satisfying $P^1(\pi_1)(p) \wedge \sim \leq P^1(\pi_2)(p)$.

Proposition 3.4. $\mathrm{ER}(P)$ *is an indexed $(\wedge, \top)$-preorder over a binary-product category, and has identity objects: if $(X, \sim)$ is an object in $\mathrm{ER}(P)^0$, then $\sim$ is an identity object on it.*

We will state an adaptation of Pasquali's result to our settings. The following definition is needed.

Definition 3.5. Let P and Q be indexed $(\wedge, \top)$-preorders with identity objects over binary-product categories. Let $F \colon P \to Q$ be an oplax-natural morphism that preserves binary products. We say F *preserves identity objects* if

$$\mathrm{Id}_{F^0(X)} \simeq (F^0(X) \times F^0(X) \xrightarrow{\cong} F^0(X \times X))^* F^1_{X \times X}(\mathrm{Id}_X)$$

for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ denote the 2-category whose 0-cells are indexed $(\wedge,\top)$-preorders with identity objects over binary-product categories, 1-cells are pseudonatural morphisms that preserve binary products, meets, tops and identity objects, and 2-cells are 2-morphisms.

Theorem 3.6. *The assignment* $P \mapsto \mathrm{ER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is right biadjoint to the inclusion 2-functor.*

4 PER-descent construction and partial identity objects

Let P be an indexed $\wedge$-preorder over a binary-product category.

Definition 4.1. A *partial equivalence relation* on an object $X \in P^0$ is an element $\sim \in P^1(X \times X)$ that is symmetric and transitive.

The *PER-descent construction* on P is the indexed preorder $\mathrm{PER}(P)$ defined in the same way as $\mathrm{ER}(P)$ but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations in P instead.

The following weakened form of identity objects will give us a result analogous to Theorem 3.6 for the PER-descent construction.

Definition 4.2. P has *partial identity objects* if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \leq (X \times X \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X), (X \times X \overset{\pi_2,\pi_2}{\to} X \times X)^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for any object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if
$$(X \times Y \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2,\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(p) \leq (X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(q)$$
then $(X \times X \times Y \overset{\pi_3,\pi_3}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge (X \times X \times Y \overset{\pi_1,\pi_2}{\to} X \times X)^*(\mathrm{PId}_X) \wedge p \leq q$,
3. each arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{PId}_X \leq (f \times f)^*(\mathrm{PId}_Y)$, and
4. $\mathrm{PId}_{X\times Y} \simeq (X \times Y \times X \times Y \overset{\pi_1,\pi_3}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \times X \times Y \overset{\pi_2,\pi_4}{\to} Y \times Y)^*(\mathrm{PId}_Y)$.

Remark 4.3. Paravirtual elimination, as the name suggests, is related to virtualisation (§5). Specifically, under the assumption of the other axioms (1., 3. and 4.), PId_X satisfies paravirtual elimination if and only if it satisfies elimination (Definition 3.1) in the virtualisation $\mathrm{Virt}(P)$ of P (Definition 5.2).

Proposition 4.4. $\mathrm{PER}(P)$ *is an indexed* $\wedge$*-preorder with partial identity object over a binary-product category, with* $\mathrm{PId}_{(X,\sim)} := \sim$.

The preservation of partial identity objects is defined in the same way as that of identity objects:

Definition 4.5. Let P and Q be indexed $\wedge$-preorders with partial identity objects over binary-product categories. Let $F\colon P \to Q$ be an oplax-natural morphism that preserves binary products. We say F *preserves partial identity objects* if
$$\mathrm{PId}_{F^0(X)} \simeq (F^0(X) \times F^0(X) \overset{\cong}{\to} F^0(X \times X))^* F^1_{X\times X}(\mathrm{PId}_X)$$
for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ denote the 2-category whose 0-cells are indexed $\wedge$-preorders with partial identity objects over binary-product categories, 1-cells are pseudonatural morphisms that preserve binary products, meets and partial identity objects, and 2-cells are 2-morphisms.

Theorem 4.6. *The assignment* $P \mapsto \mathrm{PER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ *that is right biadjoint to the forgetful 2-functor.*

In other words, the PER-descent construction is a right-biadjoint completion adding partial identity objects.

5 Virtualisation

Virtualisation is a construction that allows us to promote partial identity objects into identity objects. Its general definition involves the following structure on indexed preorders.

Definition 5.1. An indexed preorder P is *oplaxly sectioned* if each object $X \in P^0$ is *equipped* with an element $\mathrm{os}_X \in P^1(X)$ and each arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \le f^*(\mathrm{os}_Y)$.

We may regard an indexed preorder with partial identity objects as oplaxly sectioned, with $\mathrm{os}_X := (X \xrightarrow{\delta} X \times X)^*(\mathrm{PId}_X)$.

Let P be an oplaxly sectioned indexed $\wedge$-preorder.

Definition 5.2. The *virtualisation* of P is the indexed preorder $\mathrm{Virt}(P)$ given by $\mathrm{Virt}(P)^0 := P^0$ and $\mathrm{Virt}(P)^1(X) := (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\le})$ where $p \overset{\mathrm{v}}{\le} q$ if and only if $\mathrm{os}_X \wedge p \le q$.

Remark 5.3. Let $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ denote the preorder-enriched category of functors and *oplax* natural transformations. $\mathrm{Virt}(P)^1$ is in fact a Kleisli as well as Eilenberg-Moore object for the (necessarily idempotent) comonad $\mathrm{v}_P\colon P^1 \Rightarrow P^1$ in $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ given by $(\mathrm{v}_P)_X(p) := \mathrm{os}_X \wedge p$.

Proposition 5.4. $\mathrm{Virt}(P)$ *is an indexed* $(\wedge, \top)$*-preorder, with top elements given by the* os_X*. If* P *has partial identity objects, then* $\mathrm{Virt}(P)$ *has identity objects given by the* PId_X*.*

We will describe universal properties of virtualisation. The following definition is needed.

Definition 5.5. Let P and Q be oplaxly sectioned indexed preorders. An oplax-natural morphism $F\colon P \to Q$ *preserves the specified oplax section* if $\mathrm{os}_{F^0(X)} \simeq F^1_X(\mathrm{os}_X)$ for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}}$ denote the 2-category whose 0-cells are oplaxly sectioned indexed $\wedge$-preorders, 1-cells are oplax-natural morphisms that preserve meets and the specified oplax section, and 2-cells are 2-morphisms. Let $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}}$ and $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$ be the variants of $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ and $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ respectively whose 1-cells are oplax- rather than pseudonatural morphisms.

Theorem 5.6. *The assignment* $P \mapsto \mathrm{Virt}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\top}$ *as well as a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is ambidextrously biadjoint to 'the' respective inclusion 2-functor. The left-biadjoint part also holds with respect to pseudonatural morphisms.*

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Shelah's conjecture fails for higher cardinalities

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Abstract

The main goal of this paper is to generalize the results that where presented in [3] for $\aleph_1$-Kurepa trees to $\aleph_{\alpha+1}$-Kurepa trees.

We construct an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ_α, that codes $\aleph_{\alpha+1}$-Kurepa trees, for some countable α. One of the main results for its spectrum (the spectrum of a sentence is the class of all cardinals for which there exists some model of the sentence) is the following:

It is consistent that $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, that $2^{\aleph_{\alpha+1}}$ is weakly inaccessible and that the spectrum of ψ_α is equal to $[\aleph_0, 2^{\aleph_{\alpha+1}})$.

This relates to a conjecture of Shelah, that if $\aleph_{\omega_1} < 2^{\aleph_0}$ and there is a model of some $\mathcal{L}_{\omega_1,\omega}$-sentence of size $\aleph_{\omega_1}$, then there is a model of size $2^{\aleph_0}$. Shelah proves the consistency of this conjecture in [2]. This statement proves that it is consistent that there is no Hanf number below $2^{\aleph_{\alpha+1}}$ for every countable α.

There are some interesting results for the amalgamation Spectrum too (the amalgamation Spectrum is defined similarly to the Spectrum, but we also require that κ-amalgamation holds). We prove that the κ-amalgamation for $\mathcal{L}_{\omega_1,\omega}$- sentences is not absolute. More specifically we prove:

- for $\alpha > 0$ finite, it is consistent that $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ and the Amalgamation Spectrum of ψ_α is equal to $[(2^{\aleph_\alpha})^+, \aleph_{\omega_{\alpha+1}}]$.
- for $\alpha > 0$ finite, it is consistent that $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ is weakly inaccessible and the Amalgamation Spectrum of ψ_α is equal to $[(2^{\aleph_\alpha})^+, 2^{\aleph_{\alpha+1}})$.

1 Kurepa trees and $\mathcal{L}_{\omega_1,\omega}$

Firstly, we need to see some useful definitions.

Definition 1.1. *For an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, the* **spectrum** *of ϕ is the class*

$$Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } |M| = \kappa\}.$$

If Spec(ϕ) $= [\aleph_0, \kappa]$, we say that ϕ characterizes κ.
The **maximal models spectrum** *of ϕ is the class*

$$MM\text{-}Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } |M| = \kappa \text{ and } M \text{ is maximal }\}.$$

We can, also, define the **amalgamation spectrum** *of ϕ, AP-Spec(ϕ) and the* **joint embedding spectrum** *of ϕ, JEP-Spec(ϕ) as follows:*

$$AP\text{-}Spec(\phi) = \{\kappa | \phi \text{ has at least one model of size } \kappa \text{ and the models of size } \kappa \text{ satisfy the amalgamation property }\}$$
$$JEP\text{-}Spec(\phi) = \{\kappa | \phi \text{ has at least one model of size } \kappa \text{ and the models of size } \kappa \text{ satisfy the joint embedding property }\}.$$

Definition 1.2. *Assume κ is an infinite cardinal. A κ-**tree** has height κ and each level has at most $< \kappa$ elements. A κ-**Kurepa tree** is a κ-tree with at least κ^+ many branches of height κ.*

*If $\lambda \geq \kappa^+$, a (κ, λ)-**Kurepa tree** is a κ-Kurepa tree with exactly λ branches of height κ. KH(κ, λ) is the statement that there exists a (κ, λ)-Kurepa tree.*

Define $\mathcal{B}(\kappa) = sup\{\lambda | KH(\kappa, \lambda) \text{ holds }\}$.

*A **weak** κ-**Kurepa tree** is a κ-Kurepa tree, where each level has at most $\leq \kappa$ elements.*

Comment:For this paper we will assume that κ-Kurepa trees are pruned, i.e. every node is contained in a maximal branch of order type κ.

Definition 1.3. *Let $\kappa \leq \lambda$ be infinite cardinals. A sentence σ in a language with a unary predicate P admits (λ, κ), if σ has a model M such that $|M| = \lambda$ and $|P^M| = \kappa$. In this case, we will say that M is of type (λ, κ).*

Our goal, now, is to construct an $\mathcal{L}_{\omega_1,\omega}$ sentence such that every $\aleph_{\alpha+1}$-Kurepa tree (where α is countable) belongs to its spectrum.

From [1], we know the following theorem.

Theorem 1.4. *There is a first order sentence σ such that for all infinite cardinals κ, σ admits (κ^{++}, κ) iff $KH(\kappa^+, \kappa^{++})$.*

We will not present the proof for this theorem, but we are going to use some parts of the construction for σ, in order to construct the desired $\mathcal{L}_{\omega_1,\omega}$ sentence, ψ_α.

Assume that α is a countable ordinal. The vocabulary τ consists of the constants 0, $(c_n)_{n\in\omega}$, the unary symbols $L_0, L_1, ..., L_\alpha, L_{\alpha+1}$, the binary symbols $S, V, T, <_1, <_2, ..., <_\alpha, <_{\alpha+1}$ and the ternary symbols $F_0, F_1, ..., F_\alpha, G$. The idea is to build an $\aleph_{\alpha+1}$-Kurepa tree. $L_{\alpha+1}$ is a set that corresponds to the "levels" of the tree. $L_{\alpha+1}$ is linearly ordered by $<_{\alpha+1}$ and 0 is its minimum element. $L_{\alpha+1}$ may or may not have a maximum element. Every element $a \in L_{\alpha+1}$ that is not a maximum element has a successor b that satisfies $S(a, b)$. We will denote the successor of a by $S(a)$. The maximum element (which we will call m) is not a successor. For every $a \in L, V(a, \cdot)$ is the set of nodes at level a and we assume that $V(a, \cdot)$ is disjoint from all the $L_0, L_1, \ldots, L_{\alpha+1}$. If $V(a, x)$, we will say that x is at the level a and we may write $x \in V(a)$.

T is a tree ordering on $V = \bigcup_{a\in L_{\alpha+1}} V(a)$. If $T(x, y)$, then x is at some level strictly less than the level of y. If $y \in V(a)$ and $b < a$, there is some x so that $x \in V(b)$ and $T(x, y)$. If a is a limit, that is neither a successor nor 0, then two distinct elements in $V(a)$ cannot have the same predecessors. If m is the maximum element of $L_{\alpha+1}, V(m)$ is the set of maximal branches through the tree. Both "the height of T" and "the height of $L_{\alpha+1}$" refer to the order type of $(L_{\alpha+1}, <_{\alpha+1})$. We can also stipulate that the $\aleph_{\alpha+1}$-Kurepa tree is pruned.

Our goal, now is to bound the size of each L_β by $\aleph_\beta$. For the first level, we require that $\forall x(L_0(x) \leftrightarrow \bigvee_n x = c_n)$. That gives us that $|L_0| = \aleph_0$.

Each $L_\beta, \beta = 1, 2, ..., \alpha$ is linearly ordered by $<_\beta$.

In order to bound the size of $L_{\beta+1}$ by $\aleph_{\beta+1}$, we bound the size of each initial segment by $\aleph_\beta$. Our treatment is slightly different for $\beta < \alpha$ than for $\beta = \alpha$.

Let $\beta < \alpha$. For every $x \in L_{\beta+1}$ there is a surjection $F_\beta(x, \cdot, \cdot)$ from L_β to $(L_{\beta+1})_{\leq(\beta+1)x} = \{b \in L_{\beta+1} | b \leq_{(\beta+1)} x\}$. This bounds the size of each initial segment $(L_{\beta+1})_{\leq(\beta+1)x}, \beta < \alpha$ by $|L_\beta|$.

