Identity types in predicate logic

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1 Introduction

Model constructions based on equivalence relations and partial equivalence relations (PERs) are instrumental in type theory, programming language theory and categorical logic. They provide well-behaved models of intensional type theory (e.g. [7]) and polymorphism (e.g. [6]). PERs are also a key feature in the tripos-to-topos construction [5]. This work draws on Martin-Löf's identity types [2, 4] to describe universal properties of certain forms of such constructions.

Universal properties of constructions based on equivalence relations have appeared in works by Lawvere and subsequent authors on exact completions (particularly the ex/reg completion [3, 8]) at the generality of categories, and in more recent works by Maietti and Rosolini on quotient completions [10, 11] and by Pasquali on the 'elementary completion' [12] at the generality of indexed posets. The present work builds upon Pasquali's completion result and the construction that underlies it, the latter of which was in essence originally described by Maietti and Rosolini for their 'effective quotient completion' [11] and we shall refer to as the *ER-descent construction* (Definition 3.3).

Theorem 1.1 (Pasquali [12], §3)**.** *The ER-descent construction gives a right 2-adjoint to the inclusion of the 2-category of elementary doctrines and strict-natural morphisms in the 2-category of primary doctrines (= indexed* $(\wedge, \top)$*-posets over finite-product categories) and strict-natural morphisms.*

This result will be reinterpreted in terms of identity types, in the following way. In view that indexed preorders are an interpretation of many-sorted predicate logic, and many-sorted predicate logic in turn is a (extremely) truncated version of dependent type theory, we formulate an adaptation of the inductive axioms of identity types to indexed preorders, calling the resulting notion *identity objects* (Definition 3.1). It turns out that a primary doctrine has identity objects if and only if it is an elementary doctrine (Theorem 3.2). Therefore, Theorem 1.1 is telling us that the ER-descent construction is a right 2-adjoint completion that adds identity objects.

We then describe a universal property analogous to this for the *PER-descent construction* (Definition 4.1), the partial equivalence relations version of the ER-descent construction. This construction can be seen as a step in the tripos-to-topos construction (cf. [9]). The universal property is obtained by considering a suitable weakening of identity objects, called *partial identity objects* (Definition 4.2). We show that the PER-descent construction is a right-biadjoint completion that adds partial identity objects (Theorem 4.6).

Partial identity objects can be promoted to identity objects by a comonadic construction we call *virtualisation* (Definition 5.2 and Remark 5.3). It serves as another step in the tripos-to-topos construction (cf. [9]), where it in particular turns the PERs emerged from the PER-descent construction into equivalence relations. Virtualisation in fact plays a role in the definition of partial identity objects (Remark 4.3). We establish universal properties of virtualisation (Theorem 5.6). It is an ambidextrously biadjoint completion with respect to oplax-natural morphisms of indexed preorders. With respect to pseudonatural morphisms of indexed preorders, it is merely a left-biadjoint completion.

2 2-categories of indexed preorders

Let Pre, $\mathrm{Pre}^{\wedge}$ and $\mathrm{Pre}^{\wedge,\top}$ denote the category of preorders, $\wedge$-preorders and $(\wedge,\top)$-preorders respectively.

Definition 2.1. A *(strictly) indexed preorder*, $\wedge$*-preorder* and $(\wedge,\top)$*-preorder* P consists of a category P^0 and a functor $P^1\colon (P^0)^{\mathrm{op}} \to \mathrm{Pre}$, $\mathrm{Pre}^{\wedge}$ and $\mathrm{Pre}^{\wedge,\top}$ respectively.

Definition 2.2. Let P and Q be indexed preorders. An *oplax-natural morphism* $F\colon P \to Q$ consists of a functor $F^0\colon P^0 \to Q^0$ and an oplax natural transformation

$$\begin{array}{ccc} (P^0)^{\mathrm{op}} & \xrightarrow{(F^0)^{\mathrm{op}}} & (Q^0)^{\mathrm{op}} \\ & \overset{F^1}{\Rightarrow} & \\ {\scriptstyle P^1}\searrow & & \swarrow{\scriptstyle Q^1} \\ & \mathrm{Pre.} & \end{array}$$

A *pseudonatural morphism* is an oplax-natural morphism F for which F^1 is a pseudonatural transformation. A *strict-natural morphism* is an oplax-natural morphism F for which F^1 is a strict natural transformation.

