Generic groups and WAP

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Abstract

We consider the logic space of countable (enumerated) groups and show that closed subspaces corresponding to some standard classes of groups have (do not have) generic groups.

1. INTRODUCTION. The recent paper [1] considers countable groups as elements of a Polish space in the following way. An **enumerated group** is the set ω together with a multiplication function $\cdot : \omega\times\omega\rightarrow\omega$, an inversion function $^{-1} : \omega\rightarrow\omega$, and an identity element $e\in\omega$ defining a group. The space $\mathcal{G}$ of enumerated group is the closed subset of the space $\mathcal{X}=\omega^{\omega\times\omega}\times\omega^{\omega}\times\omega$ (under the product topology). If U is a universal extension of the theory of groups then the space of enumerated groups satisfying U, say $\mathcal{G}_U$, is a closed subspace of $\mathcal{G}$. Since all these spaces are Polish (separable and completely metrizable), Baire category methods can be applied.

In fact, instead of U any abstract property of groups, say P, can be considered. It naturally defines the invariant subset $\mathcal{G}_P$ of $\mathcal{G}$. Assuming that $\mathcal{G}_P$ is Polish it is studied in [1] which group-theoretic properties define comeagre subsets of $\mathcal{G}_P$. In particular, are there **generic groups** in $\mathcal{G}_P$ (i.e.groups forming a comeagre isomorphism class)? The main results of [1] state that there is no generic group in $\mathcal{G}$ and moreover, when P is the property of left orderability the space $\mathcal{G}_P$ does not have a generic group. The authors deduce this using involved arguments in the style of model-theoretic forcing.

A. Ivanov studied the problem of existence of generic objects in [2] in full generality. He introduced some general condition which is now called the weak amalgamation property (WAP) and showed that it together with the joint embedding property characterizes the existence of generics. (In [2] this condition was called the almost amalgamation property. It was considered later for automorphisms in the very influential paper [4] were the name WAP was used.)

The goal of our paper is to demonstrate that the results of [1] mentioned above can be deduced by an application of [2]. We also add some new important examples of this kind. Furthermore, the approach of [1] is extended to actions of groups on first-order structures, for example on $(\mathbb{Q},<)$. We also show how [2] works in the contrary direction, i.e. for proving of existence of generics in $\mathcal{G}_U$ for some natural group varieties U.

In this text we omit proofs. They can be found in [3]. Theorems 4 and 7 are already available to the reader at this stage. These are our new results concerning Burnside varieties. Other results below describe our approach and technical details.

General preliminaries. Fix a countable ω-categorical structure M in a language L. Let T be an expansion of $Th(M)$ in some $L\cup\bar{r}$ where $\bar{r}=(r_1,\ldots,r_\iota)$ is a sequence of additional relational/functional symbols. We assume that T is axiomatizable by sentences of the following form:

$$(\forall\bar{x})\,(\bigvee_i(\phi_i(\bar{x})\wedge\psi_i(\bar{x})),$$

where $\phi_i(\bar{x})$, is a quantifier-free formula in the language $L \cup \bar{r}$, and $\psi_i(\bar{x})$ is a first order formula of the language L. Let $\mathcal{X}_M$ be the space of all $\bar{r}$-expansions of M to models of T. This is a topological space with respect to so called **logic topology**. In the following few paragraphs we introduce some notions from [2]. They will be helpful in a description of the logic topology too.

For a tuple $\bar{a} \subset M$ we define a diagram $\phi(\bar{a})$ of $\bar{r}$ on $\bar{a}$ as follows. To every functional symbol from $\bar{r}$ we associate a partial function from $\bar{a}$ to $\bar{a}$. When r_i is a relational symbol from $\bar{r}$ we choose a formula from every pair $\{r_i(\bar{a}'), \neg r_i(\bar{a}')\}$, where $\bar{a}'$ is a subtuple from $\bar{a}$ of the corresponding length.

