Vagueness across the Type Hierarchy

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Abstract

How to characterise vagueness for entities of all types? The paper critically examines one influential proposal to this effect and then offers an alternative, according to which an entity is vague iff it (seemingly) possibly lacks a sharp boundary on some soritical series for it.

1 Introduction: The Problem of Characterising/Defining Vagueness across the Type Hierarchy

When is an entity *vague* (or, on the contrary, *precise*)? Perhaps a natural answer would be to say that a property is vague iff *it (possibly) presents borderline cases* (and precise otherwise), but such an answer is problematic on at least two counts. Firstly, it is not clear how to generalise the answer to *other types of entities* such as *e.g.* objects. Secondly, the answer *overgenerates* as it also makes vague *e.g.* the paradigmatically precise property x-is-a-geometrically-perfect-cube (for there might be concrete cubes that are borderline geometrically perfect cubes).

This paper critically examines one influential proposal for characterising vagueness across the type hierarchy and then offers an alternative. While most of the discussion will centre on the task of simply providing a *nontrivial necessary and sufficient condition* for an entity to be vague (using '*characterisation*' as a shorthand for such a condition), some remarks will also be made concerning the more ambitious task of providing an analysis of *what it is for an entity to be vague* (using '*definition*' as a shorthand for such an analysis).

2 The Rolf-Style Characterisation and Its Problems

According to an influential proposal going back at least as far as Rolf [1980] (and recently defended *e.g.* by Bacon [2018]), we should take the notion of vagueness as *primitive* for some types (say, objects and propositions) and characterise vagueness for other types by saying that an entity is vague iff *it takes at least one precise input and yields a vague output*. For example, assuming that 1 is precise and that the proposition $\langle 1$ is small$\rangle$ is vague, this *Rolf-style characterisation* correctly implies that the property x-is-small is vague.

I shall argue that the Rolf-style characterisation embodies an objectionably "*purist*" conception of vagueness. For example, consider a property ("*schbaldness*") taking any precise object x to yield, say, $\langle x$ is a number$\rangle$ (plausibly assuming that the property x-is-a-number is precise) and any vague object x to yield $\langle x$ is bald$\rangle$. *Schbaldness would seem vague*, for, say, it takes a man, Harry, whose vagueness (we may so suppose) only resides in the vagueness of where its right toe ends and who has $50,000$ hairs, to yield the vague ⟨Harry is bald⟩. If taking Harry to yield ⟨Harry is bald⟩ is sufficient for baldness to be vague (and it is!), how could it not be sufficient for schbaldness to be vague? Where else could the vagueness of ⟨Harry is schbald⟩ come from, if not from the vagueness in schbaldness (the only other entity at play is Harry, but ⟨Harry is schbald⟩ is vague for the same reason as ⟨Harry is bald⟩ is, and Harry's vagueness resides in a feature that is totally irrelevant for the vagueness of the "latter" proposition)? However, *schbaldness is precise on the Rolf-style characterisation*, for it takes any precise object x to yield the precise ⟨x is a number⟩.

This train of thought leads to the issue that, *on the Rolf-style characterisation, it is not even clear that baldness is vague*, since *objects capable of having hair on their scalp and for which therefore the question of baldness could arise are typically—and, one may well suspect, invariably—vague* (and those of them that are vague are anyway those that *paradigmatically* support the idea that baldness is vague). Typical precise objects (such as numbers, graphs, points in space *etc.*) are not objects capable of having hair on their scalp and for which therefore the question of baldness could arise, and, even granting the possibility of precise objects that are capable of having hair on their scalp and for which therefore the question of baldness could arise, such extravagant objects are certainly not necessary for supporting the idea that baldness is vague. Nor, for analogous reasons, is it clear that a paradigmatically vague object like *e.g.* Mt Athos is vague, since *properties nontrivially applying to a mountain are typically—and, one may well suspect, invariably—vague* (and those of them that are vague are anyway those that *paradigmatically* support the idea that Mt Athos is vague). For example, properties of the kind x-is-at-most-im-high paradigmatically support the idea that Mt Athos is vague, but, *pace e.g.* Bacon [2018], these are arguably vague, as manifested by the following kind of series: start with a im-high mountain with a thin protuberance rising up to $(i+1)$m, and then gradually enlarge the protuberance, eventually ending up with a $(i+1)$m-high mountain.