Terminology 2.3. Let P and Q be indexed preorders. An oplax-natural morphism $F\colon P \to Q$ *preserves binary products* if the functor F^0 preserves binary products, and *preserves meets* or *top* if each component of the oplax natural transformation F^1 does so respectively.

Definition 2.4. A *2-morphism* $\rho\colon F \to G\colon P \to Q$ between oplax morphisms of indexed preorders is a natural transformation $\rho\colon F^0 \to G^0\colon P^0 \to Q^0$ such that

$$\begin{array}{ccc} & P^1(X) & \\ {\scriptstyle F^1_X}\swarrow & & \searrow{\scriptstyle G^1_X} \\ & \leq & \\ Q^1F^0(X) & \xleftarrow[\rho_X{}^* := Q^1(\rho_X)]{} & Q^1G^0(X) \end{array} \tag{1}$$

as homomorphisms of preorders for each object $X \in P^0$.

The following defines notations for some 2-categories of indexed preorders that we will need.

Proposition 2.5. *Indexed*

a. $\wedge$*-preorders over binary-product categories,*

b. $(\wedge,\top)$*-preorders,* *c.* $(\wedge,\top)$*-preorders over binary-product categories,*

their structure-preserving

1. oplax-natural *2. pseudonatural*

morphisms, and 2-morphisms form a 2-category, denoted

1a. $\mathrm{IdxPre}^{\times,\wedge}_{\mathrm{on}}$, *1b.* $\mathrm{IdxPre}^{\wedge,\top}_{\mathrm{on}}$, *1c.* $\mathrm{IdxPre}^{\times,\wedge,\top}_{\mathrm{on}}$

2a. $\mathrm{IdxPre}^{\times,\wedge}_{\mathrm{pn}}$, *2b.* $\mathrm{IdxPre}^{\wedge,\top}_{\mathrm{pn}}$, *2c.* $\mathrm{IdxPre}^{\times,\wedge,\top}_{\mathrm{pn}}$

respectively.

3 Identity objects and ER-descent construction

Let P be an indexed $(\wedge, \top)$-preorder over a binary-product category.

Definition 3.1. An *identity object* on an object $X \in P^0$ is

1. (*formation*) an element $\mathrm{Id}_X \in P^1(X \times X)$,

such that

2. (*introduction* or *reflexivity*) $\top \leq (X \xrightarrow{\delta} X \times X)^*(\mathrm{Id}_X)$, and
3. (*elimination*) for any object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if

$$(X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(p) \leq (X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(q),$$

then $(X \times X \times Y \overset{\pi_1,\pi_2}{\to} X \times X)^*(\mathrm{Id}_X) \wedge p \leq q$.

We say P *has identity objects* if each X has an identity object.

This Martin-Löf notion of equality is equivalent to Lawvere's hyperdoctrine equality [1] as extracted by Maietti and Rosolini [10] in the notion of elementary doctrine:

Theorem 3.2. *An indexed $(\wedge, \top)$-poset over a finite-product category has identity objects if and only if it is an elementary doctrine.*

This means Pasquali's 'elementary completion' result (Theorem 1.1) is telling us that the ER-descent construction is a right 2-adjoint completion adding identity objects. Let us review this construction.

Definition 3.3. An *equivalence relation* on an object $X \in P^0$ is an element $\sim \in P^1(X \times X)$ that is

1. (*reflexive*) $\top \leq (X \xrightarrow{\delta} X \times X)^*(\sim)$
2. (*symmetric*) $\sim \leq (X \times X \overset{\pi_2,\pi_1}{\to} X \times X)^*(\sim)$, and
3. (*transitive*) $(X\times X\times X \overset{\pi_1,\pi_2}{\to} X\times X)^*(\sim) \wedge (X\times X\times X \overset{\pi_2,\pi_3}{\to} X\times X)^*(\sim) \leq (X\times X\times X \overset{\pi_1,\pi_3}{\to} X \times X)^*(\sim)$.

The *ER-descent construction* on P is the indexed preorder $\mathrm{ER}(P)$ where an object in $\mathrm{ER}(P)^0$ is a pair $(X, \sim)$ with $X \in \mathrm{Ob}(P^0)$ and $\sim \in P^1(X\times X)$ an equivalence relation, an arrow $(X, \sim_X) \to (Y, \sim_Y)$ is an arrow $f \colon X \to Y$ in P^0 satisfying $\sim_X \leq P^1(f \times f)(\sim_Y)$, and $\mathrm{ER}(P)^1(X, \sim)$ the full subpreorder of $P^1(X)$ on those elements p satisfying $P^1(\pi_1)(p) \wedge \sim \leq P^1(\pi_2)(p)$.