Let $\mathcal{B}_T$ be the set of all theories $D(\bar{a})$, $\bar{a} \subset M$, such that each of them consists of $Th(M, \bar{a}) \cup T$ together with a diagram of $\bar{r}$ on $\bar{a}$ satisfied in some $(M, \bar{\mathbf{r}}) \models T$. Since M is atomic, each element of $\mathcal{B}_T$ is determined by a formula of the form $\phi(\bar{a}) \wedge \psi(\bar{a})$, where $\psi(\bar{x})$ is a complete formula for M realized by $\bar{a}$ and $\phi(\bar{a})$ is a quantifier-free formula in the language $L \cup \bar{r}$. The corresponding $\phi(\bar{x}) \wedge \psi(\bar{x})$ is called **basic**.

For every diagram $D(\bar{a}) \in \mathcal{B}_T$ the set

$$[D(\bar{a})] = \{(M, \bar{\mathbf{r}}) \in \mathcal{X}_M \mid (M, \bar{\mathbf{r}}) \text{ satisfies } D(\bar{a})\}$$

is clopen in the logic topology. In fact the family $\{[D(\bar{a})] \mid D(\bar{a}) \in \mathcal{B}_T\}$ is usually taken as a base of this topology. It is metrizable. Each $D(\bar{a}) \in \mathcal{B}_T$ can be viewed as an expansion of M by finite relations corresponding to $\bar{r}$. When $\mathbf{r}_i$ is a functional symbol the corresponding relation is $\mathsf{Graph}(\mathbf{r}_i)$ the graph of the corresponding partial function on $\bar{a}$. We will say that $\bar{a}$ is the domain of this diagram: $\bar{a} = \mathsf{Dom}(D(\bar{a}))$.

Definition 0. *An expansion $(M, \bar{\mathbf{r}})$ is called* **generic** *if it has a comeagre isomorphism class in $\mathcal{X}_M$.*

The set $\mathcal{B}_T$ is ordered by the relation of extension: $D(\bar{a}) \subseteq D'(\bar{b})$ if $\bar{a} \subseteq \bar{b}$ and $D'(\bar{b})$ implies $D(\bar{a})$ under T (in particular, the partial functions defined in $D'(\bar{b})$ extend the corresponding partial functions defined in $D(\bar{a})$).

In these terms we formulate the definitions of JEP, AP and WAP.

- The family $\mathcal{B}_T$ has the **joint embedding property** if for any two elements $D_1, D_2 \in \mathcal{B}_T$ there is D_3 from $\mathcal{B}_T$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.
- The family $\mathcal{B}_T$ has the **amalgamation property** if for any D_0, D_1, $D_2 \in \mathcal{B}_T$ with $D_0 \subseteq D_1$ and $D_0 \subseteq D_2$ there is $D_3 \in \mathcal{B}_T$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ fixing $\mathsf{Dom}(D_0)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.
- The family $\mathcal{B}_T$ has the **weak amalgamation property** if for every $D_0 \in \mathcal{B}_T$ there is an extension $D'_0 \in \mathcal{B}_T$ such that for any $D_1, D_2 \in \mathcal{B}_T$ with $D'_0 \subseteq D_1$ and $D'_0 \subseteq D_2$ there is $D_3 \in \mathcal{B}'$ and an automorphism $\alpha \in \mathsf{Aut}(M)$ fixing $\mathsf{Dom}(D_0)$ such that $D_1 \subseteq D_3$ and $\alpha(D_2) \subseteq D_3$.

By Theorem 1.2 from [2]:

$\mathcal{X}_M$ has a generic expansion $(M, \bar{\mathbf{r}})$ if and only if the family $\mathcal{B}_T$ has JEP and WAP.

I. Enumerated groups. The basic case. The structure M is just ω and L consists of one constant symbol 1 which is interpreted by number 1. Let $\bar{r}$ consist of the binary function of multiplication $\cdot$ and a unary function $^{-1}$. Let T be the universal theory of groups with the unit 1. [1]

[1] It is not necessary to fix 1 in the language of M. We do it just for convenience of notations in Scenario **II** below.

