A Synthetic Proof of Myhill's Theorem in the Effective Topos

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Abstract

One of the first results in classical computability theory was establishing the undecidability of the halting problem. In this presentation we will prove an even stronger version in the internal logic of the effective topos; more precisely in its full subcategory $\mathrm{Mod}(\mathcal{K}_1)$ of modest sets internal to assemblies $\mathrm{Ass}(\mathcal{K}_1)$. We will do this by proving that the diagonal halting set K is creative with our new definition. Our notion of creativity is classically equivalent to Post's and Myhill's definition, but more importantly, it contains recursive content. The moral lesson is that if we do computability theory in the effective topos, the proofs turn out to be more constructive and in the spirit of what one intended to begin with.

1 Introduction

An analytic treatment of computability theory in a classical model for set theory inevitably leans heavily towards informal proof methods. They are of course partially justified by the empirical evidence provided by the works of Turing, Church and Kleene among others [6, 11, 12]. But informal methods are mainly used to avoid cumbersome details involving Gödel numbers to be able to get to the core mathematical ideas without having to deal with routine manipulations. This creates the need for a more synthetic presentation, which factors those cumbersome details into axioms.

A more suitable mathematical universe in which these ideas can be encoded turns out to be Hyland's *effective topos* $\mathcal{E}ff$ [4]. Here, all functions are recursive or computable so that no reference to an external model of computation is necessary. Synthetic or axiomatic treatment of computability theory, pioneered by Bauer among others [1], allows us for instance to talk about recursively enumerable sets as just the (effective) sets, which are enumerable. In this sense, the synthetic approach reveals the mathematical structures without the encoded 'noise'. What is more, both the objects and morphisms between them carry constructive data in the effective topos. It therefore captures the essence of computability theory in which not only the results, but also the proofs are uniformly effective.

2 Synthetic Computability Theory

The first steps in synthetic computability theory in the effective topos have been taken by Bauer [1]. In this exposition, we take a few extra steps in this direction. We briefly present the context in which our investigation is carried out. Our references are from [10, 8].

Definition 2.1. A $\mathcal{K}_1-$*valued assembly* X is a set $|X|$ together with a function $E\colon |X| \to \mathcal{P}^*(\mathbb{N})$ assigning to each $x \in |X|$ a nonempty subset Ex of $\mathbb{N}$.

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Here $\mathcal{K}_1$ refers to Kleene's first model, and in the setting of recursive realizability, we think of Ex as the set of proofs for x. Assemblies can intuitively be thought as data types with an underlying set of values $|X|$ whose elements are given machine-level representations, or in our setting, realisers Ex. The representations of the elements in the underlying set is not unique as the set of realisers are not necessarily disjoint. We therefore ask for those datatypes for which the codes uniquely determine each value. The following definition captures this idea.

Definition 2.2. An assembly is said to be a *modest set* if for all $x, x' \in |X|$,

$$x \neq x' \implies Ex \cap Ex' = \emptyset.$$

Definition 2.3. Suppose $(|X|, E)$, $(|Y|, F)$ are two $\mathcal{K}_1-$valued assemblies. A function $f\colon |X| \to |Y|$ is said to be *tracked* by an element $t \in \mathbb{N}$ if for all $x \in X$ and for all $a \in Ex$, $ta\downarrow$ and $ta \in Ff(x)$.

Following our analogy, the morphisms between assemblies are precisely the functions that can be simulated, in our case, by a partial recursive function acting on the realisers instead of the elements. Assemblies and modest sets on $\mathcal{K}_1$ together with tracked maps form a bicartesian closed category, which is finitely complete and cocomplete with a natural numbers object $N \coloneqq (\mathbb{N}, E)$, $En \coloneqq \{ n \}$. We denote these categories $\mathrm{Ass}(\mathcal{K}_1)$ and $\mathrm{Mod}(\mathcal{K}_1)$ respectively.

There are close connections between fragments of a certain logic and particular classes of categories. In fact, the internal language of a cartesian closed category is simply typed λ-calculus, where the objects of the category serve as basic types and morphisms as basic terms [7]. What is more, we are able to write down formulae of intuitionistic higher-order logic, which readily have the intended meaning in $\mathcal{E}\!f\!f$. We will use a suitable internal language without much reference hereafter.