Proposition 3.4. $\mathrm{ER}(P)$ *is an indexed $(\wedge, \top)$-preorder over a binary-product category, and has identity objects: if $(X, \sim)$ is an object in $\mathrm{ER}(P)^0$, then $\sim$ is an identity object on it.*

We will state an adaptation of Pasquali's result to our settings. The following definition is needed.

Definition 3.5. Let P and Q be indexed $(\wedge, \top)$-preorders with identity objects over binary-product categories. Let $F \colon P \to Q$ be an oplax-natural morphism that preserves binary products. We say F *preserves identity objects* if

$$\mathrm{Id}_{F^0(X)} \simeq (F^0(X) \times F^0(X) \overset{\cong}{\to} F^0(X \times X))^* F^1_{X\times X}(\mathrm{Id}_X)$$

for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ denote the 2-category whose 0-cells are indexed $(\wedge,\top)$-preorders with identity objects over binary-product categories, 1-cells are pseudonatural morphisms that preserve binary products, meets, tops and identity objects, and 2-cells are 2-morphisms.

Theorem 3.6. *The assignment* $P \mapsto \mathrm{ER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is right biadjoint to the inclusion 2-functor.*

4 PER-descent construction and partial identity objects

Let P be an indexed $\wedge$-preorder over a binary-product category.

Definition 4.1. A *partial equivalence relation* on an object $X \in P^0$ is an element $\sim\, \in P^1(X \times X)$ that is symmetric and transitive.

The *PER-descent construction* on P is the indexed preorder $\mathrm{PER}(P)$ defined in the same way as $\mathrm{ER}(P)$ but with as objects in $\mathrm{PER}(P)^0$ *partial* equivalence relations in P instead.

The following weakened form of identity objects will give us a result analogous to Theorem 3.6 for the PER-descent construction.

Definition 4.2. P has *partial identity objects* if each object $X \in P^0$ is *equipped* with an element $\mathrm{PId}_X \in P^1(X \times X)$, such that

1. (*partial reflexivity*) $\mathrm{PId}_X \leq (X \times X \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X), (X \times X \overset{\pi_2,\pi_2}{\to} X \times X)^*(\mathrm{PId}_X)$,
2. (*paravirtual elimination*) for any object $Y \in P^0$ and elements $p, q \in P^1(X \times X \times Y)$, if
$$(X \times Y \overset{\pi_1,\pi_1}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \overset{\pi_2,\pi_2}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge$$
$$(X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(p) \leq (X \times Y \overset{\delta\times Y}{\to} X \times X \times Y)^*(q)$$
then $(X \times X \times Y \overset{\pi_3,\pi_3}{\to} Y \times Y)^*(\mathrm{PId}_Y) \wedge (X \times X \times Y \overset{\pi_1,\pi_2}{\to} X \times X)^*(\mathrm{PId}_X) \wedge p \leq q$,
3. each arrow $f : X \to Y$ in P^0 satisfies $\mathrm{PId}_X \leq (f \times f)^*(\mathrm{PId}_Y)$, and
4. $\mathrm{PId}_{X\times Y} \simeq (X \times Y \times X \times Y \overset{\pi_1,\pi_3}{\to} X \times X)^*(\mathrm{PId}_X) \wedge (X \times Y \times X \times Y \overset{\pi_2,\pi_4}{\to} Y \times Y)^*(\mathrm{PId}_Y)$.

Remark 4.3. Paravirtual elimination, as the name suggests, is related to virtualisation (§5). Specifically, under the assumption of the other axioms (1., 3. and 4.), PId_X satisfies paravirtual elimination if and only if it satisfies elimination (Definition 3.1) in the virtualisation $\mathrm{Virt}(P)$ of P (Definition 5.2).

Proposition 4.4. $\mathrm{PER}(P)$ *is an indexed* $\wedge$*-preorder with partial identity object over a binary-product category, with* $\mathrm{PId}_{(X,\sim)} \coloneqq \sim$.