Each element of $\mathcal{B}_T$ consists of a tuple $\bar{c} \subset \omega$ containing 1 and partial functions for $\cdot$ and $^{-1}$ on $\bar{c}$. The space $\mathcal{X}_M$ corresponding to T is the logic space of all countable groups, where the unit is fixed. *We emphasize that this is exactly the space $\mathcal{G}$ of enumerated groups from [1].*

If instead of T above one takes a universal extension of the theory of groups, say $\bar{T}$, then the corresponding space of enumerated groups is the closed subspace $\mathcal{G}_{\bar{T}} \subseteq \mathcal{G}$.

In the paragraph below we will have several sorts in M. The sort just described will always occur. We will denote it by Gp. In particular if $(M, \bar{\mathbf{r}}) \in \mathcal{X}_M$ then we denote by $\mathsf{Gp}_{\bar{\mathsf{r}}}$ the group defined by $\bar{\mathbf{r}}$ on this sort.

II. The case of an action. Let M_0 be an atomic structure of some language L_0. Define M to be M_0 with an additional sort ω called Gp and the constant symbol 1 interpreted by 1. The symbols $\bar{r}$ include $\cdot$, $^{-1}$ and a new symbol ac for a function $M_0 \times \mathsf{Gp} \to M_0$. The theory T contains the universal axioms of groups on Gp with 1. We also add the axioms for an action:

$$\mathsf{ac}(x, z_1 \cdot z_2) = \mathsf{ac}(\mathsf{ac}(x, z_1), z_2) \ \ , \ \ \mathsf{ac}(x, 1) = x.$$

The space $\mathcal{X}_M$ is the logic space of all countable expansions of M where the group structure is defined on Gp, with a fixed unit 1 and an action ac .

It is natural to add the universal axioms that ac preserves the structure of M_0. In this way we obtain the space of actions on M_0 by automorphisms.

2. SEEKING GENERICS. The following definition concerns cases **I,II**.

Definition 1. *Let $\bar{c}$ be a subtuple of $\bar{c}'$ and $D(\bar{c}') \in \mathcal{B}_T$. We say that $D(\bar{c}')$ is* **t-isolating** *(term-isolating) for $\bar{c}$ if for any two members of $[D(\bar{c}')]$ the groups $\langle \bar{c} \cap \mathsf{Gp} \rangle$ coincide (on the sets of group words of the alphabet $\bar{c} \cap \mathsf{Gp}$).*

In (basic) case **I** let G be a group which is defined on the sort Gp and $G = \langle \bar{c} \rangle$ for some $\bar{c} \subseteq \omega$ (it is not assumed that $G = \omega$). We say that G is *t-isolated* by $D(\bar{c}')$ if $D(\bar{c}')$ is t-isolating for $\bar{c}$ and G is the corresponding $\langle \bar{c} \rangle$.

Definition 2. (Case **I**) *We say that an abstract group G is* **t-isolated** *if it has a Gp-copy, say $\langle \bar{c} \rangle$, which is isolated by some $D(\bar{c}')$ for $\bar{c}$ where $\bar{c}' \subset \langle \bar{c} \rangle$.*

We say that t-isolated diagrams are *dense* in $\mathcal{B}_T$ if any $D_0(\bar{c}) \in \mathcal{B}_T$ extends to some $D(\bar{c}')$ which is t-isolating for $\bar{c}$ and $\bar{c}' \subset \langle \bar{c} \rangle$.

The following proposition is our tool for seeking generics.