3 The Lack-of-Sharp-Boundary Characterisation and Its Developments

Turning now to my favoured alternative, let a *soritical series* for an entity be a series along a dimension relevant for the entity's *presence* (*i.e.*, depending on the entity's type, its *existence* (in the case of *objects*) or *truth* (in the case of *propositions*) or *occurrence* (in the case of *properties*) *etc.*), where at the start the entity is clearly *present* while at the end it is clearly *not present*, and where each successive case in the series represents *only a tiny worsening* of the conditions for the entity's presence. Further, let an entity *lack a sharp boundary* on a soritical series for it iff, *for no pair of adjacent cases in the series, the entity is present in one and not present in the other*. Then, the *same* characterisation of vagueness that many have thought to apply for properties can be defended to apply for all other types as well: just as a property is vague iff it *(seemingly) possibly lacks a sharp boundary on some soritical series for it*, so is any entity of any other type. I'd propose the version with 'seemingly'—understood *epistemically*, rather than *psychologically*, in terms of *prima facie justification—as a characterisation*, whereas, within the nontransitive system to be mentioned in the remainder of this section, I'd propose the version without 'seemingly' *as a definition*. It's true that, in the case of *e.g.* objects, for different cases, the (seeming) possible lack of a sharp boundary is realised on different dimensions (spatial, temporal, mereological *etc.*) and, for each particular case, good judgement is needed to set up a compelling soritical series for it manifesting such lack, but so it is also in the case of properties (because of their pervasive multidimensionality). And it's in fact extremely plausible that, on this understanding of soritical series, while there is no possible soritical series for x-is-a-geometrically-perfect-cube where that property (seemingly) lacks a sharp boundary, there are possible soritical series for x-is-schbald where that property (seemingly) lacks a sharp boundary.

Let's see how a *nontransitive* logic can be so developed as to satisfy the definition of vagueness as lack of a sharp boundary. The most promising family of nontransitive logics I know of that does this is the family of *tolerant logics* I've first introduced in Zardini [2008a]; [2008b]. The logics are defined *semantically*, in particular *lattice-theoretically*. Say that a *$\mathcal{T}$-structure* $\mathfrak{S}$ is a 6ple $\langle U_{\mathfrak{S}}, V_{\mathfrak{S}}, \preceq_{\mathfrak{S}}, D_{\mathfrak{S}}, \mathsf{tol}_{\mathfrak{S}}, O_{\mathfrak{S}}\rangle$, where:

- $U_{\mathfrak{S}}$ is a nonempty set of objects (the *universe of discourse*);
- $V_{\mathfrak{S}}$ is a nonempty set of objects (the *values*);
- $\preceq_{\mathfrak{S}}$ is a partial ordering on $V_{\mathfrak{S}}$ such that, for every $X \subseteq V_{\mathfrak{S}}$, the greatest lower bound glb of X and the least upper bound lub of X exist ($\preceq_{\mathfrak{S}}$ corresponds to a *complete* lattice);
- $D_{\mathfrak{S}}$ is a nonempty subset of $V_{\mathfrak{S}}$ (the *designated* values);
- $\mathsf{tol}_{\mathfrak{S}}$ is a function from $V_{\mathfrak{S}}$ into the powerset pow of $V_{\mathfrak{S}}$ (the *tolerance* function);
- $O_{\mathfrak{S}}$ is a nonempty set of operations on $V_{\mathfrak{S}}$ with, in particular, $\{\mathsf{neg}_{\mathfrak{S}}, \mathsf{imp}_{\mathfrak{S}}\} \subseteq O_{\mathfrak{S}}$.

Without going into details, we assume a *standard first-order language* so that $\mathcal{T}$-structures can be used to evaluate its sentences *via* a *model*- and *assignment*-relative *valuation* function val (where conjunction and universal generalisation are interpreted as glb, disjunction and particular generalisation are interpreted as lub, negation as neg and implication as imp).

