Shelah's conjecture fails for higher cardinalities

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Abstract

The main goal of this paper is to generalize the results that where presented in [3] for $\aleph_1$-Kurepa trees to $\aleph_{\alpha+1}$-Kurepa trees.

We construct an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ_α, that codes $\aleph_{\alpha+1}$-Kurepa trees, for some countable α. One of the main results for its spectrum (the spectrum of a sentence is the class of all cardinals for which there exists some model of the sentence) is the following:

It is consistent that $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, that $2^{\aleph_{\alpha+1}}$ is weakly inaccessible and that the spectrum of ψ_α is equal to $[\aleph_0, 2^{\aleph_{\alpha+1}})$.

This relates to a conjecture of Shelah, that if $\aleph_{\omega_1} < 2^{\aleph_0}$ and there is a model of some $\mathcal{L}_{\omega_1,\omega}$-sentence of size $\aleph_{\omega_1}$, then there is a model of size $2^{\aleph_0}$. Shelah proves the consistency of this conjecture in [2]. This statement proves that it is consistent that there is no Hanf number below $2^{\aleph_{\alpha+1}}$ for every countable α.

There are some interesting results for the amalgamation Spectrum too (the amalgamation Spectrum is defined similarly to the Spectrum, but we also require that κ-amalgamation holds). We prove that the κ-amalgamation for $\mathcal{L}_{\omega_1,\omega}$- sentences is not absolute. More specifically we prove:

- for $\alpha > 0$ finite, it is consistent that $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ and the Amalgamation Spectrum of ψ_α is equal to $[(2^{\aleph_\alpha})^+, \aleph_{\omega_{\alpha+1}}]$.
- for $\alpha > 0$ finite, it is consistent that $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ is weakly inaccessible and the Amalgamation Spectrum of ψ_α is equal to $[(2^{\aleph_\alpha})^+, 2^{\aleph_{\alpha+1}})$.

1 Kurepa trees and $\mathcal{L}_{\omega_1,\omega}$

Firstly, we need to see some useful definitions.

Definition 1.1. *For an $\mathcal{L}_{\omega_1,\omega}$ sentence ϕ, the* **spectrum** *of ϕ is the class*

$$Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } |M| = \kappa\}.$$

If Spec(ϕ) $= [\aleph_0, \kappa]$, we say that ϕ characterizes κ.
The **maximal models spectrum** *of ϕ is the class*

$$MM\text{-}Spec(\phi) = \{\kappa | \exists M \models \phi \text{ and } |M| = \kappa \text{ and } M \text{ is maximal } \}.$$

We can, also, define the **amalgamation spectrum** *of ϕ, AP-Spec(ϕ) and the* **joint embedding spectrum** *of ϕ, JEP-Spec(ϕ) as follows:*

$$AP\text{-}Spec(\phi) = \{\kappa | \phi \text{ has at least one model of size } \kappa \text{ and the models of size } \kappa \text{ satisfy the amalgamation property } \}$$

$$JEP\text{-}Spec(\phi) = \{\kappa | \phi \text{ has at least one model of size } \kappa \text{ and the models of size } \kappa \text{ satisfy the joint embedding property } \}.$$

Definition 1.2. *Assume κ is an infinite cardinal. A κ-**tree** has height κ and each level has at most $<\kappa$ elements. A κ-**Kurepa tree** is a κ-tree with at least κ^+ many branches of height κ.*

*If $\lambda \geq \kappa^+$, a (κ,λ)-**Kurepa tree** is a κ-Kurepa tree with exactly λ branches of height κ. KH(κ,λ) is the statement that there exists a (κ,λ)-Kurepa tree.*

Define $\mathcal{B}(\kappa) = sup\{\lambda | KH(\kappa,\lambda) \text{ holds }\}$.

*A **weak** κ-**Kurepa tree** is a κ-Kurepa tree, where each level has at most $\leq \kappa$ elements.*

Comment:For this paper we will assume that κ-Kurepa trees are pruned, i.e. every node is contained in a maximal branch of order type κ.

Definition 1.3. *Let $\kappa \leq \lambda$ be infinite cardinals. A sentence σ in a language with a unary predicate P admits (λ,κ), if σ has a model M such that $|M| = \lambda$ and $|P^M| = \kappa$. In this case, we will say that M is of type (λ,κ).*

Our goal, now, is to construct an $\mathcal{L}_{\omega_1,\omega}$ sentence such that every $\aleph_{\alpha+1}$-Kurepa tree (where α is countable) belongs to its spectrum.

From [1], we know the following theorem.

Theorem 1.4. *There is a first order sentence σ such that for all infinite cardinals κ, σ admits (κ^{++}, κ) iff $KH(\kappa^+,\kappa^{++})$.*

We will not present the proof for this theorem, but we are going to use some parts of the construction for σ, in order to construct the desired $\mathcal{L}_{\omega_1,\omega}$ sentence, ψ_α.