At limit stages we take L_β as the union of the previous L_γ. The linear order on limit stages is not relevant to the linear orders in the previous stages.

Finally, for every $x \in L_{\alpha+1}$, that is not the maximum element, there is a surjection $F_\alpha(x, \cdot, \cdot)$ from L_α to $(L_{\alpha+1})_{\leq(\alpha+1)x}$ and another surjection $G(x, \cdot, \cdot)$ from L_α to $V(x)$. This bounds the size of $(L_{\alpha+1})_{\leq(\alpha+1)x}$ and the size of every $V(x)$, which is not maximal level, by $|L_\alpha|$.

Observation: Defining the $F_\alpha(x,\cdot,\cdot)$, we demand that x is not the maximum element of $L_{\alpha+1}$. We don't have the same restriction for the rest of the F_β's. That difference plays an important role throughout the rest of the paper.

This construction gives us that for all $\beta = 1,2,...,\alpha+1, |L_\beta| \leq \aleph_\beta$ and for all non maximal levels $|V(x)| \leq \aleph_\alpha$.

So, our desired $\mathcal{L}_{\omega_1,\omega}$ sentence, ψ_α is the conjuction of all the above requirements.

Definition 1.5. *A* $(\kappa-\lambda)$-**Kurepa tree***, where $\lambda \geq \kappa$, is a tree of height κ, each level has at most $\leq \lambda$ elements with at least λ^+ branches of height κ. A $(\kappa-\kappa)$-Kurepa tree is a weak κ-Kurepa tree.*

The dividing line for models of ψ to code $\aleph_{\alpha+1}$-Kurepa trees is the size of $L_{\alpha+1}$. By definition, every initial segment of $L_{\alpha+1}$ has size at most $\aleph_\alpha$. If in addition $|L_{\alpha+1}| = \aleph_{\alpha+1}$, then we can embed $\omega_{\alpha+1}$ cofinally into $L_{\alpha+1}$. Hence, every model of ψ of size $\geq \aleph_{\alpha+2}$ and for which $|L_{\alpha+1}| = \aleph_{\alpha+1}$, codes an $\aleph_{\alpha+1}$-Kurepa tree.

Let $\mathbf{K}$ be the collection of all models of ψ, equipped with the substructure relation. I.e. for $M, N \in \mathbf{K}, M \prec_{\mathbf{K}} N$ if $M \subset N$.

Now, I present some interesting results and theorems, without their proofs.

Lemma 1.6. *If $M \prec_{\mathbf{K}} N$, then*

1. $L_0^M = L_0^N$
2. L_1^M *is initial segment of* L_1^N
3. *For $1 \leq \beta \leq \alpha$, if $|L_\gamma^M| = \aleph_\gamma$, for every $\gamma \leq \beta$, then $L_\beta^M = L_\beta^N$*
4. *For $1 \leq \beta \leq \alpha$, if $|L_\gamma^M| = \aleph_\gamma$, for every $\gamma \leq \beta$, then $L_{\beta+1}^M$ is an initial segment of $L_{\beta+1}^N$*
5. *If $|L_\beta^M| = \aleph_\beta$, for every $\beta \leq \alpha$, then $V^M(x) = V^N(x)$, for every non maximal $x \in L_{\alpha+1}^M$*
6. *the tree ordering is preserved*

Corollary 1.7. *If $M \prec_{\mathbf{K}} N$, then*

1. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha$ and $L_{\alpha+1}^M = L_{\alpha+1}^N$, then N differs from M only in the maximal branches it contains.*
2. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha+1$ and $L_{\alpha+1}^N$ is a strict end extension of $L_{\alpha+1}^M$, then $L_{\alpha+1}^M$ does not have a maximum element and $L_{\alpha+1}^N$ is one point end extension of $L_{\alpha+1}^M$.*
3. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha$, $L_{\alpha+1}^M$ has a maximum element and $L_{\alpha+1}^N$ is a strict end extension of $L_{\alpha+1}^M$, then $|M| = \aleph_\alpha$.*

Proposition 1.8. *$(\mathbf{K}, \prec_{\mathbf{K}})$ is an Abstract Elementary Class (AEC) with countable Lowenheim-Skolem number.*

Theorem 1.9. *The spectrum of ψ is characterized by the following properties:*

1. $[\aleph_0, \aleph_\alpha^{\aleph_0}] \subseteq Spec(\psi)$ *and* $\aleph_{\alpha+1} \in Spec(\psi)$.
2. *if there exists a $(\mu-\lambda)$-Kurepa tree, where $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, with κ cofinal branches, then $[\aleph_0, \kappa] \subseteq Spec(\psi)$.*
3. *if there exists an $\aleph_{\alpha+1}$-Kurepa tree with κ cofinal branches, then $[\aleph_0, \kappa] \subseteq Spec(\psi)$.*

4. *no cardinal belongs to Spec(ψ) except those required by (1)-(2)-(3). I.e. if ψ has a model of size κ, then $\kappa \in [\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ or there exists a $(\mu - \lambda)$-Kurepa tree with κ cofinal branches or there exists an $\aleph_{\alpha+1}$-Kurepa tree with κ cofinal branches.*

Theorem 1.10. *The maximal models Spectrum of ψ is characterized by the following:*

1. *ψ has maximal model of size $\aleph_{\alpha+1}$*
2. *If $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}$, for some $\aleph_0 \leq \lambda \leq \aleph_\alpha$, then ψ has maximal model of size $\lambda^{\aleph_0}$*
3. *If there exists an $(\mu - \aleph_\alpha)$-Kurepa tree, $\mu \geq \aleph_1$, with exactly κ cofinal branches, then ψ has maximal model in κ*
4. *If there exists an $\aleph_{\alpha+1}$-Kurepa tree with exactly κ cofinal brnches, then ψ has maximal model in κ*
5. *ψ has maximal models only on those cardinalities required by (1)-(4).*

Corollary 1.11. 1. *If there are no $(\mu - \lambda)$-Kurepa trees and no $\aleph_{\alpha+1}$-Kurepa trees, then Spec(ψ) $= [\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ and $MM - Spec(\psi) = \{\lambda^{\aleph_0} | \aleph_0 \leq \lambda \leq \aleph_\alpha$ and $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}\} \cup \{\aleph_{\alpha+1}\}$.*

2. *If $\mathcal{B}(\aleph_{\alpha+1})$ is a maximum, i.e. there is an $\aleph_{\alpha+1}$-Kurepa tree of size $\mathcal{B}(\aleph_{\alpha+1})$ and there are no $(\mu - \lambda)$-Kurepa trees for $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, then ψ characterizes $max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}, \mathcal{B}(\aleph_{\alpha+1})\}$.*

3. *If $\mathcal{B}(\aleph_{\alpha+1})$ is not a maximum and there are no $(\mu - \lambda)$-Kurepa trees for $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, then Spec($\psi$) equals $[\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ or $[\aleph_0, \mathcal{B}(\aleph_{\alpha+1}))$, whichever is greater. Moreover, ψ has maximal models in $\aleph_{\alpha+1}, \lambda^{\aleph_0}$, if it is $\geq \aleph_{\alpha+1}$ and in cofinally many cardinalities below $\mathcal{B}(\aleph_{\alpha+1})$.*

Theorem 1.12. 1. *$(\mathbf{K}, \prec_{\mathbf{K}})$ fails JEP in all cardinals.*

2.
   - *If $\alpha < \omega$, then $(\mathbf{K}, \prec_{\mathbf{K}})$ satisfies AP for all cardinals $> 2^{\aleph_\alpha}$ that belong to Spec(ψ), but fails AP in every cardinal $\leq 2^{\aleph_\alpha}$.*
   - *If $\alpha \geq \omega$, then $(\mathbf{K}, \prec_{\mathbf{K}})$ fails AP in all cardinalities.*

2 Consistency results

Theorem 2.1. *It is consistent with ZFC that $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}} = \mathcal{B}(\aleph_{\alpha+1}) < 2^{\aleph_{\alpha+1}}$ and there exists an $\aleph_{\alpha+1}$-Kurepa tree with $\aleph_{\omega_{\alpha+1}}$-many cofinal branches.*

Theorem 2.2. *From a Mahlo cardinal, it is consistent with ZFC that $2^{\aleph_\alpha} < \mathcal{B}(\aleph_{\alpha+1}) = 2^{\aleph_{\alpha+1}}$, for every $\kappa < 2^{\aleph_{\alpha+1}}$ there is an $\aleph_{\alpha+1}$-Kurepa tree with at least κ-many maximal branches, but no $\aleph_{\alpha+1}$-Kurepa tree has $2^{\aleph_{\alpha+1}}$-many maximal branches.*

Corollary 2.3. *For every α countable ordinal, there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ that it is consistent with ZFC that:*

1. *ψ characterizes $max\{\aleph_{\alpha+1}, \aleph_\alpha^{\aleph_0}\}$*
2. *$2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ and ψ characterizes $\aleph_{\omega_{\alpha+1}}$*

3. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $Spec(\psi) = [\aleph_0, 2^{\aleph_{\alpha+1}})$

4. $MM - Spec(\psi) = \{\lambda^{\aleph_0} | \aleph_0 \leq \lambda \leq \aleph_\alpha$ *and* $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}\} \cup \{\aleph_{\alpha+1}\}$

5. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $MM - Spec(\psi)$ *is a cofinal subset of* $[\aleph_{\alpha+1}, 2^{\aleph_{\alpha+1}})$

   If, in addition α *is finite, then it is also consistent that*

6. $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ *and* $AP - Spec(\psi) = (2^{\aleph_\alpha}, \aleph_{\omega_{\alpha+1}}]$

7. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $AP - Spec(\psi) = (2^{\aleph_\alpha}, 2^{\aleph_{\alpha+1}})$

Finally, throughout the paper there are some interesting open questions that have been risen:

Open Question 1. *Is the negation of Shelah's conjecture consistent with ZFC?*

Open Question 2. *Is* $\aleph_1$*-amalgamation for* $\mathcal{L}_{\omega_1,\omega}$*-sentences absolute for models of ZFC?*

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Fuzzy Semantics for the Language of Precise Truth*

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Abstract

This short paper investigates the prospects of designing semantically satisfactory fuzzy models for the formal language of precise truth. We start by showing that this language fails to admit fuzzy models based on Kronecker-Delta semantics for sharp truth-predications, and then we explore some alternative semantic possibilities.

1 Species of Truth Predicates

In his work on the topic of vagueness, Smith [Smi08] made an important logico-philosophical distinction between two kinds of truth predicates:

1. The global truth-predicate T with the property that the semantic value of any truth predication $\mathsf{T}(\overline{\ulcorner\varphi\urcorner})$ matches the semantic value of the underlying sentence φ, i.e. $[\![\mathsf{T}(\overline{\ulcorner\varphi\urcorner})]\!] = [\![\varphi]\!]$.

2. The family $\{\, \mathsf{T_i} \mid i \in [0,1] \,\}$ of *indexed* truth-predicates that we use in order to say that a sentence has a *specific* degree of truth—e.g. that it is true to degree 0.54, written as $\mathsf{T_{0.54}}(\overline{\ulcorner\varphi\urcorner})$.

The formal semantics literature contains many non-classical ways in which one can successfully add the basic symbol T to the (object-)language of arithmetic, viz. $\mathcal{L}_{\mathsf{PA}}$ (e.g. [Kri75]). We shall now add more symbols to $\mathcal{L}_{\mathsf{PA}}$ in order to enhance its expressive powers, so that precise truth-predications can be articulated. Let $\mathcal{L}_{\mathsf{T}}^{\infty} := \mathcal{L}_{\mathsf{PA}} \cup \{\, \mathsf{T_i} \mid i \in [0,1] \,\}$.[1] Ideally, the semantics for precise truth-predications should be governed by the Kronecker-Delta function $\delta : [0,1]^2 \to \{\, 0,1 \,\}$ given by:

$$\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

More explicitly, if $\mathsf{T_r}(\overline{\ulcorner\varphi\urcorner})$ is an atomic precise truth-predication for some $r \in [0,1]$, we want that:

$$[\![\mathsf{T_r}(\overline{\ulcorner\varphi\urcorner})]\!] = \begin{cases} 1 & \text{if } r = [\![\varphi]\!] \\ 0 & \text{otherwise} \end{cases}$$

*I would like to thank Johannes Stern for many insightful discussions. This paper is based on Section 6.5 of my PhD thesis, and my research was supported by an ERC Starting Grant (TRUST, Grant No. 803684).

[1] Of course, we can avoid making our language uncountable. We can restrict our indexes to range over $\mathbb{Q} \cap [0,1]$ and use a two-place predicate T with the property that $\mathsf{T}(\overline{\mathsf{n}}, \overline{\ulcorner\varphi\urcorner})$ is (perfectly) true iff φ has the n^{th} rational number—in the canonical enumeration of the countable set $\mathbb{Q}$—as its truth-degree. In other words, when we're writing $\mathsf{T_i}(\overline{\ulcorner\varphi\urcorner})$, this can be seen as shorthand for $\mathsf{T}(\overline{\#\mathsf{i}}, \overline{\ulcorner\varphi\urcorner})$, where $\#i \in \mathbb{N}$ is i's code. For simplicity, in this paper we'll carry out our formal investigation as if everything were real-valued.

2 δ-Semantics for $\mathcal{L}^{\infty}_{\mathsf{T}}$

Under this picture, there are *prima facie* no obvious obstacles in providing semantic values for some sentences of interest. For example, consider the following lemma, that we shall prove, which concerns the semantic value of the sentence that denies its bivalence:

Lemma 2.1 (The Bivalence Denier)**.** The $\mathcal{L}^{\infty}_{\mathsf{T}}$-sentence which says about itself that it is a counterexample to the principle of bivalence can only be perfectly false (regardless of the truth-structure underpinning the fuzzy semantics).