Now, the following is a nice fact: the category of modest sets $\mathrm{Mod}(\mathcal{K}_1)$ can be regarded as a category internal to assemblies $\mathrm{Ass}(\mathcal{K}_1)$ which is internally complete [5]. For what this kind of internalization means in a more general context see [3]. We will use this fact in order to carry on our investigation in these categories. We point out a few objects and facts that form the main ingredients of our results:

- While the subobject classifier Ω of $\mathcal{E}\!f\!f$ is itself not an object of $\mathrm{Ass}(\mathcal{K}_1)$, two of its subobjects of interest are: the object of *decidable* truth-values 2 with the underlying set $\{ p \in \omega \mid p \vee \neg p \}$, which up to isomorphism is the assembly $(\{ 0, 1 \}, E)$ with $E0 \coloneqq \{ 0 \}, E1 \coloneqq \{ 1 \}$ [10, §3.2.7], and the object of *semidecidable* truth-values with the underlying set $\Sigma \coloneqq \{ p \in \Omega \mid \exists f{:}N^N(p \leftrightarrow (\exists n(f(n) = 0))) \}$, which up to isomorphism is the assembly $(\{ 0, 1 \}, E)$ with $E0 \coloneqq \overline{K}$ and $E1 \coloneqq K$, where K denotes the diagonal halting set [10, Proposition 3.2.27]. Both are clearly modest, however the latter shows that truth and falsehood in this sense are recursively inseperable.

- There is indeed an one-to-one correspondence between the decidable subobjects of X and morphisms $X \to 2$. In particular, in $\mathcal{E}\!f\!f$ the Cantor space 2^N is the object of decidable subobjects of N. Recall that these are precisely the subsets of $\mathbb{N}$ that posess a recursive characteristic function, $2^N \cong (R, E)$, where $R \coloneqq \{ f\colon \mathbb{N} \to 2 \mid f \text{ is recurisve} \}$ and $Ef \coloneqq \{ e \mid e \text{ is Gödel number for } f \}$. In $\mathcal{E}\!f\!f$, 2^N and the space of functions N^N are isomorphic [10, Proposition 3.2.26]. Similarly, there is a one-to-one correspondence between semidecidable subobjects of N and tracked maps $N \to \Sigma$. The subobject $\Sigma \rightarrowtail \Omega$ is called the semidecidable subobject classifier because of the following isomorphism: $\Sigma^N \cong (RE, W)$, where $RE \coloneqq \{ R \subseteq \mathbb{N} \mid R \text{is recursively enumerable} \}$ and $WR \coloneqq \{ e \mid R = W_e \}$ [10, Proposition 3.2.28].

- The Σ-partial functions $N \to N_\perp$ are the synthetic analogue of partial recursive functions in the effective topos whose domains are precisly the semidecidable subobjects of N, for details see [1, §4]. This is part of a more general construction called lifting monads [2].
- We take for granted a pairing and unpairing isomorphism $N \times N \to N$. There exists an enumeration $\phi\colon N \twoheadrightarrow N_\perp^N$ such that $\forall \psi{:}N_\perp^N \exists e{:}N \phi(e) = \psi$, which together with pairing yields an enumeration $\phi_2\colon N \twoheadrightarrow N_\perp^{(N^2)}$ such that $\phi_2(e)(a,b) = \phi(e)(\langle a, b\rangle)$. We can continue the pattern to get a epimorphism ϕ_k for every natural number k. There is also an enumeration $W\colon N \twoheadrightarrow \Sigma^N$ such that $\forall A{:}\Sigma^N \exists e{:}N W_e = A$ [1, §4].
- The principle of countable choice (ACC), $\forall n{:}N \exists x{:}X R(n,x) \to \exists \alpha{:}(X^N) \forall n{:}N R(n, \alpha(n))$ holds for every object X of $\mathcal{E}ff$ [10, Corollary 3.2.9].

2.1 Basic synthetic results

The various results in the two coming sections emerged as an ongoing joint work with J.M.E. Hyland. Unless otherwise stated, to the best of our knowledge these results have not appeared in the literature.

Theorem 2.4. *In $\mathcal{E}ff$, the s-m-n theorem holds:*

$$\exists s_n^m{:}N^{(N^{m+1})} \forall e, y_1, \ldots, y_m{:}N \lambda \overline{x}.\phi_{m+n}(e)(\overline{y}, \overline{x}) = \phi_n(s(e, \overline{y}))$$

The crux of the argument of *s-m-n* lies within the ACC, and the two following results are a direct application of it, much like the classical case.