Assume that α is a countable ordinal. The vocabulary τ consists of the constants 0, $(c_n)_{n\in\omega}$, the unary symbols $L_0, L_1, ..., L_\alpha, L_{\alpha+1}$, the binary symbols $S, V, T, <_1, <_2, ..., <_\alpha, <_{\alpha+1}$ and the ternary symbols $F_0, F_1, ..., F_\alpha, G$. The idea is to build an $\aleph_{\alpha+1}$-Kurepa tree. $L_{\alpha+1}$ is a set that corresponds to the "levels" of the tree. $L_{\alpha+1}$ is linearly ordered by $<_{\alpha+1}$ and 0 is its minimum element. $L_{\alpha+1}$ may or may not have a maximum element. Every element $a \in L_{\alpha+1}$ that is not a maximum element has a successor b that satisfies $S(a,b)$. We will denote the successor of a by $S(a)$. The maximum element (which we will call m) is not a successor. For every $a \in L, V(a,\cdot)$ is the set of nodes at level a and we assume that $V(a,\cdot)$ is disjoint from all the $L_0, L_1, \dots, L_{\alpha+1}$. If $V(a,x)$, we will say that x is at the level a and we may write $x \in V(a)$.

T is a tree ordering on $V = \bigcup_{a\in L_{\alpha+1}} V(a)$. If $T(x,y)$, then x is at some level strictly less than the level of y. If $y \in V(a)$ and $b < a$, there is some x so that $x \in V(b)$ and $T(x,y)$. If a is a limit, that is neither a successor nor 0, then two distinct elements in $V(a)$ cannot have the same predecessors. If m is the maximum element of $L_{\alpha+1}, V(m)$ is the set of maximal branches through the tree. Both "the height of T" and "the height of $L_{\alpha+1}$" refer to the order type of $(L_{\alpha+1}, <_{\alpha+1})$. We can also stipulate that the $\aleph_{\alpha+1}$-Kurepa tree is pruned.

Our goal, now is to bound the size of each L_β by $\aleph_\beta$. For the first level, we require that $\forall x(L_0(x) \leftrightarrow \bigvee_n x = c_n)$. That gives us that $|L_0| = \aleph_0$.

Each $L_\beta, \beta = 1, 2, ..., \alpha$ is linearly ordered by $<_\beta$.

In order to bound the size of $L_{\beta+1}$ by $\aleph_{\beta+1}$, we bound the size of each initial segment by $\aleph_\beta$. Our treatment is slightly different for $\beta < \alpha$ than for $\beta = \alpha$.

Let $\beta < \alpha$. For every $x \in L_{\beta+1}$ there is a surjection $F_\beta(x,\cdot,\cdot)$ from L_β to $(L_{\beta+1})_{\leq_{(\beta+1)}x} = \{b \in L_{\beta+1} | b \leq_{(\beta+1)} x\}$. This bounds the size of each initial segment $(L_{\beta+1})_{\leq_{(\beta+1)}x}, \beta < \alpha$ by $|L_\beta|$.

At limit stages we take L_β as the union of the previous L_γ. The linear order on limit stages is not relevant to the linear orders in the previous stages.

Finally, for every $x \in L_{\alpha+1}$, that is not the maximum element, there is a surjection $F_\alpha(x,\cdot,\cdot)$ from L_α to $(L_{\alpha+1})_{\leq_{(\alpha+1)}x}$ and another surjection $G(x,\cdot,\cdot)$ from L_α to $V(x)$. This bounds the size of $(L_{\alpha+1})_{\leq_{(\alpha+1)}x}$ and the size of every $V(x)$, which is not maximal level, by $|L_\alpha|$.

Observation: Defining the $F_\alpha(x,\cdot,\cdot)$, we demand that x is not the maximum element of $L_{\alpha+1}$. We don't have the same restriction for the rest of the F_β's. That difference plays an important role throughout the rest of the paper.

This construction gives us that for all $\beta = 1, 2, ..., \alpha+1, |L_\beta| \leq \aleph_\beta$ and for all non maximal levels $|V(x)| \leq \aleph_\alpha$.

So, our desired $\mathcal{L}_{\omega_1,\omega}$ sentence, ψ_α is the conjuction of all the above requirements.