Proof. The bivalence denier, τ, asserts that its semantic value is neither 0 nor 1. Let τ be a fixed point of the open formula $\neg(\mathsf{T}_0(\mathsf{x}) \vee \mathsf{T}_1(\mathsf{x}))$. By the semantic definition of a fixed point, it follows that $[\![\tau]\!] = [\![\neg(\mathsf{T}_0(\overline{\ulcorner\tau\urcorner}) \vee \mathsf{T}_1(\overline{\ulcorner\tau\urcorner}))]\!]$. Suppose $[\![\tau]\!] \in (0,1)$. Then $[\![\neg(\mathsf{T}_0(\overline{\ulcorner\tau\urcorner}) \vee \mathsf{T}_1(\overline{\ulcorner\tau\urcorner}))]\!] = f_\neg(f_\vee(\delta([\![\tau]\!],0), \delta([\![\tau]\!],1))) = f_\neg(f_\vee(0,0)) = f_\neg(0) = 1$. Hence we cannot assign a truth degree strictly between 0 and 1 to τ. Now suppose that $[\![\tau]\!] \in \{0,1\}$. In this case we have that $[\![\neg(\mathsf{T}_0(\overline{\ulcorner\tau\urcorner}) \vee \mathsf{T}_1(\overline{\ulcorner\tau\urcorner}))]\!] = f_\neg(f_\vee(\delta([\![\tau]\!],0), \delta([\![\tau]\!],1))) = f_\neg(1) = 0$. Thus, truth-value 1 is discounted and 0 is the only possibility. □

That being said, some unfortunate news are due. Even though the Kronecker-Delta semantics for $\mathcal{L}^{\infty}_{\mathsf{T}}$ seems promising with respect to a multitude of sentences, we can mathematically prove the negative result that $\mathcal{L}^{\infty}_{\mathsf{T}}$ has no models at all. The result resembles in many respects Tarski's [Tar56] classical argument:

Theorem 2.2 (The Undefinability of Precise Truth)**.** There are no fuzzy models $\mathcal{M}$ of language $\mathcal{L}^{\infty}_{\mathsf{T}}$.[2]

Proof. Suppose there is a model $\mathcal{M}$ of our language, where the semantics of the indexed truth-predicates is guided by the Kronecker-Delta proposal. For any $y \in [0,1]$, let λ_{y} be $\neg\mathsf{T}_{\mathsf{y}}(\mathsf{x})$'s liar sentence. To show that $\mathcal{M}$ cannot exist, we just need to show that there is at least one number r in the unit interval such that there's no truth-degree that can be assigned to λ_{r}.

Let's start by checking what happens to λ_1. Given the semantics of the indexed truth-predicates (and the workings of the generalised negation function), it follows that $[\![\neg\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] \in \{0,1\}$, which means that λ_1 itself can only be interpreted as 0 or 1. Now, if $[\![\lambda_1]\!] = 1$, then $\delta([\![\lambda_1]\!],1) = 1$, so $[\![\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] = 1$, which in turn means that $[\![\neg\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] = 0$. Since $\neg\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})$ and λ_1 must have matching semantic values, this is impossible.

On the other hand, if $[\![\lambda_1]\!] = 0$, then $\delta([\![\lambda_1]\!],1) = 0$, so $[\![\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] = 0$, which means that $[\![\neg\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] = 1$. Just as in the last case, this cannot obtain. In conclusion, there cannot be any fuzzy models of the entire language $\mathcal{L}^{\infty}_{\mathsf{T}}$ because there is at least one uninterpretable symbol of $\mathcal{L}^{\infty}_{\mathsf{T}}$—and T_1 serves as an explicit example. □

3 Alternative Fuzzy Semantics for $\mathcal{L}^{\infty}_{\mathsf{T}}$

Perhaps the Kronecker-Delta semantics that we relied on is overly punishing of close mismatches of values. Under this brand of semantics, if some sentence φ has semantic value r,

[2] Hájek, Paris and Shepherdson [HPS00] prove a result in this vicinity, but theirs is slightly different than ours. In their paper, they show that the standard model of arithmetic, $\mathcal{N}$, cannot be extended to a model of $\mathsf{PAT}_{\text{Ł}\forall}$. There are no indexed truths in their framework—it's only about a global, disquotational truth-predicate T. They have also shown that theory $\mathsf{PAT}_{\text{Ł}\forall}$ is actually consistent, but it immediately becomes inconsistent if one attempts to extend it with truth-theoretic axioms which say that T commutes with connectives.

then for any small $\varepsilon > 0$ and $s \in (r-\varepsilon, r+\varepsilon) \setminus \{r\}$, the precise truth-predication $\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi\urcorner})$ will have semantic value 0. This seems too harsh. For a concrete example, suppose that:

$$[\![\varphi]\!] = 0.67583 \text{ for some } \varphi \in \mathsf{Sent}_{\mathcal{L}_\mathsf{T}^\infty}$$

Then, using our semantics, the following assignment will obtain:

$$[\![\mathsf{T}_{0.67584}(\overline{\ulcorner\varphi\urcorner})]\!] = 0$$

even though, roughly speaking, the indexed-predicate $\mathsf{T}_{0.67584}$ "got it right"—the error is just 0.00001. It seems reasonable to suggest that the proper semantic value of $\mathsf{T}_{0.67584}(\overline{\ulcorner\varphi\urcorner})$ ought to be some number $s \in (1-\varepsilon, 1)$ for some very small $\varepsilon > 0$. One way of accomplishing this might be to suggest that, if $\varphi_\mathsf{r} \in \mathsf{Sent}_{\mathcal{L}_\mathsf{T}^\infty}$ is a sentence with semantic value $r \in [0,1]$, then:

$$[\![\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})]\!] = \gamma^{d(s,r)}$$

where $\gamma > 0$ is some tiny, epsilonic number, e.g. Liouville's constant (or any other small quantity), and $d : \mathbb{R}^2 \to R_{\geq 0}$ is the ordinary distance function on the reals, defined as follows:

$$d(x,y) = \begin{cases} y - x & \text{if } x \leq y \\ x - y & \text{otherwise} \end{cases}$$

This semantic framework has the following upshots:

- When $s = r$, then $[\![\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})]\!] = \gamma^0 = 1$.
- When $s \approx r$, then $[\![\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})]\!] \approx 1$.

Thus, unlike the Kronecker-Delta semantics, where indexed truth predications can only take Boolean values, we now allow for fuzzy semantic values for precise statements such as $\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})$. That being said, this fuzzified framework makes some odd predictions of its own. For instance, this framework makes it impossible for any precise truth-predication to be perfectly false, since 0 is not in the range of function $f(x,y) \coloneqq \gamma^{d(x,y)}$. This means that even statements that attribute perfect truth to outright falsities, e.g. $\mathsf{T}_1(\overline{\ulcorner 2+2=5\urcorner})$, will turn out *partially true*. This seems seriously problematic.

What if, instead of imposing $[\![\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})]\!] = \gamma^{d(s,r)}$, we designed our semantics to assign the product $s \times r$ as the semantic value of $\mathsf{T}_\mathsf{s}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})$? This sounds like a natural suggestion, but it comes with some other problematic predictions. For example, the sentence $\mathsf{T}_\mathsf{r}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})$ should be a paradigmatic example of a perfectly true sentence, since it says that sentence φ_r has semantic value r, which it does.[3] However, the product-semantics gets this wrong, since for any $r \in (0,1)$, we have that $[\![\mathsf{T}_\mathsf{r}(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})]\!] = r^2$, which is strictly less than 1. Another problem arises when we consider positive truth-predications of perfectly false sentences, e.g. $\mathsf{T}_\mathsf{r}(\overline{\ulcorner\varphi_0\urcorner})$, or perfectly false predications of partially true sentences, e.g. $\mathsf{T}_0(\overline{\ulcorner\varphi_\mathsf{r}\urcorner})$. With respect to the former case: if $r \approx 0$, the product-semantics makes the wrong assignment $[\![\mathsf{T}_\mathsf{r}(\overline{\ulcorner\varphi_0\urcorner})]\!] = 0$, when in fact it should be the case that $[\![\mathsf{T}_\mathsf{r}(\overline{\ulcorner\varphi_0\urcorner})]\!] \approx 1$.[4]

[3] In particular, $[\![\mathsf{T}_{\frac{1}{2}}(\overline{\ulcorner\lambda\urcorner})]\!]$ should arguably be 1, where λ is the liar sentence with $[\![\lambda]\!] = \frac{1}{2}$.

[4] The truth-predication *correctly* states that the perfectly false sentence φ_0 has a truth-value that is extremely

4 Modulus Semantics for $\mathcal{L}_{\mathsf{T}}^{\infty}$

Hence, we must look for a new binary function f to underpin our semantics. In light of the discussion above, it seems reasonable to impose that the function $f : [0,1]^2 \to [0,1]$ such that $[\![\mathsf{T}_{\mathsf{x}}(\ulcorner\varphi_{\mathsf{y}}\urcorner)]\!] = f(x,y)$ should obey the following desiderata:

- f ought to be a continuous function.
- $f(1,0) = f(0,1) = 0$.
- $f(x,x) = 1$ for all $x \in [0,1]$.
- If $x \approx y$, then $f(x,y) \approx 1$.
- If $d(x,y) \approx 1$, then $f(x,y) \approx 0$.

We will denote the distance between x and y, viz. $d(x,y)$ via the usual modulus notation, i.e. $|x-y|$. The cleanest function $f : [0,1]^2 \to [0,1]$ which obeys all of these properties is:

$$f(x,y) = 1 - |x-y|$$

Under the modulus semantics for $\mathcal{L}_{\mathsf{T}}^{\infty}$, we do not have the same obstacle with respect to the truth-value of the fixed point of $\neg\mathsf{T}_1$.

Theorem 4.1 (Perfect Truth and Modulus Semantics)**.** The $\mathcal{L}_{\mathsf{T}}^{\infty}$ sentence which says about itself that it is not perfectly true can only have fuzzy equilibriums as semantic values, i.e. fixed points of the truth-function for negation.

Proof. Let λ_1 be the fixed point of $\neg\mathsf{T}_1$. Then $[\![\lambda_1]\!] = [\![\neg\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!] = f_{\neg}([\![\mathsf{T}_1(\overline{\ulcorner\lambda_1\urcorner})]\!]) = f_{\neg}(1 - |1 - [\![\lambda_1]\!]|) = f_{\neg}([\![\lambda_1]\!])$. Thus, depending on the negation truth-function that one chooses, the semantic value of λ_1 will need to be a fuzzy equilibrium.

□

There are a handful of choices for the truth-functions of our usual connectives. With respect to the foregoing theorem, the choice of the negation function will directly impact the fuzzy equilibriums that can serve as the semantic value of λ_1, which should be a value in $[0,1]$ such that $f_{\neg}([\![\lambda_1]\!]) = [\![\lambda_1]\!]$.

This immediately discounts the Gödel and Product semantics for $\mathcal{L}_{\mathsf{T}}^{\infty}$, because it is impossible for λ_1 to have a semantic value in $[0,1]$ such that $[\![\lambda_1]\!] = f_{\neg}^{G}([\![\lambda_1]\!])$ or $[\![\lambda_1]\!] = f_{\neg}^{P}([\![\lambda_1]\!])$. The proof is straightforward. Both $f_{\neg}^{G}$ and $f_{\neg}^{P}$ are identical to the function g which returns 1 on argument 0 and returns 0 on any other positive argument in the unit interval. There's no value in $[0,1]$ that $[\![\lambda_1]\!]$ can have such that $g([\![\lambda_1]\!]) = [\![\lambda_1]\!]$, because if $[\![\lambda_1]\!] = 0$, then $g([\![\lambda_1]\!]) = 1$ and if $1 \geq [\![\lambda_1]\!] > 0$, then $g([\![\lambda_1]\!]) = 0$.

The only contender left amongst the canonical fuzzy systems is $Ł_{\aleph_1}$ semantics, because the function $f_{\neg}^{Ł} : [0,1] \to [0,1]$ demonstrably admits a unique fixed point, since it is a decreasing continuous function from a real interval to itself. This fixed point happens to be $\frac{1}{2}$ and our semantics for $\mathcal{L}_{\mathsf{T}}^{\infty}$ ought to be designed such that $[\![\lambda_1]\!]$ gets assigned this value.

close to 0. Thus, its overall value should be extremely close to 1, and yet it actually happens to be as far as possible from 1.

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On aggregation in parametric array theories

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Abstract

We prove an NP upper bound on a theory of integer-indexed integer-valued arrays that extends combinatory array logic with the ability to express sums of elements. The decision procedure that we give is based on observations obtained from our analysis of the theory of power structures.

1 Introduction

Many applications of computer science to operations research and software engineering require some form of constraint solving technology. We focus in the satisfiability modulo theories (SMT) framework which was intensively developed in the first decade of the century, leveraging progress in the architecture of propositional satisfiability solvers [19, 21, 8].

SMT addresses the satisfiability problem of fragments of first-order theories that are quantifier-free or have a small number of quantifier alternations. In fortunate occasions, this restriction makes the satisfiability problem NP-complete. In such cases, it is possible to reduce the satisfiability problem of the fragments to the satisfiability problem of propositional logic in polynomial time. Some theories supported using such reduction include real numbers, integers, lists, arrays, bit vectors, and strings [2, 15].

This work analyses the structure of a well-known fragment of the quantifer-free theory of arrays. In the SMT framework, arrays are conceived as indexed homogeneous collections of elements from some fixed domain. This is in contrast to other data-structures, like lists, which can only be accessed with recursive operators. The popularity of arrays stems from the fact that they can be used to model many abstractions useful in applications such as programming [6, 30], databases [12, 9], model checkers [11], memory models [5] or quantum circuits [4].

Several theories of arrays in the literature express essentially the same concepts under different syntactic appearances [29, 7, 13, 1]. As a consequence, a systematic classification of these theories is becoming increasingly difficult. This results in duplicated engineering efforts. It has been argued [18, Lecture 19] that some of these redundancies could be avoided by adopting a semantic perspective on the study of SMT theories.

Our results show that the semantic approach is fruitful in the area of decision procedures for theories of arrays. We demonstrated in [26, 27] how, by fixing a model of such theories, we are able to reconstruct and extend the celebrated combinatory array logic fragment [3]. In this paper, we further show how these observations can be extended to support summation constraints. Our methodology is inspired in the model theory of power structures [20, 10], which we adapt from the first-order to the quantifier-free setting, which is the one relevant for applications to SMT.

[*]Research supported by the Swiss NSF Project P500PT_222338

2 First-order model theory

We start reviewing some notions from first-order model theory.