Theorem 2.5. *In $\mathcal{E}ff$, the Fixed point theorem holds cf. [1, Corollary 4.24]:*

$$\forall f{:}N^N \exists n{:}N \phi(f(n)) = \phi(n)$$

Theorem 2.6. *In $\mathcal{E}ff$, the Second recursion theorem holds:*

$$\forall f{:}N^{(N^2)} \exists n{:}N^N \forall x{:}N \phi(f(n(x), x)) = \phi(n(x))$$

2.2 Synthetic Myhill's theorem

In this section we establish our main theorem, namely that creativeness and completeness conincide in the effective topos. We will also show that $K{:}\Sigma^N$ is undecidable in the strong sense that it is creative. This version is even stronger than its classical counterpart as we shall see. We begin by stating the weak version, which is well known, and the argument mimics the classical one.

Proposition 2.7. *In $\mathcal{E}ff$, the set K is undecidable, that is*

$$\forall R{:}2^N R \neq K.$$

Let us first consider Myhill's characterisation of a creative set in our setting to see why it fails to be valid in the effective topos. The statement that K is creative would read as follows,

$$\forall A{:}\Sigma^N[\exists n{:}A \cap K \vee \exists n(n{:}A \cup K \to \perp)]. \tag{1}$$

Now consider $A = \emptyset$, then the first assertion of the disjunction is false and the second is true, while if $A = N$ then the situation is reversed. Recall, however, that Σ truth-values are recursively inseperable. Thus the above statement is asking us to do too much 'work'. Next, we provide a definition that implies Myhill's characterization and is classically equivalent to it. It is a version, which has as much constructive information as possible.

Definition 2.8 (Hyland)**.** In $\mathcal{E}\!f\!f$, a set $A{:}\Sigma^N$ is creative if $\exists u{:}N_\perp^N \forall e{:}N$

(i) $\exists n{:}W_e \cap A \quad \vee \quad u(e){:}N$;

(ii) $u(e){:}W_e \cup A \to \exists n{:}W_e \cap A$.

Remark 2.9. We have that N is regarded as a Σ-subobject of $N_\perp$ via the pullback

$$\begin{array}{ccc} N & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ N_\perp & \longrightarrow & \Sigma \end{array}$$

so that $u(x){:}N$ in the above definition means $u(x){\downarrow}$.

The following proposition establishes that our characterization coincides with the standard ones in the classical world. The proof is a matter of fiddling with the logic and the forward direction uses the intuitionistically invalid De Morgan's law $\neg(p \wedge q) \to \neg p \vee \neg q$. As a result, our definition is constructively stronger.

Proposition 2.10. *A set C is creative if and only if there exists a unary partial function u such that for all x,*

(i) $\exists n \in W_x \cap A \quad \vee \quad u(x){\downarrow}$;

(ii) $u(x) \in W_x \cup A \implies \exists n \in W_x \cap A$.

Definition 2.11. In $\mathcal{E}\!f\!f$, the set $A{:}\Sigma^N$ is complete if and only if $\forall B{:}\Sigma^N \exists f{:}N^N B = \ f^{-1}(A)$.

Note that $f^{-1}(A){:}\Sigma^N$ whenever $A{:}\Sigma^N$ with characteristic morphism $A \circ f$. We are now in a position of establishing that $K : \Sigma^N$ is creative according to our definition. This forms part of the key argument in our theorem stated below.

Proposition 2.12. *In $\mathcal{E}\!f\!f$, the set K is creative.*

Based on the full force of the discussions above, we can conclude this section with our main result:

Theorem 2.13. *(Synthetic Myhill's theorem) In $\mathcal{E}\!f\!f$, a set A is creative if and only if A is complete.*

3 Conclusion and future work

The *s-m-n* theorem while being an important result in the classical world, has turned out to be a simple application of the axiom of countable choice in the effective topos. This structure was previously not present in the classical informal or formal proof. We showed that K being creative is a straightforward fact, despite the fact that our definition of creativeness is constructively stronger. Indeed, we have demonstrated that non-trivial facts about computability theory find their home in the effective topos. This synthetic version of Myhill's theorem is only one example. In general, the synthetic results made no explicit reference to Gödel encoding or Turing machines, and the proofs were couched in purely set theoretic terms. As Andrej Bauer puts it "we just [did] ordinary math–in an extraordinary universe" [1].

There are various directions we can explore from here. Originally, the project started with looking at a paper by Moschovakis [9], where Myhill's theorem appeared among the applications of the Second recursion theorem. Another interesting result there concering partial recursive functionals is the Kreisel-Lacombe-Shoenfield-Ceiten theorem. As we did not use the full force of the effective topos, such a development would show how higher-order computability results appear here.

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