Definition 1.5. *A $(\kappa-\lambda)$-**Kurepa tree**, where $\lambda \geq \kappa$, is a tree of height κ, each level has at most $\leq \lambda$ elements with at least λ^+ branches of height κ. A $(\kappa-\kappa)$-Kurepa tree is a weak κ-Kurepa tree.*

The dividing line for models of ψ to code $\aleph_{\alpha+1}$-Kurepa trees is the size of $L_{\alpha+1}$. By definition, every initial segment of $L_{\alpha+1}$ has size at most $\aleph_\alpha$. If in addition $|L_{\alpha+1}| = \aleph_{\alpha+1}$, then we can embed $\omega_{\alpha+1}$ cofinally into $L_{\alpha+1}$. Hence, every model of ψ of size $\geq \aleph_{\alpha+2}$ and for which $|L_{\alpha+1}| = \aleph_{\alpha+1}$, codes an $\aleph_{\alpha+1}$-Kurepa tree.

Let $\mathbf{K}$ be the collection of all models of ψ, equipped with the substructure relation. I.e. for $M, N \in \mathbf{K}, M \prec_{\mathbf{K}} N$ if $M \subset N$.

Now, I present some interesting results and theorems, without their proofs.

Lemma 1.6. *If $M \prec_{\mathbf{K}} N$, then*

1. $L_0^M = L_0^N$
2. L_1^M *is initial segment of* L_1^N
3. *For $1 \leq \beta \leq \alpha$, if $|L_\gamma^M| = \aleph_\gamma$, for every $\gamma \leq \beta$, then $L_\beta^M = L_\beta^N$*
4. *For $1 \leq \beta \leq \alpha$, if $|L_\gamma^M| = \aleph_\gamma$, for every $\gamma \leq \beta$, then $L_{\beta+1}^M$ is an initial segment of $L_{\beta+1}^N$*
5. *If $|L_\beta^M| = \aleph_\beta$, for every $\beta \leq \alpha$, then $V^M(x) = V^N(x)$, for every non maximal $x \in L_{\alpha+1}^M$*
6. *the tree ordering is preserved*

Corollary 1.7. *If $M \prec_{\mathbf{K}} N$, then*

1. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha$ and $L_{\alpha+1}^M = L_{\alpha+1}^N$, then N differs from M only in the maximal branches it contains.*
2. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha+1$ and $L_{\alpha+1}^N$ is a strict end extension of $L_{\alpha+1}^M$, then $L_{\alpha+1}^M$ does not have a maximum element and $L_{\alpha+1}^N$ is one point end extension of $L_{\alpha+1}^M$.*
3. *If $|L_\beta^M| = \aleph_\beta$ for every $\beta \leq \alpha$, $L_{\alpha+1}^M$ has a maximum element and $L_{\alpha+1}^N$ is a strict end extension of $L_{\alpha+1}^M$, then $|M| = \aleph_\alpha$.*

Proposition 1.8. *$(\mathbf{K}, \prec_{\mathbf{K}})$ is an Abstract Elementary Class (AEC) with countable Lowenheim-Skolem number.*

Theorem 1.9. *The spectrum of ψ is characterized by the following properties:*

1. $[\aleph_0, \aleph_\alpha^{\aleph_0}] \subseteq Spec(\psi)$ *and* $\aleph_{\alpha+1} \in Spec(\psi)$.
2. *if there exists a $(\mu-\lambda)$-Kurepa tree, where $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, with κ cofinal branches, then $[\aleph_0, \kappa] \subseteq Spec(\psi)$.*
3. *if there exists an $\aleph_{\alpha+1}$-Kurepa tree with κ cofinal branches, then $[\aleph_0, \kappa] \subseteq Spec(\psi)$.*

4. *no cardinal belongs to Spec(ψ) except those required by (1)-(2)-(3). I.e. if ψ has a model of size κ, then $\kappa \in [\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ or there exists a $(\mu - \lambda)$-Kurepa tree with κ cofinal branches or there exists an $\aleph_{\alpha+1}$-Kurepa tree with κ cofinal branches.*

Theorem 1.10. *The maximal models Spectrum of ψ is characterized by the following:*

1. *ψ has maximal model of size $\aleph_{\alpha+1}$*
2. *If $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}$, for some $\aleph_0 \leq \lambda \leq \aleph_\alpha$, then ψ has maximal model of size $\lambda^{\aleph_0}$*
3. *If there exists an $(\mu - \aleph_\alpha)$-Kurepa tree, $\mu \geq \aleph_1$, with exactly κ cofinal branches, then ψ has maximal model in κ*
4. *If there exists an $\aleph_{\alpha+1}$-Kurepa tree with exactly κ cofinal brnches, then ψ has maximal model in κ*
5. *ψ has maximal models only on those cardinalities required by (1)-(4).*

Corollary 1.11. 1. *If there are no $(\mu - \lambda)$-Kurepa trees and no $\aleph_{\alpha+1}$-Kurepa trees, then $Spec(\psi) = [\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ and $MM - Spec(\psi) = \{\lambda^{\aleph_0} | \aleph_0 \leq \lambda \leq \aleph_\alpha$ and $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}\} \cup \{\aleph_{\alpha+1}\}$.*