A **first-order language** is one whose logical symbols are $\neg, \wedge, \vee, \forall$ and $\exists$, whose terms are either variables, constants or function symbols applied to terms and whose formulas are either atomic (relation symbols applied to terms) or general (atomic formulas and inductively, from formulas A, B, we get new formulas $\neg A, A \wedge B$ and $A \vee B$ and from a formula A and a variable symbol x we get the new formulas $\exists x.A$ and $\forall x.A$).

A variable in a formula is free if there is no occurrence of a quantifier binding the variable name on the path of the syntax tree of the formula reaching the occurrence of the variable. A formula without free variables is a **sentence**. A **first-order theory** is a set of sentences written in some first-order language.

A **first-order structure** $\mathcal{A}$ over a first-order language L is a tuple with four components: a set A called the domain of A; a set of elements of A corresponding to the constant symbols of L; for each positive integer n, a set of n-ary relations on A (i.e. subsets of A^n), each of which is named by one or more n-ary relation symbols of L and for each positive integer n, a set of n-ary operations on A (i.e maps from A^n to A), each of which is named by one or more n-ary function symbols of L. The mapping assigning each first-order symbol of L to its corresponding interpretation in $\mathcal{A}$ is denoted $\cdot^{\mathcal{A}}$. This function is extended to work on terms, i.e. the application of function symbols to constants, variables or other terms, by requiring that $(f(s_1, \ldots, s_n))^{\mathcal{A}} = f^{\mathcal{A}}(s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}})$.

Let ϕ be a sentence in a first-order language L and let $\cdot^{\mathcal{A}}$ be an interpretation of the symbols of L in the structure $\mathcal{A}$. The **sentence ϕ is satisfied in the structure** $\mathcal{A}$, written $\mathcal{A} \models \phi$, if the following conditions apply.

- If ϕ is the atomic sentence $R(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n$ are terms of L then $\mathcal{A} \models \phi$ if and only if $(s_1^{\mathcal{A}}, \ldots, s_n^{\mathcal{A}}) \in R^{\mathcal{A}}$.
- $\mathcal{A} \models \neg\phi$ if and only if it is not true that $\mathcal{A} \models \phi$.
- $\mathcal{A} \models \phi_1 \wedge \phi_2$ if and only if $\mathcal{A} \models \phi_1$ and $\mathcal{A} \models \phi_2$.
- $\mathcal{A} \models \phi_1 \vee \phi_2$ if and only if $\mathcal{A} \models \phi_1$ or $\mathcal{A} \models \phi_2$.
- If ϕ is the sentence $\forall y.\psi(y)$ then $\mathcal{A} \models \phi$ if and only if for all elements b of A, $\mathcal{A} \models \psi(b)$.
- If ϕ is the sentence $\exists y.\psi(y)$ then $\mathcal{A} \models \phi$ if and only if there is at least one element b of A such that $\mathcal{A} \models \psi(b)$.

Let Ax be a set of first-order sentences. We define the relation $Ax \models \phi$ which holds if and only if for every structure $\mathcal{A}$, if $\mathcal{A} \models ax$ for each sentence $ax \in Ax$ then $\mathcal{A} \models \phi$.

The **axiomatic theory** defined by a set of axioms Ax is $Th(Ax) = \{\phi | Ax \models \phi\}$.

The **semantic theory** of a structure $\mathcal{A}$ is the set $Th(\mathcal{A}) := \{\phi \mid \mathcal{A} \models \phi\}$.

When studying the sets $Th(\mathcal{A})$ and $Th(Ax)$ we may assume the sentences are in prenex normal form. A **prenex normal form** of a first-order formula F is a first-order formula consisting of a string of quantifiers (called the prefix of the formula) followed by a quantifier-free formula (known as the matrix of the formula) which is equivalent to F. It is well-known that there is a polynomial time algorithm transforming sentences of a first-order theory into to equivalent sentences in prenex normal form.

The **existential fragment of the first-order theory** T, denoted $Th_{\exists^*}(T)$,is the subset of sentences in T whose prefix in prenex normal form is purely existential. We write $Th_{\exists^*}(Ax)$ if T is axiomatically specified and $Th_{\exists^*}(\mathcal{A})$ if T is semantically specified.

3 Array theories

The theory of arrays T_A is defined as a first-order theory with three sorts: A for arrays, I for indices and E for elements of arrays. It has one "read" function symbol $\cdot[\cdot] : A \times I \to E$, one "write" function symbol $\cdot\langle\cdot\triangleleft\cdot\rangle : A \times I \times E \to A$ and includes the equality relation symbol $\cdot = \cdot$ for indices and elements. The theory is described axiomatically as the sets of sentences satisfying axioms Ax of the following form [2]. $=$ is axiomatised as a reflexive, symmetric and transitive relation. Array read is assumed to be a congruence relation, i.e. $\forall a, i, j.i = j \to a[i] = a[j]$. Finally, there are axioms relating the read and write operations $\forall a, v, i, j.i = j \to a\langle i \triangleleft v\rangle[j] = v$ and $\forall a, v, i, j.i \neq j \to a\langle i \triangleleft v\rangle[j] = a[j]$.

The quantifier-free fragment of T_A is the set of formulas that can be written without any use of quantifiers. Our goal is to decide which quantifier-free formulas are satisfiable. The satisfiable quantifier-free formulas correspond precisely to set of formulas in $Th_{\exists^*}(Ax)$.

Proposition 1. *The existential closure of the satisfiable formulas in the quantifier-free fragment of T_A is the set $Th_{\exists^*}(Ax)$. Conversely, if we drop the existential prefixes in $Th_{\exists^*}(Ax)$, we obtain the satisfiable formulas of the quantifier-free fragment of T_A.*

Proof. The existential closure of a satisfiable formula in the quantifier-free fragment of T_A is, by definition, in $Th_{\exists^*}(Ax)$. A formula in $Th_{\exists^*}(Ax)$ is true by definition. Converting it to prenex normal form and dropping the existential quantifier prefix leaves a formula of the quantifier-free fragment of T_A. □

Many works, starting with [29], consider an extension of the theory T_A with axioms of the form $R(a_1, \dots, a_n) \leftrightarrow \forall i.R(a_1(i), \dots, a_n(i))$ which says that some relation holds on a tuple of array variables $a_1, \dots, a_n$ if and only it holds at each component. One example is the extensionality axiom $\forall a, b.a = b \leftrightarrow (\forall i.a[i] = b[i])$. In [26], we observed that several fragments extending combinatory array logic [7] can be described semantically as the theory of a power structure [20]. More precisely, we showed the following results.

Definition 2. *The generalised power $\mathcal{P}(\mathcal{M}, I)$ of the combinatory array logic fragment is a structure whose carrier set is the set M^I of functions from the index set I to the carrier set of the structure of the array elements M and whose relations are interpreted as sets of the form*

$$\{(a_1, \dots, a_n) \in (M^I)^n | \Phi(S_1, \dots, S_k)\}$$

where Φ is a Boolean algebra expression over $\mathcal{P}(I)$ using the symbols $\subseteq$, $\cup$, $\cap$ or $\cdot^c$ and each set variable S is interpreted as $S = \{i \in I | \theta(a_1(i), \dots, a_n(i))\}$ where θ is a formula in the theory of the elements.

Theorem 3. *The quantifier-free formulas of combinatory array logic can be encoded in polynomial time as sentences in the theory of the generalised power $\mathcal{P}(\mathcal{A}, I)$ in a way that preserves satisfiability of the formulas.*

Theorem 4. *The theory $Th_{\exists^*}(\mathcal{P}(\mathcal{A}, I))$ can be decided in* NP *even when the algebra of indices $\mathcal{P}(I)$ includes a cardinality operator and the language includes linear arithmetic constraints on the cardinality constraints.*

Our goal in this note is to generalise this result to summation constraints over the array (function symbols) variables. Interestingly, to preserve decidability one has to disallow constants in the element theory specifications θ.

4 Decision Procedure

Our first step is defining the input language to be decided.

Definition 5. *The theory of generalised powers with sums consists of formulae of the form*

$$F(S_1, \dots, S_k, \overline{\sigma}) \wedge \bigwedge_{i=1}^{k} S_i = \{n \in I | \varphi_i(\overline{c}(n))\} \wedge \overline{\sigma} = \sum (\!| \overline{c}(n) \mid \varphi_0(\overline{c}(n)) |\!) \tag{1}$$

where F is a formula from Boolean algebra of sets, $\varphi_0, \dots, \varphi_k$ are formulae in the existential fragment of Presburger arithmetic and $\overline{c}$ is a tuple of arrays of natural numbers. We will refer to the first conjunct of this formula as the Boolean algebra term, to the second conjunct as the set interpretations and to the third conjunct as the multiset interpretations.

There are some differences between Definition 5 and [23, Definition 2.1]. [23, Definition 2.1] has a quantifier-free Presburger arithmetic formula instead of the Boolean algebra term F. Second, the term $\forall e.F$ corresponds to our set interpretations. Third, the term $(u_1, \dots, u_n) = \sum_{e \in E}(t_1, \dots, t_n)$ corresponds to our multiset interpretation. It should be noted that the indices in our setting range over the natural numbers and not over a finite set E as in [23].

An important observation is that the definition does not allow free variables to be shared between the three conjuncts. In fact, if we allowed such shared constants, the resulting fragment would have an undecidable satisfiability problem.

Corollary 6. *The satisfiability of formulas of the form*

$$F(S_1, \dots, S_k, \overline{\sigma}, \overline{f}) \wedge \bigwedge_{i=1}^{k} S_i = \{n \in I | \varphi_i(\overline{c}(n), \overline{f})\} \wedge \overline{\sigma} = \sum (\!| \overline{c}(n) \mid \varphi_0(\overline{c}(n), \overline{f}) |\!) \tag{2}$$

is undecidable.

Proof. By reduction from Hilbert's tenth problem [17]. One can encode in this theory the addition of two natural numbers using the formula F which is in Boolean algebra of sets with cardinalities and thus includes quantifier-free Presburger arithmetic. Multiplication $z = xy$ can be encoded by imposing the array $\overline{c}$ to be equal to the constant x in each position, have length y and sum up to z. □

Let us now describe the main steps of the decision procedure for the theory in Definition 5.

Elimination of terms in Boolean algebra with Cardinalities. To eliminate these constraints, we introduce k array variables $c_1, \dots, c_k$ and we rewrite the Boolean algebra expressions and cardinality constraints in terms of set interpretations and summation constraints. See the appendix for further details.

As a result of this phase, we obtain a formula of the form:

$$\psi(\overline{\sigma}) \wedge \bigwedge_{i=1}^{k} I = \{n \in I | \varphi_i(\overline{c}(n))\} \wedge \overline{\sigma} = \sum (\!| \overline{c}(n) \mid \varphi_0(\overline{c}(n)) |\!) \tag{3}$$

where ψ is a quantifier-free Presburger arithmetic formula and all the Boolean algebra and cardinality constraints has been translated into set interpretations and summation constraints.

Elimination of the Set Interpretations. The next step in the decision procedure is to eliminate the set interpretation term. However, in the form of Formula 3, this is particularly simple. Formula 3 is equivalent to:

$$\psi(\overline{\sigma}) \wedge \overline{\sigma} = \sum (\!|\overline{c}(n) \mid \bigwedge_{i=0}^{k} \varphi_i(\overline{c}(n))|\!) \tag{4}$$

It thus remains to remove the summation operator.

Elimination of the Summation Operator. The next step is to rewrite sums to a star operator introduced in [24]. Given a set A, the set A^* is defined as:

$$A^* = \{u \mid \exists N \geq 0, x_1, \ldots, x_N \in A.u = \sum_{i=1}^{N} x_i\}$$

Proposition 7 (Multiset elimination)**.** *The formula*

$$\exists \overline{\sigma}, \overline{c}.\psi(\overline{\sigma}) \wedge \overline{\sigma} = \sum_{n \in \mathbb{N}} (\!|\overline{c}(n) \mid \varphi(\overline{c}(n))|\!) \tag{5}$$

and the formula

$$\exists \overline{\sigma}.\psi(\overline{\sigma}) \wedge \overline{\sigma} \in \{\overline{k} \mid \varphi(\overline{k})\}^* \tag{6}$$

are equivalent.

The argument needs to be adapted from Theorem 2.4 of [25] since both our index and element set are infinite. The details are given in the appendix.

The next step is to eliminate the star operator introduced in Proposition 7. To do so, one could use [25, Theorem 2.23] which shows that if Formula 6 is satisfiable then it also has a solution that can be written with a polynomial number of bits. We adapt this result to the case where we consider explicit integer exponents in the sets. That is we consider given a set A and an integer $m \in \mathbb{N}$, the set A^m defined as $A^m = \{u \mid x_1, \ldots, x_m \in A.u = \sum_{i=1}^{m} x_i\}$. The reason to do this is that when mixing summation and other kinds constraints such as in [27], we need to *synchronise* the cardinality constraints of the combined theory with the cardinality constraints arising from the number of addends used in the sums.

Definition 8. *LIA with sum cardinalities, denoted LIA^{card}, is the theory consisting of formulas of the form $F_0 \wedge \bigwedge_{i=1}^{n} u \in \{x \mid F_i(x)\}^{x_i}$ where F_0 and F are quantifier-free Presburger arithmetic formulae.*

Proposition 9. *LIA^{card} is in* NP*.*

A detailed proof is given in the appendix.

5 Conclusion

Despite the numerous works that are dedicated to the theory of arrays and its variations, it remains a challenge to provide a comprehensive classification of array theories according to the computational complexity of their satisfiability problem and their expressive power. This paper shows that even classical theories such as the combinatory array logic fragment can be optimised with respect to both metrics. An interesting extension that we leave open is to support combinatory array logic with sums and different element sorts.

Acknowledgements.

The author wishes to express his gratitude to Viktor Kunčak for useful suggestions on the presentation of the results of this paper.

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Towards an abstract theory of definitions

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Contents

1 Introduction

In [8], Tarski[1] laid down a series of axioms aiming to characterise a primitive notion of *consequence* and showed that, by means of this notion only, other metalogical concepts could be defined; among them the concepts of *theory*, *logical equivalence*, *consistency*, *completeness*. Stemming from Tarski's seminal work, the study of abstract consequence relations — motivated by their logical interpretations — has developed into a mature and active field of research (see, for instance, Martin & Pollard's book [4]).