The preservation of partial identity objects is defined in the same way as that of identity objects:

Definition 4.5. Let P and Q be indexed $\wedge$-preorders with partial identity objects over binary-product categories. Let $F : P \to Q$ be an oplax-natural morphism that preserves binary products. We say F *preserves partial identity objects* if
$$\mathrm{PId}_{F^0(X)} \simeq (F^0(X) \times F^0(X) \overset{\cong}{\to} F^0(X \times X))^* F^1_{X\times X}(\mathrm{PId}_X)$$
for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ denote the 2-category whose 0-cells are indexed $\wedge$-preorders with partial identity objects over binary-product categories, 1-cells are pseudonatural morphisms that preserve binary products, meets and partial identity objects, and 2-cells are 2-morphisms.

Theorem 4.6. *The assignment* $P \mapsto \mathrm{PER}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge} \to \mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ *that is right biadjoint to the forgetful 2-functor.*

In other words, the PER-descent construction is a right-biadjoint completion adding partial identity objects.

5 Virtualisation

Virtualisation is a construction that allows us to promote partial identity objects into identity objects. Its general definition involves the following structure on indexed preorders.

Definition 5.1. An indexed preorder P is *oplaxly sectioned* if each object $X \in P^0$ is *equipped* with an element $\mathrm{os}_X \in P^1(X)$ and each arrow $f\colon X \to Y$ in P^0 satisfies $\mathrm{os}_X \leq f^*(\mathrm{os}_Y)$.

We may regard an indexed preorder with partial identity objects as oplaxly sectioned, with $\mathrm{os}_X \coloneqq (X \xrightarrow{\delta} X \times X)^*(\mathrm{PId}_X)$.

Let P be an oplaxly sectioned indexed $\wedge$-preorder.

Definition 5.2. The *virtualisation* of P is the indexed preorder $\mathrm{Virt}(P)$ given by $\mathrm{Virt}(P)^0 \coloneqq P^0$ and $\mathrm{Virt}(P)^1(X) \coloneqq (\mathrm{U}_{\mathrm{Set}} P^1(X), \overset{\mathrm{v}}{\leq})$ where $p \overset{\mathrm{v}}{\leq} q$ if and only if $\mathrm{os}_X \wedge p \leq q$.

Remark 5.3. Let $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ denote the preorder-enriched category of functors and *oplax* natural transformations. $\mathrm{Virt}(P)^1$ is in fact a Kleisli as well as Eilenberg-Moore object for the (necessarily idempotent) comonad $\mathrm{v}_P\colon P^1 \Rightarrow P^1$ in $[(P^0)^{\mathrm{op}}, \mathrm{Pre}^{\wedge}]_{\mathrm{o}}$ given by $(\mathrm{v}_P)_X(p) \coloneqq \mathrm{os}_X \wedge p$.

Proposition 5.4. $\mathrm{Virt}(P)$ *is an indexed* $(\wedge, \top)$*-preorder, with top elements given by the* os_X*. If* P *has partial identity objects, then* $\mathrm{Virt}(P)$ *has identity objects given by the* PId_X*.*

We will describe universal properties of virtualisation. The following definition is needed.

Definition 5.5. Let P and Q be oplaxly sectioned indexed preorders. An oplax-natural morphism $F\colon P \to Q$ *preserves the specified oplax section* if $\mathrm{os}_{F^0(X)} \simeq F^1_X(\mathrm{os}_X)$ for every object $X \in P^0$.

Let $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}}$ denote the 2-category whose 0-cells are oplaxly sectioned indexed $\wedge$-preorders, 1-cells are oplax-natural morphisms that preserve meets and the specified oplax section, and 2-cells are 2-morphisms. Let $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}}$ and $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$ be the variants of $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\mathrm{PId}}$ and $\mathrm{IdxPre}_{\mathrm{pn}}^{\times,\wedge,\top,\mathrm{Id}}$ respectively whose 1-cells are oplax- rather than pseudonatural morphisms.

Theorem 5.6. *The assignment* $P \mapsto \mathrm{Virt}(P)$ *extends to a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\mathrm{os}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\wedge,\top}$ *as well as a 2-functor* $\mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\mathrm{PId}} \to \mathrm{IdxPre}_{\mathrm{on}}^{\times,\wedge,\top,\mathrm{Id}}$ *that is ambidextrously biadjoint to 'the' respective inclusion 2-functor. The left-biadjoint part also holds with respect to pseudonatural morphisms.*

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