2. *If $\mathcal{B}(\aleph_{\alpha+1})$ is a maximum, i.e. there is an $\aleph_{\alpha+1}$-Kurepa tree of size $\mathcal{B}(\aleph_{\alpha+1})$ and there are no $(\mu - \lambda)$-Kurepa trees for $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, then ψ characterizes $max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}, \mathcal{B}(\aleph_{\alpha+1})\}$.*

3. *If $\mathcal{B}(\aleph_{\alpha+1})$ is not a maximum and there are no $(\mu - \lambda)$-Kurepa trees for $\aleph_1 \leq \mu \leq \lambda \leq \aleph_\alpha$, then $Spec(\psi)$ equals $[\aleph_0, max\{\aleph_\alpha^{\aleph_0}, \aleph_{\alpha+1}\}]$ or $[\aleph_0, \mathcal{B}(\aleph_{\alpha+1}))$, whichever is greater. Moreover, ψ has maximal models in $\aleph_{\alpha+1}, \lambda^{\aleph_0}$, if it is $\geq \aleph_{\alpha+1}$ and in cofinally many cardinalities below $\mathcal{B}(\aleph_{\alpha+1})$.*

Theorem 1.12. 1. *$(\mathbf{K}, \prec_{\mathbf{K}})$ fails JEP in all cardinals.*

2. - *If $\alpha < \omega$, then $(\mathbf{K}, \prec_{\mathbf{K}})$ satisfies AP for all cardinals $> 2^{\aleph_\alpha}$ that belong to $Spec(\psi)$, but fails AP in every cardinal $\leq 2^{\aleph_\alpha}$.*
   - *If $\alpha \geq \omega$, then $(\mathbf{K}, \prec_{\mathbf{K}})$ fails AP in all cardinalities.*

2 Consistency results

Theorem 2.1. *It is consistent with ZFC that $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}} = \mathcal{B}(\aleph_{\alpha+1}) < 2^{\aleph_{\alpha+1}}$ and there exists an $\aleph_{\alpha+1}$-Kurepa tree with $\aleph_{\omega_{\alpha+1}}$-many cofinal branches.*

Theorem 2.2. *From a Mahlo cardinal, it is consistent with ZFC that $2^{\aleph_\alpha} < \mathcal{B}(\aleph_{\alpha+1}) = 2^{\aleph_{\alpha+1}}$, for every $\kappa < 2^{\aleph_{\alpha+1}}$ there is an $\aleph_{\alpha+1}$-Kurepa tree with at least κ-many maximal branches, but no $\aleph_{\alpha+1}$-Kurepa tree has $2^{\aleph_{\alpha+1}}$-many maximal branches.*

Corollary 2.3. *For every α countable ordinal, there exists an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ that it is consistent with ZFC that:*

1. *ψ characterizes $max\{\aleph_{\alpha+1}, \aleph_\alpha^{\aleph_0}\}$*
2. *$2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ and ψ characterizes $\aleph_{\omega_{\alpha+1}}$*

3. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $Spec(\psi) = [\aleph_0, 2^{\aleph_{\alpha+1}})$
4. $MM - Spec(\psi) = \{\lambda^{\aleph_0} | \aleph_0 \leq \lambda \leq \aleph_\alpha$ *and* $\lambda^{\aleph_0} \geq \aleph_{\alpha+1}\} \cup \{\aleph_{\alpha+1}\}$
5. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $MM - Spec(\psi)$ *is a cofinal subset of* $[\aleph_{\alpha+1}, 2^{\aleph_{\alpha+1}})$

   If, in addition α *is finite, then it is also consistent that*

6. $2^{\aleph_\alpha} < \aleph_{\omega_{\alpha+1}}$ *and* $AP - Spec(\psi) = (2^{\aleph_\alpha}, \aleph_{\omega_{\alpha+1}}]$
7. $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, $2^{\aleph_{\alpha+1}}$ *is weakly inaccessible and* $AP - Spec(\psi) = (2^{\aleph_\alpha}, 2^{\aleph_{\alpha+1}})$

Finally, throughout the paper there are some interesting open questions that have been risen:

Open Question 1. *Is the negation of Shelah's conjecture consistent with ZFC?*

Open Question 2. *Is* $\aleph_1$*-amalgamation for* $\mathcal{L}_{\omega_1,\omega}$*-sentences absolute for models of ZFC?*

References

[1] C. C. Chang and H. J. Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[2] Saharon Shelah. Borel sets with large squares. *Fund. Math.*, 159(1):1–50, 1999.

[3] Dima Sinapova and Ioannis Souldatos. Kurepa trees and spectra of $L_{\omega_1,\omega}$-sentences. *Arch. Math. Logic*, 59(7-8):939–956, 2020.