Later, in [9], Tarski initiated a study of the notion of *definition* showing important analogies with the abstract approach taken in [8]:

> In the methodology of the deductive sciences two groups of concepts occur which, although rather remote from one another in content, nevertheless show considerable analogies, if we consider their role in the construction of deductive theories, as well as the inner relations between concepts within each of the two groups themselves. To the first group belong such concepts as 'axiom', 'derivable sentence' (or 'theorem'), 'rule of inference', 'proof', to the second — 'primitive (undefined) concept' (or 'primitive term'), 'definable concept', 'rule of definition', 'definition'. A far-reaching parallelism can be established between the concepts of the two groups: the primitive concepts correspond to the axioms, the defined concepts to the derivable sentences, the process and rules of definition to the process and rules of proof. [10, p. 296].

According to Pogorzelski & Surma review of an English translation[2] of [9],

> "Paper X belongs to those papers of Tarski which have organized a certain branch of metalogic and established some of its fundamental notions. It deals with syntactic definability of terms, and together with some earlier results concerning the concept of semantic definability [...] it establishes the foundations of the theory of definability of terms. X is a natural extension of Tarski's papers [...] on the notion of consequence, since it establishes for terms a number of notions analogous to those

[1] An earlier exposition, without proofs, of the results collected in [8] appeared in [7].

[2] The English translation of [9] by J. H. Woodger is included in [10] and referred to as "paper X" of the collection.

which are fundamental for propositional expressions. We find in X the definitions of such notions as: equivalence of sets of terms, closure of a set of terms (an analogue of the concept of the system), system of primitive terms (an analogue of the concept of the set of axioms), the notion of independent and complete set of terms (the last notion relativized to an arbitrary but fixed set of sentences) [...] Results contained in X virtually exhausted syntactic problems of definability". [5, p. 104].

Even though the intimate connections of [9] with [8] are emphasised by Tarski himself, the study on definitions is developed in [9] within a framework which is considerably less 'abstract' than that assumed in [8] in order to study the notion of consequence: The former presupposes an internal structure of the sentences that distinguish variables and extra-logical terms, as the minimum setting for speaking about "the definability and the mutual independence of concepts" [10, p. 296]. By contrast, in the present paper my aim is that of establishing the fundamentals of an abstract theory of definitions in the same framework of Tarski's [8], by taking an arbitrary notion of consequence as the only primitive concept.

Reasons for undertaking the above project are mainly the same as those advanced by Tarski for an abstract study of the notion of consequence, namely, the wish of reaching the highest level of generality, by establishing the fundamental properties of concepts which are common to special meta-disciplines, and of applicability to specific deductive disciplines understood as instances of the abstract notion. In particular, an abstract theory of definitions might be applied to a realm of objects, for instance, propositions, which lack the internal structure of the sentences of a fully formalised language.

2 The classical theory of definitions

An abstract theory of definitions aims to define in terms of an abstract consequence relation some notions which intend to capture analogue concepts studied by theories of definitions within formalised languages and logic. Therefore we start by briefly recalling the fundamentals of the most developed and uncontroversial of such theories: The classical theory of definitions for first-order languages.

Let $\mathcal{L}$ be a first-order language with identity. We will use the same symbol $\mathcal{L}$ also to denote the set of all sentences (closed formulæ) of the language. The symbol $\vdash$ denotes the relation of (classical) logical consequence between sets of sentences of $\mathcal{L}$ and sentences of $\mathcal{L}$, equivalently defined, by the completeness theorem, either in terms of rules of inference or in terms of models.

We assume, for simplicity, that among the non-logical constants of $\mathcal{L}$ there is a unary predicate $\mathbf{P}$ we want to define in terms of the other non-logical constants of $\mathcal{L}$. We denote by $\mathcal{L}^-$ the set of sentences of the sublanguage of $\mathcal{L}$ built from the same non-logical constants of $\mathcal{L}$, except $\mathbf{P}$.

Let Σ be any set of sentences of $\mathcal{L}$. Let $\Sigma^- = \Sigma \cap \mathcal{L}^-$. We understand the sentences which are in Σ but not in Σ^- as axioms added to the base theory Σ^- in order to define the predicate $\mathbf{P}$. The classical theory of definitions[3] has that the set of sentences Σ is a correct definition of $\mathbf{P}$ (in terms of the base theory Σ^-) iff Σ has both the following properties:

- *Non-creativity*[4]: Every sentence of $\mathcal{L}^-$ which is provable from Σ is already provable from Σ^-.

[3] The two notions, described below, of "non-creativity" and "eliminability" are first explicitly introduced (under a different terminology) as criteria for a correct definition in [9, fn. 3]. According to Hodges [3, p. 105], Suppes' [6] is probably the first place where the two criteria are "paired as the conditions for a sound definition".

[4] An alternative name for the non-creativity property is *(syntactic) conservativeness*.

- *Eliminability*[5]: Every formula ϕ of $\mathcal{L}$ is provably equivalent in Σ to a formula ϕ^- of $\mathcal{L}^-$.

Moreover, the classical theory of definitions, via Beth's theorem, establishes that the two conditions of non-creativity and eliminability are jointly equivalent to the semantic condition of *determinability*: Every model of Σ^- has one and only one expansion to a model of Σ[6].

3 An abstract theory of definitions

We now turn to Tarski's [8] abstract setting. We work with an arbitrary non-empty set A and with a primitive notion of consequence between elements of A. In the primarily intended interpretation — the above-sketched classical theory of definitions — the set A is replaced by the set of all sentences of $\mathcal{L}$, however, in the abstract setting no properties of A are assumed and its elements can be taken to be sentences as well as any other kind of "unstructured" entities such as, for instance, propositions. For the notion of consequence — primarily interpreted by the relation of classical first-order logical consequence — it is customary to start with the properties characterising a generic notion of "closure", to which further axioms can be added to model more specific intended situations.

Officially, a *consequence relation* on A is a relation $\models$ between subsets Φ and elements ϕ of A satisfying the following properties:

- $\{\phi\} \models \phi$ (*reflexivity*).
- $\Phi \subseteq \Phi' \Rightarrow \forall\phi\,(\Phi \models \phi \Rightarrow \Phi' \models \phi)$ (*monotonicity*).
- $\Phi \models \Psi \wedge \Psi \models \phi \Rightarrow \Phi \models \phi$ (*transitivity*).

Given a consequence relation $\models$ on A we define:

- $\mathsf{Thm}_{\models}(\Phi) = \{\phi \in A \mid \Phi \models \phi\}$.
- $\mathsf{C}_{\models} = \{\Phi \subseteq A \mid \mathsf{Thm}_{\models}(\Phi) = \Phi\}$.

We omit the index $\models$ in $\mathsf{Thm}_{\models}$ and $\mathsf{C}_{\models}$ (and in subsequent similarly defined objects) when it is clear from the context. The members of $\mathsf{Thm}(\Phi)$ are called the *theorems* of Φ (under the consequence relation $\models$). The members of C are called the *theories* of $\models$.

The map $\Phi \mapsto \mathsf{Thm}(\Phi)$ is a *closure operator on* A, i.e., is a function from $\mathcal{P}(A)$ to $\mathcal{P}(A)$ which is monotone, progressive and idempotent. The family C of subsets of A is a *closure system on* A, i.e., $A \in \mathsf{C}$ and for every non-empty family $\mathcal{F} \subseteq \mathsf{C}$ the intersection $\bigcap \mathcal{F}$ belongs to C.

Following Tarski, we say that a subset Φ of A is *consistent* iff there exists $\phi \in A$ such that $\Phi \not\models \phi$. We say that Φ is *maximal consistent* iff Φ is maximal with respect to inclusion in the family of all consistent subsets of A. We denote by U the family (possibly empty) of all maximal consistent subsets of A.

Definitions in first-order logic assume that the full object language $\mathcal{L}$ is split into two subsets: The set of the sentences of $\mathcal{L}$ in which the distinguished predicate $\mathbf{P}$ occurs and the set of the sentences in which $\mathbf{P}$ does not occur, the latter denoted by $\mathcal{L}^-$. Analogously, we assume that the abstract setting is endowed with a distinguished subset A^- of A. We denote by $\models^-$ the

[5] An alternative name for the eliminability property is *(logical) definability*.

[6] By removing from determinability its existence claim, we obtain the uniqueness condition on Σ which in literature is frequently called *implicit definability*: Every model of Σ^- has at most one expansion to a model of Σ.

consequence relation $\models$ restricted to subsets and elements of A^-, which turns out to be itself a closure relation.

Since A^- has to play the role of a "sub-language" of A, it is reasonable to assume A^- to have some degree of "closure". We assume A^- to be *closed under classical negation*, a technical condition which corresponds to the intuitive requirement for a language of being closed under negation and which implies its inconsistency. This is enough to prove that the theories of $\models^-$ are exactly the intersections with $\mathcal{L}^-$ of theories of $\mathcal{L}$.

We can give the following abstract counterpart of corresponding notions involved in the classical theory of definitions. Let X be any subset of A, and let $X^- = X \cap A^-$. We say that

- W is a *syntactic definition* iff W has the properties (with respect to A^-) of non-creativity and *abstract eliminability*, namely, every element $\phi \in A$ is equivalent in W to an element ϕ^- of A^-.
- W is a *relative definition* iff (a) for every consistent subset X of A^- such that $W^- \subseteq X$, the set $X \cup W$ is consistent, and (b) for every maximal (in A^-) consistent subset X of A^- such that $W^- \subseteq X$, the set $X \cup W$ is maximal consistent.
- W is a *semantic definition* iff for every maximal (in A^-) consistent subset X of A^- such that $W^- \subseteq X$, there exists one and only one maximal consistent set U such that $X \cup W \subseteq U$.

The above-mentioned abstract notions of definitions are motivated as follows. The property of non creativity *verbatim* translates from the first-order to the abstract setting. The property of abstract eliminability is a straightforward weakening of the property of first-order eliminability, which we can call, in the first-order context, *sentential eliminability.* Sentential eliminability is the property we obtain from first-order eliminability by replacing the existence of a correspondent equivalent *formula* in the base language for every *formula*, with the existence of a correspondent equivalent *sentence* in the base language for every *sentence*. The property of being a relative definition can be stated *verbatim* in the first-order context and turns out to be equivalent to the conjunction of non-creativity and (first-order) eliminability. Finally, the notion of semantic definition is an abstract counterpart of the first-order notion of determinability: The talk about models is replaced by talk about maximal consistent sets by observing that, in the first-order context, a set of sentences is maximal consistent if and only if is the set of all sentences which are true in a model.

The virtue of the three notions of definition above introduced is that they can be formulated in terms of just an arbitrary consequence relation $\models$ on A and a subset A^- of A, and that they looks as natural counterparts of well-known first-order notions. However, we can say little about the mutual relationships between the three notions in the general case. Even worse, without further assumptions on the consequence relation $\models$ the existence of maximal consistent set is not granted, hence the notions of relative and semantic definition can trivialise.

For these reasons, we need to specify further the relation of consequence and the sublanguage we are dealing with in order to study how the corresponding notions of definitions behave. As a matter of example we can consider the notion of *Henkin consequence*. Recall that a non-empty family $\mathcal{S}$ of subsets of A is a *closure base* for $\models$ iff for every subset Φ and element ϕ of A,

$$\Phi \models \phi \Leftrightarrow \forall Z \in \mathcal{S}\,(\Phi \subseteq Z \Rightarrow \phi \in Z).$$

We say that a closure base $\mathcal{S}$ for $\models$ *has exclusion negation* iff for every $\phi \in A$ there exists $\phi' \in A$ such that

$$\forall Z \in \mathcal{S}\,(\phi \in Z \Leftrightarrow \phi' \notin Z).$$

Finally, we say that $\models$ is *Henkin* iff there exists a closure base for $\models$ having exclusion negation.

Some useful consequences of $\models$ being Henkin:

1. The family U of all maximal consistent sets is not empty and is the unique closure base for $\models$ having exclusion negation.
2. The family U^- of all maximal (in A^-) consistent sets is formed by the intersections with $\mathcal{L}^-$ of the members of U.

The above-mentioned properties allow us to prove the following

Thm 3.1. *Let $\models$ be a Henkin consequence relation on the non-empty set A and let A^- be a non-empty subset of A closed under classical negation. Then, a non-empty subset W of A is a relative definition iff is a semantic definition.*

Moreover, in the first-order context, Theorem 3.1 leads to the following "sentential" version of Beth's theorem, which equates sentential eliminability with a natural weakening of first-order implicit definability:

Thm 3.2. *For a first-order theory Σ, the* sentential *eliminability property is equivalent to the following model-theoretic property: Any two expansions of elementarily equivalent models of Σ^- to models of Σ are elementarily equivalent.*

Finally, I conjecture that, by exploiting the abstract version of Craig's interpolation lemma given in [2], we can prove in the abstract setting that, under the hypotheses of Theorem 3.1, if the Henkin consequence relation $\models$ on A satisfies the further conditions of compactness and weak conjunction[7], then a non-empty subset W of A is a syntactic definition iff is a semantic definition.

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[7] Cfr. [1] for the weak conjunction property.

A Synthetic Proof of Myhill's Theorem in the Effective Topos

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Abstract

One of the first results in classical computability theory was establishing the undecidability of the halting problem. In this presentation we will prove an even stronger version in the internal logic of the effective topos; more precisely in its full subcategory $\mathrm{Mod}(\mathcal{K}_1)$ of modest sets internal to assemblies $\mathrm{Ass}(\mathcal{K}_1)$. We will do this by proving that the diagonal halting set K is creative with our new definition. Our notion of creativity is classically equivalent to Post's and Myhill's definition, but more importantly, it contains recursive content. The moral lesson is that if we do computability theory in the effective topos, the proofs turn out to be more constructive and in the spirit of what one intended to begin with.

1 Introduction

An analytic treatment of computability theory in a classical model for set theory inevitably leans heavily towards informal proof methods. They are of course partially justified by the empirical evidence provided by the works of Turing, Church and Kleene among others [6, 11, 12]. But informal methods are mainly used to avoid cumbersome details involving Gödel numbers to be able to get to the core mathematical ideas without having to deal with routine manipulations. This creates the need for a more synthetic presentation, which factors those cumbersome details into axioms.