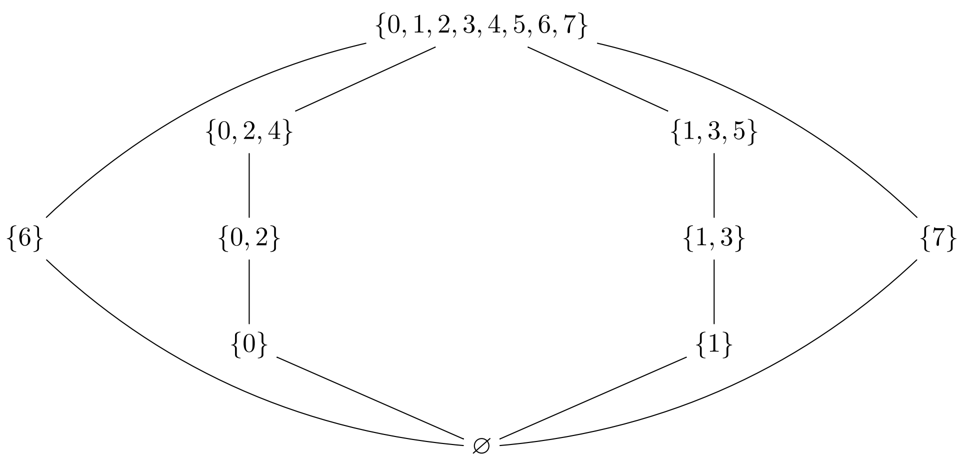
Now, given the richness of $\mathcal{T}$-structures, and in particular given tol, we can use D to generate another set T of interesting values (the *tolerated* values), by setting, for every $\mathcal{T}$-structure $\mathfrak{S}$, $T_{\mathfrak{S}} = \bigcup_{d \in D_{\mathfrak{S}}} \mathsf{tol}_{\mathfrak{S}}(d)$. Following in particular Zardini [2008a]; [2008b]; [2015]; [2019], we can interpret designated values to be those values that, when possessed by a sentence, model the fact that *that sentence can safely be used as a premise in further reasoning*, while we can interpret tolerated values to be those values that, when possessed by a sentence, model the fact that, *although that sentence can safely be accepted (possibly as a conclusion of previous reasoning), it might not be the case that it can safely be used as a premise in further reasoning*. In a slogan, while designated values are "*very good*" values, tolerated values are "*good enough*" values. With designated and tolerated values in place, and given the interpretation just sketched of what they amount to, it is very natural to extract from $\mathcal{T}$-structures of kind $\mathbf{X}$ the corresponding, typically nontransitive, consequence relation:

($\mathrm{TC}^{\mathbf{X}}$) Δ is an $\mathbf{X}$-consequence of Γ ($\Gamma \vdash_{\mathbf{X}} \Delta$) iff, for every $\mathcal{T}$-structure $\mathfrak{S}$ of kind $\mathbf{X}$, for every model $\mathfrak{M}$ and assignment ass on $\mathfrak{S}$, if, for every $\varphi \in \Gamma$, $\mathsf{val}_{\mathfrak{M},\mathsf{ass}}(\varphi) \in D_{\mathfrak{S}}$, then, for some $\psi \in \Delta$, $\mathsf{val}_{\mathfrak{M},\mathsf{ass}}(\psi) \in T_{\mathfrak{S}}$.

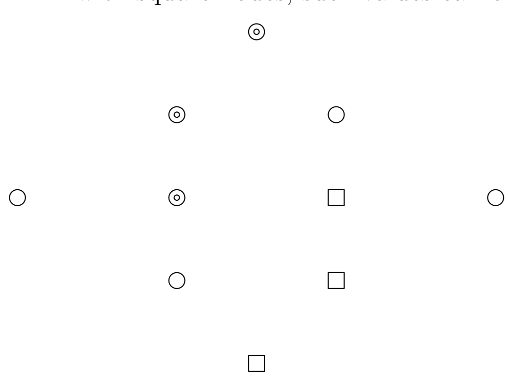
Obviously, given the extreme liberality of $\mathcal{T}$-structures, we need to restrict to fairly specific kinds in order for ($\mathrm{TC}^{\mathbf{X}}$) to deliver interesting enough logics. Here is a particularly nice restriction. Let a $\mathcal{T}$-structure $\mathfrak{S}$ be of kind $\mathbf{C}$ iff:

- $V_{\mathfrak{S}}$ is representable as: $\{X : X \in \mathsf{pow}(\{i : i \leq 7\})$ and, if $X \neq \{i : i \leq 7\}$, either, [[for every $i \in X$, i is even] and, [for every i and j, if $i \in X$ and ≤ 4, and j is even and $< i$, $j \in X$] and, [for every i and $j \in X$, $|i - j| < 6$]] or, [[for every $i \in X$, i is odd] and, [for every i and j, if $i \in X$ and ≤ 5, and j is odd and $< i$, $j \in X$] and, [for every i and $j \in X$, $|i - j| < 6$]]$\}$;

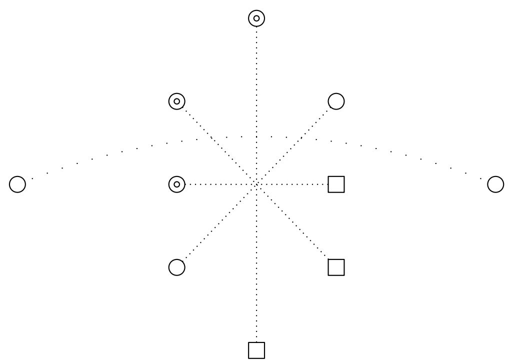
- $\preceq_{\mathfrak{S}}$ is representable as: $\{\langle X, Y\rangle : X \subseteq Y\}$. Thus, $V_{\mathfrak{S}}$ and $\preceq_{\mathfrak{S}}$ jointly constitute the lattice depicted by the following Hasse diagram:



- $D_{\mathfrak{S}}$ and $\mathsf{tol}_{\mathfrak{S}}$ determine that, indicating designated values with doubly circular nodes, tolerated but not designated values with simply circular nodes and not tolerated values with square nodes, such values can be depicted as:



- $\mathsf{neg}_{\mathfrak{S}}$ is such that, indicating it with pointed edges, it can be depicted as:



- $\mathsf{imp}_{\mathfrak{S}}$ is such that, for every $v, w \in V_{\mathfrak{S}}$, $\mathsf{imp}_{\mathfrak{S}}(v, w) = \mathsf{neg}_{\mathfrak{S}}(\mathsf{glb}(v, \mathsf{neg}_{\mathfrak{S}}(w)))$.

It's easy to check that transitivity of logical consequence does not hold in the tolerant logic **C** resulting from (TC$^{\mathbf{C}}$) (for example, $\varphi, \varphi \rightarrow \psi \vdash_{\mathbf{C}} \psi$—and so $\psi, \psi \rightarrow \chi \vdash_{\mathbf{C}} \chi$—holds, but $\varphi, \varphi \rightarrow \psi, \psi \rightarrow \chi \vdash_{\mathbf{C}} \chi$ does not) and that, indeed, lack of a sharp boundary of *e.g.* a property is consistent in **C** with the property's having both positive and negative cases on the same soritical series (for example, letting Bi be short for 'A man with i hairs is bald', $B0, \neg B100,000, \neg\exists i(Bi \,\&\, \neg Bi + 1) \vdash_{\mathbf{C}} \varnothing$ does not hold: for instance, consider a **C**-model $\mathfrak{M}$ such that, for every i $[i : 1 \leq i \leq 35,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{0, 1, 2, 3, 4, 5, 6, 7\}$; for every i $[i : 35,001 \leq i \leq 45,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{0, 2, 4\}$; for every i $[i : 45,001 \leq i \leq 55,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{6\}$; for every i $[i : 55,001 \leq i \leq 65,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \{1\}$; for every i $[i : 65,001 \leq i \leq 100,000]$, $\mathsf{val}_{\mathfrak{M}}(Bi) = \varnothing$). The construction can naturally be generalised to model the lack of a sharp boundary for other entities such as objects, propositions, connectives *etc.*

4 Conclusion: A Single Nonprimitive Notion of Vagueness Irreducibly Realised across the Type Hierarchy

In conclusion, on this view, there is one single nonprimitive notion of vagueness—(seeming) possible lack of a sharp boundary—that gets realised in different irreducible ways among and within different types, as opposed to the Rolf-style characterisation, on which there are primitive separate notions of vagueness for certain types to which vagueness of all other types is reduced (plus, as indicated, the proposed characterisation can be turned into a much more satisfying definition than the Rolf-style one).

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