A more suitable mathematical universe in which these ideas can be encoded turns out to be Hyland's *effective topos* $\mathcal{E}ff$ [4]. Here, all functions are recursive or computable so that no reference to an external model of computation is necessary. Synthetic or axiomatic treatment of computability theory, pioneered by Bauer among others [1], allows us for instance to talk about recursively enumerable sets as just the (effective) sets, which are enumerable. In this sense, the synthetic approach reveals the mathematical structures without the encoded 'noise'. What is more, both the objects and morphisms between them carry constructive data in the effective topos. It therefore captures the essence of computability theory in which not only the results, but also the proofs are uniformly effective.

2 Synthetic Computability Theory

The first steps in synthetic computability theory in the effective topos have been taken by Bauer [1]. In this exposition, we take a few extra steps in this direction. We briefly present the context in which our investigation is carried out. Our references are from [10, 8].

Definition 2.1. A $\mathcal{K}_1$*–valued assembly* X is a set $|X|$ together with a function $E\colon |X| \to \mathcal{P}^*(\mathbb{N})$ assigning to each $x \in |X|$ a nonempty subset Ex of $\mathbb{N}$.

*This work was done as part of a Bachelor's thesis project completed at Lund University, Lund, Sweden.

Here $\mathcal{K}_1$ refers to Kleene's first model, and in the setting of recursive realizability, we think of Ex as the set of proofs for x. Assemblies can intuitively be thought as data types with an underlying set of values $|X|$ whose elements are given machine-level representations, or in our setting, realisers Ex. The representations of the elements in the underlying set is not unique as the set of realisers are not necessarily disjoint. We therefore ask for those datatypes for which the codes uniquely determine each value. The following definition captures this idea.

Definition 2.2. An assembly is said to be a *modest set* if for all $x, x' \in |X|$,

$$x \neq x' \implies Ex \cap Ex' = \emptyset.$$

Definition 2.3. Suppose $(|X|, E)$, $(|Y|, F)$ are two $\mathcal{K}_1$−valued assemblies. A function $f\colon |X| \to |Y|$ is said to be *tracked* by an element $t \in \mathbb{N}$ if for all $x \in X$ and for all $a \in Ex$, $ta\downarrow$ and $ta \in Ff(x)$.

Following our analogy, the morphisms between assemblies are precisely the functions that can be simulated, in our case, by a partial recursive function acting on the realisers instead of the elements. Assemblies and modest sets on $\mathcal{K}_1$ together with tracked maps form a bicartesian closed category, which is finitely complete and cocomplete with a natural numbers object $N := (\mathbb{N}, E)$, $En := \{ n \}$. We denote these categories $\mathrm{Ass}(\mathcal{K}_1)$ and $\mathrm{Mod}(\mathcal{K}_1)$ respectively.

There are close connections between fragments of a certain logic and particular classes of categories. In fact, the internal language of a cartesian closed category is simply typed λ-calclulus, where the objects of the category serve as basic types and morphisms as basic terms [7]. What is more, we are able to write down formulae of intuitionistic higher-order logic, which readily have the intended meaning in $\mathcal{E}\!f\!f$. We will use a suitable internal language without much reference hereafter.

Now, the following is a nice fact: the category of modest sets $\mathrm{Mod}(\mathcal{K}_1)$ can be regarded as a category internal to assemblies $\mathrm{Ass}(\mathcal{K}_1)$ which is internally complete [5]. For what this kind of internalization means in a more general context see [3]. We will use this fact in order to carry on our investigation in these categories. We point out a few objects and facts that form the main ingredients of our results:

- While the subobject classifier Ω of $\mathcal{E}\!f\!f$ is itself not an object of $\mathrm{Ass}(\mathcal{K}_1)$, two of its subobjects of interest are: the object of *decidable* truth-values 2 with the underlying set $\{ p \in \omega \mid p \vee \neg p \}$, which up to isomorphism is the assembly $(\{ 0, 1 \}, E)$ with $E0 := \{ 0 \}, E1 := \{ 1 \}$ [10, §3.2.7], and the object of *semidecidable* truth-values with the underlying set $\Sigma := \{ p \in \Omega \mid \exists f{:}N^N(p \leftrightarrow (\exists n(f(n) = 0))) \}$, which up to isomorphism is the assembly $(\{ 0, 1 \}, E)$ with $E0 := \overline{K}$ and $E1 := K$, where K denotes the diagonal halting set [10, Proposition 3.2.27]. Both are clearly modest, however the latter shows that truth and falsehood in this sense are recursively inseperable.
- There is indeed an one-to-one correspondence between the decidable subobjects of X and morphisms $X \to 2$. In particular, in $\mathcal{E}\!f\!f$ the Cantor space 2^N is the object of decidable subobjects of N. Recall that these are precisely the subsets of $\mathbb{N}$ that posess a recursive characteristic function, $2^N \cong (R, E)$, where $R := \{ f\colon \mathbb{N} \to 2 \mid f \text{ is recurisve} \}$ and $Ef := \{ e \mid e \text{ is Gödel number for } f \}$. In $\mathcal{E}\!f\!f$, 2^N and the space of functions N^N are isomorphic [10, Proposition 3.2.26]. Similarly, there is a one-to-one correspondence between semidecidable subobjects of N and tracked maps $N \to \Sigma$. The subobject $\Sigma \rightarrowtail \Omega$ is called the semidecidable subobject classifier because of the following isomorphism: $\Sigma^N \cong (RE, W)$, where $RE := \{ R \subseteq \mathbb{N} \mid R \text{is recursively enumerable} \}$ and $WR := \{ e \mid R = W_e \}$ [10, Proposition 3.2.28].

- The Σ-partial functions $N \to N_\perp$ are the synthetic analogue of partial recursive functions in the effective topos whose domains are precisly the semidecidable subobjects of N, for details see [1, §4]. This is part of a more general construction called lifting monads [2].
- We take for granted a pairing and unpairing isomorphism $N \times N \to N$. There exists an enumeration $\phi\colon N \twoheadrightarrow N_\perp^N$ such that $\forall \psi{:}N_\perp^N \exists e{:}N \phi(e) = \psi$, which together with pairing yields an enumeration $\phi_2\colon N \twoheadrightarrow N_\perp^{(N^2)}$ such that $\phi_2(e)(a,b) = \phi(e)(\langle a,\ b\rangle)$. We can continue the pattern to get a epimorphism ϕ_k for every natural number k. There is also an enumeration $W\colon N \twoheadrightarrow \Sigma^N$ such that $\forall A{:}\Sigma^N \exists e{:}N W_e = A$ [1, §4].
- The principle of countable choice (ACC), $\forall n{:}N \exists x{:}X R(n,x) \to \exists \alpha{:}(X^N) \forall n{:}N R(n, \alpha(n))$ holds for every object X of $\mathcal{E}ff$ [10, Corollary 3.2.9].

2.1 Basic synthetic results

The various results in the two coming sections emerged as an ongoing joint work with J.M.E. Hyland. Unless otherwise stated, to the best of our knowledge these results have not appeared in the literature.

Theorem 2.4. *In $\mathcal{E}ff$, the s-m-n theorem holds:*

$$\exists s_n^m{:}N^{(N^{m+1})} \forall e, y_1, \ldots, y_m{:}N \lambda \overline{x}.\phi_{m+n}(e)(\overline{y}, \overline{x}) = \phi_n(s(e, \overline{y}))$$

The crux of the argument of *s-m-n* lies within the ACC, and the two following results are a direct application of it, much like the classical case.

Theorem 2.5. *In $\mathcal{E}ff$, the Fixed point theorem holds cf. [1, Corollary 4.24]:*

$$\forall f{:}N^N \exists n{:}N \phi(f(n)) = \phi(n)$$

Theorem 2.6. *In $\mathcal{E}ff$, the Second recursion theorem holds:*

$$\forall f{:}N^{(N^2)} \exists n{:}N^N \forall x{:}N \phi(f(n(x), x)) = \phi(n(x))$$

2.2 Synthetic Myhill's theorem

In this section we establish our main theorem, namely that creativeness and completeness conincide in the effective topos. We will also show that $K{:}\Sigma^N$ is undecidable in the strong sense that it is creative. This version is even stronger than its classical counterpart as we shall see. We begin by stating the weak version, which is well known, and the argument mimics the classical one.

Proposition 2.7. *In $\mathcal{E}ff$, the set K is undecidable, that is*

$$\forall R{:}2^N R \neq K.$$

Let us first consider Myhill's characterisation of a creative set in our setting to see why it fails to be valid in the effective topos. The statement that K is creative would read as follows,

$$\forall A{:}\Sigma^N [\exists n{:}A \cap K \vee \exists n(n{:}A \cup K \to \bot)]. \tag{1}$$

Now consider $A = \emptyset$, then the first assertion of the disjunction is false and the second is true, while if $A = N$ then the situation is reversed. Recall, however, that Σ truth-values are recursively inseperable. Thus the above statement is asking us to do too much 'work'. Next, we provide a definition that implies Myhill's characterization and is classically equivalent to it. It is a version, which has as much constructive information as possible.

Definition 2.8 (Hyland)**.** In $\mathcal{E}ff$, a set $A{:}\Sigma^N$ is creative if $\exists u{:}N_\perp^N \forall e{:}N$

(i) $\exists n{:}W_e \cap A \quad \vee \quad u(e){:}N$;

(ii) $u(e){:}W_e \cup A \to \exists n{:}W_e \cap A$.

Remark 2.9. We have that N is regarded as a Σ-subobject of $N_\perp$ via the pullback

$$\begin{array}{ccc} N & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ N_\perp & \longrightarrow & \Sigma \end{array}$$

so that $u(x){:}N$ in the above definition means $u(x)\downarrow$.

The following proposition establishes that our characterization coincides with the standard ones in the classical world. The proof is a matter of fiddling with the logic and the forward direction uses the intuitionistically invalid De Morgan's law $\neg(p \wedge q) \to \neg p \vee \neg q$. As a result, our definition is constructively stronger.

Proposition 2.10. *A set C is creative if and only if there exists a unary partial function u such that for all x,*

(i) $\exists n \in W_x \cap A \quad \vee \quad u(x)\downarrow$;

(ii) $u(x) \in W_x \cup A \implies \exists n \in W_x \cap A$.

Definition 2.11. In $\mathcal{E}ff$, the set $A{:}\Sigma^N$ is complete if and only if $\forall B{:}\Sigma^N \exists f{:}N^N B = f^{-1}(A)$.

Note that $f^{-1}(A){:}\Sigma^N$ whenever $A{:}\Sigma^N$ with characteristic morphism $A \circ f$. We are now in a position of establishing that $K : \Sigma^N$ is creative according to our definition. This forms part of the key argument in our theorem stated below.

Proposition 2.12. *In $\mathcal{E}ff$, the set K is creative.*

Based on the full force of the discussions above, we can conclude this section with our main result:

Theorem 2.13. *(Synthetic Myhill's theorem) In $\mathcal{E}ff$, a set A is creative if and only if A is complete.*

3 Conclusion and future work

The *s-m-n* theorem while being an important result in the classical world, has turned out to be a simple application of the axiom of countable choice in the effective topos. This structure was previously not present in the classical informal or formal proof. We showed that K being creative is a straightforward fact, despite the fact that our definition of creativeness is constructively stronger. Indeed, we have demonstrated that non-trivial facts about computability theory find their home in the effective topos. This synthetic version of Myhill's theorem is only one example. In general, the synthetic results made no explicit reference to Gödel encoding or Turing machines, and the proofs were couched in purely set theoretic terms. As Andrej Bauer puts it "we just [did] ordinary math–in an extraordinary universe" [1].

There are various directions we can explore from here. Originally, the project started with looking at a paper by Moschovakis [9], where Myhill's theorem appeared among the applications of the Second recursion theorem. Another interesting result there concering partial recursive functionals is the Kreisel-Lacombe-Shoenfield-Ceiten theorem. As we did not use the full force of the effective topos, such a development would show how higher-order computability results appear here.

Acknowledgements

I want to express my sincerest gratitude towards J.M.E. Hyland for introducing me to this subject and his invaluable guidance. This research work was also supported by the Philippa Fawcette Internship Programme and the Erasmus+ Traineeship Programme.

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Exploring Aristotelian Syllogistic in First-Order Logic: An Overview of the History and Reality of Ontological Commitments

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The purpose of the paper is to answer the question of what additional existential premises are needed in order to render Aristotelian syllogisms provable in First-Order Logic and to give an overview of how the issue of ontological commitments in syllogistic was handled throughout the history. In contemporary discussions concerning the history of logic, there is a widespread assumption that the Aristotelian syllogistic, as it is the case with the modern formal logic, did not allow for the use of empty terms. The issue is however much more complex, and ontological commitments of syllogistic were discussed extensively throughout the history. In the paper, we relate these discussions, covering four main areas: the logic of Aristotle, Arabic logic, medieval European logic, and later discussions up till the emergence of modern formal logic, as well as provide our own view of the issue.

The question of empty terms and existential import in syllogistic became first apparent when Jan Łukasiewicz [12] claimed that Aristotle supposes all terms used in syllogisms to be non-empty. This view seems to be an orthodox way of interpreting up till now and was held by many scholars [14, p. 7], [22, p. 144], [24, p. 343-4]. Kneales in their *Development of Logic* state that "In order to justify Aristotle's doctrine as a whole it is necessary, then, to presuppose that he assumed application of *all* [original emphasis] the general terms with which he dealt." [11, p. 60]. The case is also true for Patzig, who writes that "The expression 'one must examine the set of subject (predicate, contrary) terms of S(P)' clearly presupposes that in each case these sets have at least one member." [19, p. 6].

However, this viewpoint was never held by Aristotle himself, nor it was explicitly addressed in any of his works, let alone those concerned with syllogistic. Malink [13, p. 82] says that "the question of whether or not an individual falls under a term seems to be irrelevant in Prior Analytics 1.1-22.". In fact, a view that allows for the emptiness of terms when interpreting Aristotle is getting more and more advocates, and the discussion is ongoing [18]. The nonemptiness assumption, in turn, is said to be forced by attempts to render Aristotle using the modern notation [16, p. 74]. Scholars that opt for this view mostly refer to fragments from *Prior* and *Posterior Analytics* where Aristotle is speaking about a "goat-stag" as a syllogistc term [3, p. 243], [13, 81]. For example: "(...) you may know what the account or the name signifies when I say goat-stag, but it is impossible to know what a goat-stag is (...)" (*Posterior Analytics* 92b6-8). As a goat-stag is a nonexistent, from its presence it is then argued that Aristotle must have been aware of such a possibility and thus his theory have to account for empty terms as well. In this context, Wedin [25, p. 179] is also quoting *Categories* 13b12-36, where Aristotle states that both "Socrates is sick" and "Socrates is healthy" are false in case Socrates does not exist, but "Socrates is sick" and "Socrates is not sick" become opposites in that case. From this it is argued that Aristotle is claiming the existence of a subject as a truth condition and thus must be aware that additional existential premises are required for some statements to be true.

The comments above cannot be easily generalized to syllogistic, but they point out the fact that Aristotle was in fact aware of the possibility of empty terms being taken into consideration. The question of how Aristotle originally intended his syllogistic to treat empty terms remains open. Here, it will suffice to say that he does not make any explicit statements about it and (non)emptiness does not yet emerge as an issue. Nevertheless, when it comes to modern discussions, we can observe a tendency leading from the one-sidedness of first interpretations to a more nuanced view.

Whatever might be said about Aristotle, empty terms were widely discussed both in Arabic and in medieval European logic. The first one acknowledged to explicitly talk about the existential import is Al-Farabi [8, p. 39], although in his Syllogism, he does not talk about empty terms at all, and the discussion is confined to categorical statements. Avicenna continues to explicitly talk about existential assumptions [17, p. 142]. He also does some explicit remarks on syllogistic and require that negative propositions in syllogisms have an existential import as well [3, p. 293]. Moreover, the existence he talks about is not restricted to real existence as in Al-Farabi, but apart from existence *in re*, existence *in intellectu* is also considered [7, p. 90], and this line of thought continues also in works of Averroes [4, p. 361].

In Europe, historically speaking, the question of existential import was not addressed explicitly until the rise of nominalism, with William of Ockham being the first one to pronounce it [6, p. 420]. In general, the discussion of empty terms was virtually nonexistent before the nineteenth century [18]. Early Scholastics, such as Peter of Spain, have never considered it neither with respect to categorical statements nor to syllogistic [6, p. 417].

Ockham maintains that the truth conditions of both affirmation and denial are disjunctive, with an existence of a subject and a predication for the first, and a lack of those for the latter [1, p. 392-3]. He does not consider existence *in intellectu* and requires every subject of an affirmative statement to exist *in re* [3, p. 302]. The thing worth mentioning about him is that his treatment of existential assumptions tends to be conditional at times – when talking about the *dictum de omni et nullo*, he states that for an example affirmative sentence to be true, its subject need not always exist, but it suffices only that the sentence is true whenever it does exist [23, p. 42]. Thus, he can be viewed as a precursor of the modern notion of the universe of interpretation, with objects existing externally being the only possible interpretation. Buridan alike requires the subject of every true affirmative sentence to exist *in re* and makes this claim more explicit [9, p. 26]. He is, however, sceptical about Ockham's conditional approach, and the existence of a subject is to be read verbatim [23, p. 42]. From both points of view, talking about nonactual beings existing somehow is forbidden and being regarded as a nonsense [10, p. 159].

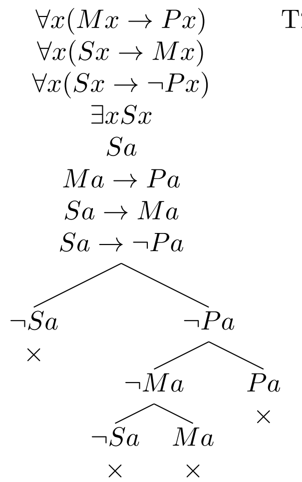
Thus, even if we agree on the tacit existential assumption in Aristotle, this view was certainly not held by the medieval logicians above, by which empty terms were discussed fervently [10, p. 143]. The Ars Logica by John of St. Thomas can be regarded as the culmination of this trend. John is holding the doctrine of existential import developed by nominalists, and his theory is greatly resembling the modern notion of the universe of interpretation, with things *in intellectu* considered as well as those existing in the past or in the future [6, p. 420]. And while he was talking only about categorical sentences, Leibniz was the first one to allow for terms to be systematically interpreted as things *in intellectu* in his syllogistic [15, p. 292].

Nevertheless, serious and detailed logical investigations of existential import were altogether abandoned in the third decade of the sixteenth century, and the discussions ceased [2, p. 147]. Some authors point out that the invention of Venn diagrams in 1881 helped to make the issue more explicit again [26, p. 416], which would be at least intuitively true with respect to syllogistic as well. Certainly, the development of Boolean algebra sparked a renewed interest, with such authorities as Peirce and Russell speaking up [26, p. 416], although they comment only on existential import in general, without making reference to syllogistic. Boole himself refrains from making any direct comparison between his system and the one of Aristotle [5, p. 226]. The first one acknowledged to state that Aristotle's syllogistic requires its terms to be non-empty was Śleszyński [21], and the widespread popularity of this view stems from the works of Łukasiewicz [20, p. 1-2].

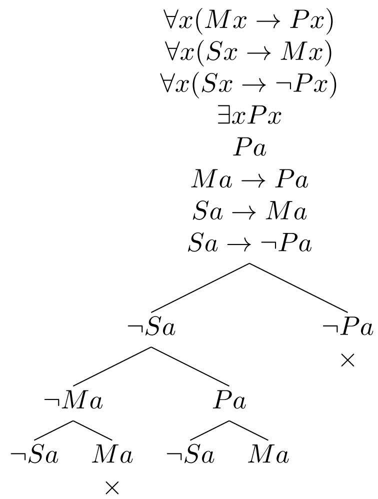
Thus, the historical development of the issue of empty terms is twofold. First, we can observe a rising awareness of the empty terms as an issue that needs to be covered – irrelevant in Aristotle, present in the Middle Ages, and substantial in the modern interpretations of Aristotle's work. The difference is that up till Łukasiewicz it was discussed either with respect to the validity of the Logical Square, as in the medieval and early modern period, or with respect to categorical statements in general, as when the Boolean algebra emerged, and only with the works of Łukasiewicz the discussion turned to syllogistic as such. Secondly, in parallel with the above, the development of the notion of the universe of interpretation can be traced, beginning with the works of Ockham and getting more and more pronounced, with Leibniz being the first one to allow for the *in intellectu* interpretation, and Boolean algebra stating the idea explicitly.

Building upon this rich historical background we may now turn to our main research question, which is this: If we consider a proof of a given syllogism, are there any additional existential premises required, besides the premises of the syllogisms itself, in order to prove it in First-Order Logic?

T1:



T2:



There are different answers to this question possible, which allow dividing valid Aristotelian syllogisms into three groups. To exemplify their members, we use syllogisms of the first figure:

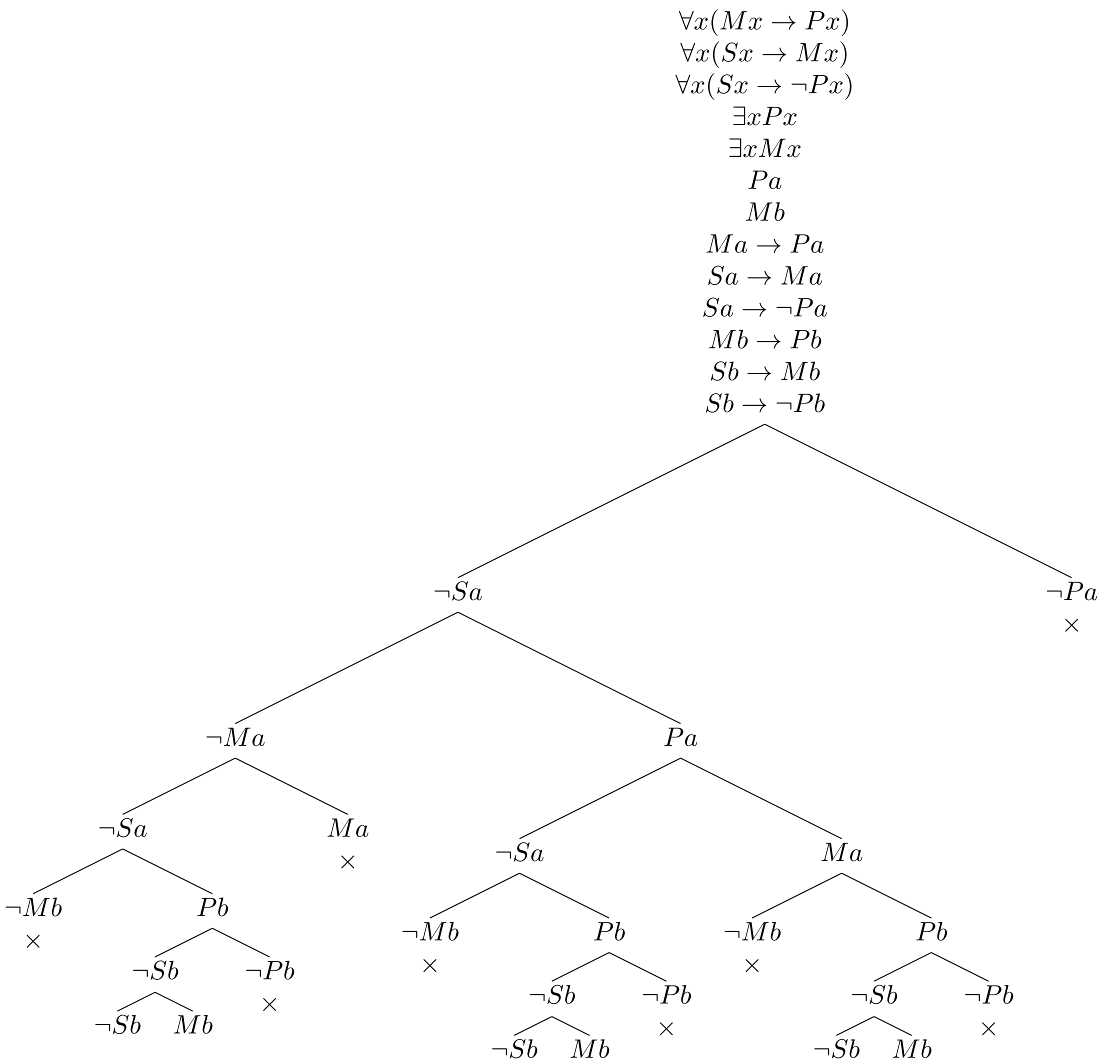
- G1 No additional premises are needed, as no special ontological commitments are required; this is the case for syllogisms in which both premises and the conclusion are general sentences (*Barbara*, *Celarent*).
- G2 No additional premises are needed, as required ontological commitments are addressed by a particular premise (*Darii*, *Ferio*).
- G3 Additional existential premise is needed, as required ontological commitments are not warranted by the premises of the syllogisms (*Barbari*, *Celaront*).

Furthermore, for the syllogisms in the third group, only one extra existential premise is necessary to prove them in First-Order Logic. Specifically, to demonstrate these syllogisms, we only need to assume the non-emptiness of one of the three terms that make up the syllogism.

Let us consider a member of the G3 group, the syllogism *Barbari* (for simplicity, we use analytic tableaux as the proof method). T1 above is an analytic tableau for *Barbari*, employing one additional existential premise, $\exists xSx$. It is easily seen that without it, the tableau will not close, while with it the tableau does close, thus forming a proof of the syllogism in question. If we add to the *Barbari*'s original premises any other existential premise, the tableau will not close; T2 is an example involving the premise $\exists xPx$.

Moreover, only addition of the $\exists xSx$ premise allows to prove *Barbari* in FOL. Consider T3, the unsuccesfull attempt, with two existential premises added ($\exists xPx$, $\exists xMx$):

T3:



The same holds for all the other syllogisms in the G3 group. However, it is not always the minor terms that needs to be non-empty in order to prove a G3 syllogism: this depends on the figure.

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Vagueness across the Type Hierarchy

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Abstract

How to characterise vagueness for entities of all types? The paper critically examines one influential proposal to this effect and then offers an alternative, according to which an entity is vague iff it (seemingly) possibly lacks a sharp boundary on some soritical series for it.

1 Introduction: The Problem of Characterising/Defining Vagueness across the Type Hierarchy

When is an entity *vague* (or, on the contrary, *precise*)? Perhaps a natural answer would be to say that a property is vague iff *it (possibly) presents borderline cases* (and precise otherwise), but such an answer is problematic on at least two counts. Firstly, it is not clear how to generalise the answer to *other types of entities* such as *e.g.* objects. Secondly, the answer *overgenerates* as it also makes vague *e.g.* the paradigmatically precise property x-is-a-geometrically-perfect-cube (for there might be concrete cubes that are borderline geometrically perfect cubes).

This paper critically examines one influential proposal for characterising vagueness across the type hierarchy and then offers an alternative. While most of the discussion will centre on the task of simply providing a *nontrivial necessary and sufficient condition* for an entity to be vague (using '*characterisation*' as a shorthand for such a condition), some remarks will also be made concerning the more ambitious task of providing an analysis of *what it is for an entity to be vague* (using '*definition*' as a shorthand for such an analysis).

2 The Rolf-Style Characterisation and Its Problems

According to an influential proposal going back at least as far as Rolf [1980] (and recently defended *e.g.* by Bacon [2018]), we should take the notion of vagueness as *primitive* for some types (say, objects and propositions) and characterise vagueness for other types by saying that an entity is vague iff *it takes at least one precise input and yields a vague output.* For example, assuming that 1 is precise and that the proposition $\langle 1 \text{ is small}\rangle$ is vague, this *Rolf-style characterisation* correctly implies that the property x-is-small is vague.

I shall argue that the Rolf-style characterisation embodies an objectionably "*purist*" conception of vagueness. For example, consider a property ("*schbaldness*") taking any precise object x to yield, say, $\langle x \text{ is a number}\rangle$ (plausibly assuming that the property x-is-a-number is precise) and any vague object x to yield $\langle x \text{ is bald}\rangle$. *Schbaldness would seem vague*, for, say, it takes a man, Harry, whose vagueness (we may so suppose) only resides in the vagueness of where its right toe ends and who has 50,000 hairs, to yield the vague ⟨Harry is bald⟩. If taking Harry to yield ⟨Harry is bald⟩ is sufficient for baldness to be vague (and it is!), how could it not be sufficient for schbaldness to be vague? Where else could the vagueness of ⟨Harry is schbald⟩ come from, if not from the vagueness in schbaldness (the only other entity at play is Harry, but ⟨Harry is schbald⟩ is vague for the same reason as ⟨Harry is bald⟩ is, and Harry's vagueness resides in a feature that is totally irrelevant for the vagueness of the "latter" proposition)? However, *schbaldness is precise on the Rolf-style characterisation*, for it takes any precise object x to yield the precise ⟨x is a number⟩.

This train of thought leads to the issue that, *on the Rolf-style characterisation, it is not even clear that baldness is vague*, since *objects capable of having hair on their scalp and for which therefore the question of baldness could arise are typically—and, one may well suspect, invariably—vague* (and those of them that are vague are anyway those that *paradigmatically* support the idea that baldness is vague). Typical precise objects (such as numbers, graphs, points in space *etc.*) are not objects capable of having hair on their scalp and for which therefore the question of baldness could arise, and, even granting the possibility of precise objects that are capable of having hair on their scalp and for which therefore the question of baldness could arise, such extravagant objects are certainly not necessary for supporting the idea that baldness is vague. Nor, for analogous reasons, is it clear that a paradigmatically vague object like *e.g.* Mt Athos is vague, since *properties nontrivially applying to a mountain are typically—and, one may well suspect, invariably—vague* (and those of them that are vague are anyway those that *paradigmatically* support the idea that Mt Athos is vague). For example, properties of the kind x-is-at-most-im-high paradigmatically support the idea that Mt Athos is vague, but, *pace e.g.* Bacon [2018], these are arguably vague, as manifested by the following kind of series: start with a im-high mountain with a thin protuberance rising up to $(i + 1)$m, and then gradually enlarge the protuberance, eventually ending up with a $(i + 1)$m-high mountain.

3 The Lack-of-Sharp-Boundary Characterisation and Its Developments

Turning now to my favoured alternative, let a *soritical series* for an entity be a series along a dimension relevant for the entity's *presence* (*i.e.*, depending on the entity's type, its *existence* (in the case of *objects*) or *truth* (in the case of *propositions*) or *occurrence* (in the case of *properties*) *etc.*), where at the start the entity is clearly *present* while at the end it is clearly *not present*, and where each successive case in the series represents *only a tiny worsening* of the conditions for the entity's presence. Further, let an entity *lack a sharp boundary* on a soritical series for it iff, *for no pair of adjacent cases in the series, the entity is present in one and not present in the other.* Then, the *same* characterisation of vagueness that many have thought to apply for properties can be defended to apply for all other types as well: just as a property is vague iff it *(seemingly) possibly lacks a sharp boundary on some soritical series for it*, so is any entity of any other type. I'd propose the version with 'seemingly'—understood *epistemically*, rather than *psychologically*, in terms of *prima facie justification—as a characterisation*, whereas, within the nontransitive system to be mentioned in the remainder of this section, I'd propose the version without 'seemingly' *as a definition.* It's true that, in the case of *e.g.* objects, for different cases, the (seeming) possible lack of a sharp boundary is realised on different dimensions (spatial, temporal, mereological *etc.*) and, for each particular case, good judgement is needed to set up a compelling soritical series for it manifesting such lack, but so it is also in the case of properties (because of their pervasive multidimensionality). And it's in fact extremely plausible that, on this understanding of soritical series, while there is no possible soritical series for x-is-a-geometrically-perfect-cube where that property (seemingly) lacks a sharp boundary, there are possible soritical series for x-is-schbald where that property (seemingly) lacks a sharp boundary.

Let's see how a *nontransitive* logic can be so developed as to satisfy the definition of vagueness as lack of a sharp boundary. The most promising family of nontransitive logics I know of that does this is the family of *tolerant logics* I've first introduced in Zardini [2008a]; [2008b]. The logics are defined *semantically*, in particular *lattice-theoretically*. Say that a $\mathcal{T}$*-structure* $\mathfrak{S}$ is a 6ple $\langle U_{\mathfrak{S}}, V_{\mathfrak{S}}, \preceq_{\mathfrak{S}}, D_{\mathfrak{S}}, \mathsf{tol}_{\mathfrak{S}}, O_{\mathfrak{S}}\rangle$, where:

- $U_{\mathfrak{S}}$ is a nonempty set of objects (the *universe of discourse*);
- $V_{\mathfrak{S}}$ is a nonempty set of objects (the *values*);
- $\preceq_{\mathfrak{S}}$ is a partial ordering on $V_{\mathfrak{S}}$ such that, for every $X \subseteq V_{\mathfrak{S}}$, the greatest lower bound glb of X and the least upper bound lub of X exist ($\preceq_{\mathfrak{S}}$ corresponds to a *complete* lattice);
- $D_{\mathfrak{S}}$ is a nonempty subset of $V_{\mathfrak{S}}$ (the *designated* values);
- $\mathsf{tol}_{\mathfrak{S}}$ is a function from $V_{\mathfrak{S}}$ into the powerset pow of $V_{\mathfrak{S}}$ (the *tolerance* function);
- $O_{\mathfrak{S}}$ is a nonempty set of operations on $V_{\mathfrak{S}}$ with, in particular, $\{\mathsf{neg}_{\mathfrak{S}}, \mathsf{imp}_{\mathfrak{S}}\} \subseteq O_{\mathfrak{S}}$.

Without going into details, we assume a *standard first-order language* so that $\mathcal{T}$-structures can be used to evaluate its sentences *via* a *model*- and *assignment*-relative *valuation* function val (where conjunction and universal generalisation are interpreted as glb, disjunction and particular generalisation are interpreted as lub, negation as neg and implication as imp).

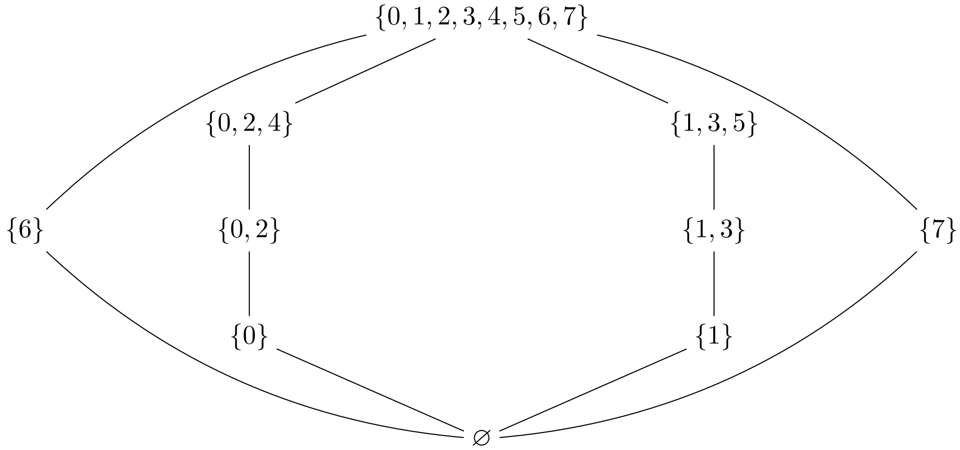
Now, given the richness of $\mathcal{T}$-structures, and in particular given tol, we can use D to generate another set T of interesting values (the *tolerated* values), by setting, for every $\mathcal{T}$-structure $\mathfrak{S}$, $T_{\mathfrak{S}} = \bigcup_{d \in D_{\mathfrak{S}}} \mathsf{tol}_{\mathfrak{S}}(d)$. Following in particular Zardini [2008a]; [2008b]; [2015]; [2019], we can interpret designated values to be those values that, when possessed by a sentence, model the fact that *that sentence can safely be used as a premise in further reasoning*, while we can interpret tolerated values to be those values that, when possessed by a sentence, model the fact that, *although that sentence can safely be accepted (possibly as a conclusion of previous reasoning), it might not be the case that it can safely be used as a premise in further reasoning.* In a slogan, while designated values are "*very good*" values, tolerated values are "*good enough*" values. With designated and tolerated values in place, and given the interpretation just sketched of what they amount to, it is very natural to extract from $\mathcal{T}$-structures of kind $\mathbf{X}$ the corresponding, typically nontransitive, consequence relation:

(TC$^{\mathbf{X}}$) Δ is an $\mathbf{X}$-consequence of Γ ($\Gamma \vdash_{\mathbf{X}} \Delta$) iff, for every $\mathcal{T}$-structure $\mathfrak{S}$ of kind $\mathbf{X}$, for every model $\mathfrak{M}$ and assignment ass on $\mathfrak{S}$, if, for every $\varphi \in \Gamma$, $\mathsf{val}_{\mathfrak{M},\mathsf{ass}}(\varphi) \in D_{\mathfrak{S}}$, then, for some $\psi \in \Delta$, $\mathsf{val}_{\mathfrak{M},\mathsf{ass}}(\psi) \in T_{\mathfrak{S}}$.

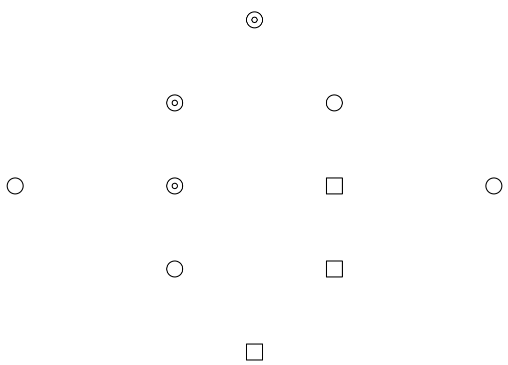
Obviously, given the extreme liberality of $\mathcal{T}$-structures, we need to restrict to fairly specific kinds in order for (TC$^{\mathbf{X}}$) to deliver interesting enough logics. Here is a particularly nice restriction. Let a $\mathcal{T}$-structure $\mathfrak{S}$ be of kind $\mathbf{C}$ iff:

- $V_{\mathfrak{G}}$ is representable as: $\{X : X \in \mathsf{pow}(\{i : i \leq 7\})$ and, if $X \neq \{i : i \leq 7\}$, either, [[for every $i \in X$, i is even] and, [for every i and j, if $i \in X$ and ≤ 4, and j is even and $< i$, $j \in X$] and, [for every i and $j \in X$, $|i - j| < 6$]] or, [[for every $i \in X$, i is odd] and, [for every i and j, if $i \in X$ and ≤ 5, and j is odd and $< i$, $j \in X$] and, [for every i and $j \in X$, $|i - j| < 6$]]$\}$;

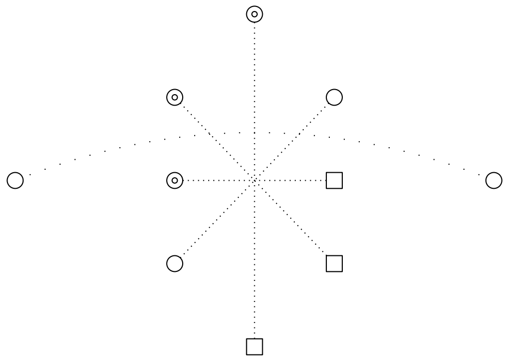
- $\preceq_{\mathfrak{G}}$ is representable as: $\{\langle X, Y\rangle : X \subseteq Y\}$. Thus, $V_{\mathfrak{G}}$ and $\preceq_{\mathfrak{G}}$ jointly constitute the lattice depicted by the following Hasse diagram:



- $D_{\mathfrak{G}}$ and $\mathsf{tol}_{\mathfrak{G}}$ determine that, indicating designated values with doubly circular nodes, tolerated but not designated values with simply circular nodes and not tolerated values with square nodes, such values can be depicted as:



- $\mathsf{neg}_{\mathfrak{G}}$ is such that, indicating it with pointed edges, it can be depicted as:



- $\mathsf{imp}_{\mathfrak{S}}$ is such that, for every $v, w \in V_{\mathfrak{S}}$, $\mathsf{imp}_{\mathfrak{S}}(v, w) = \mathsf{neg}_{\mathfrak{S}}(\mathsf{glb}(v, \mathsf{neg}_{\mathfrak{S}}(w)))$.

It's easy to check that transitivity of logical consequence does not hold in the tolerant logic **C** resulting from ($\text{TC}^{\mathbf{C}}$) (for example, $\varphi, \varphi \to \psi \vdash_{\mathbf{C}} \psi$—and so $\psi, \psi \to \chi \vdash_{\mathbf{C}} \chi$—holds, but $\varphi, \varphi \to \psi, \psi \to \chi \vdash_{\mathbf{C}} \chi$ does not) and that, indeed, lack of a sharp boundary of *e.g.* a property is consistent in **C** with the property's having both positive and negative cases on the same soritical series (for example, letting Bi be short for 'A man with i hairs is bald', $B0, \neg B100,000, \neg\exists i(Bi \,\&\, \neg Bi + 1) \vdash_{\mathbf{C}} \varnothing$ does not hold: for instance, consider a **C**-model $\mathfrak{M}$ such that, for every i $[i : 1 \leq i \leq 35,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{0, 1, 2, 3, 4, 5, 6, 7\}$; for every i $[i : 35,001 \leq i \leq 45,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{0, 2, 4\}$; for every i $[i : 45,001 \leq i \leq 55,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{6\}$; for every i $[i : 55,001 \leq i \leq 65,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{1\}$; for every i $[i : 65,001 \leq i \leq 100,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \varnothing$). The construction can naturally be generalised to model the lack of a sharp boundary for other entities such as objects, propositions, connectives *etc.*

4 Conclusion: A Single Nonprimitive Notion of Vagueness Irreducibly Realised across the Type Hierarchy

In conclusion, on this view, there is one single nonprimitive notion of vagueness—(seeming) possible lack of a sharp boundary—that gets realised in different irreducible ways among and within different types, as opposed to the Rolf-style characterisation, on which there are primitive separate notions of vagueness for certain types to which vagueness of all other types is reduced (plus, as indicated, the proposed characterisation can be turned into a much more satisfying definition than the Rolf-style one).

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