Proposition 3. *Under the circumstances of case* **I** *assume that t-isolated diagrams are dense in $\mathcal{B}_T$. Then WAP for $\mathcal{B}_T$ is equivalent to the following property:*

> *any t-isolated G_0 can be extended to a t-isolated G_1 such that any two t-isolated extensions of G_1 can be amalgamated over G_0.*

Let us note that every finite group $F = \langle \bar{c} \rangle$ is t-isolated. Using this we can conclude that *if all groups satisfying T are residually finite then t-isolated diagrams are dense in $\mathcal{B}_T$. Furthermore, then t-isolated groups satisfying T are exactly finite ones.*

Under the scenario of case **I** abelian groups form a closed subspace of $\mathcal{X}_M$. Proposition 3 gives an easy proof that this subspace has a generic group, a result proved in [5]. The following situation is more complicated.

Theorem 4. *Let $c, p \in \omega \setminus \{0, 1\}$ and p be prime $> \mathsf{max}(2, c)$. The closed subspace $\mathcal{G}_{c_{nil}p} \subseteq \mathcal{G}$ of all* **nilpotent groups of degree** *c* **and of exponent** *p has a generic group.*

3. STRONG UNDECIDABILITY IMPLIES NON-WAP. We will assume below the setup of one of the cases **I - II**, where T is a universal theory including group axioms for the sort Gp.

The following proposition is a generalization of the theorem of Kuznetsov that a recursively presented simple group has decidable word problem.

Proposition 5. *Assume that $D(\bar{c}) \in \mathcal{B}_T$ and an extension $D'(\bar{c}') \in \mathcal{B}_T$ is t-isolating for $\bar{c}$. Let $(M, \bar{\mathbf{r}}) \in [D'(\bar{c}')]$. Assume that $D'(\bar{c}')$ is a computably enumerable set. Then in $(M, \bar{\mathbf{r}})$ the elements of $\bar{c}$ which belong to the group sort $\mathsf{Gp}_{\bar{\mathsf{r}}}$ generate a recursively presented group with decidable word problem.*

Our main tool for proving the absence of generics in $\mathcal{G}_T$ is as follows.

Theorem 6. *Assume that $D(\bar{c}) \in \mathcal{B}_T$ (where $\bar{c}$ includes all distinguished elements of M) satisfies the property that every extension $D'(\bar{c}')$ is computably enumerable and for every $(M, \bar{\mathbf{r}}) \in [D'(\bar{c}')]$ the group $\langle \bar{c} \cap \mathsf{Gp}_{\bar{\mathsf{r}}} \rangle$ (defined in $(M, \bar{\mathbf{r}})$) has undecidable word problem.*

Then $D(\bar{c})$ does not have an extension required by WAP for $\mathcal{B}_T$.

We now give one of our applications of Theorem 6 for concrete spaces of enumerated groups.

Theorem 7. *There is a constant C such that for every odd integer $n \geq C$, the space $\mathcal{G}_{exp.n}$ of all groups of exponent n does not have a generic group.*

Remark 8. A similar approach works for the logic spaces of semigroups and rings. We prove that there is neither generic semigroup nor generic associative ring.

Generics over rationals. Consider Scenario **II**, where $M_0 = (\mathbb{Q}, <)$. Define M to be M_0 with the additional sort Gp and the constant symbol 1 interpreted by $1 \in \omega$. The symbols $\bar{r}$ are $\cdot$, ${}^{-1}$ and ac for an action $M_0 \times \mathsf{Gp} \to M_0$ by automorphisms.

The theory T contains the universal axioms of groups on the sort Gp with the unit 1 and the axioms for an action by automorphisms. The space $\mathcal{X}_M$ is the logic space of all countable expansions of M where the group structure is defined on Gp, with a fixed unit 1 and an action ac . We denote it by $\mathcal{X}_{(\mathbb{Q},<)}$.

Theorem 9. *The space $\mathcal{X}_{(\mathbb{Q},<)}$ does not have a generic action.*

References

[1] S. Kunnawalkam Elayavalli I. Goldbring and Y. Lodha. Generic algebraic properties in spaces of enumerated groups. Trans. Amer. Math. Soc. 376, 6245 – 6282, 2023.

[2] A. Ivanov. Generic expansions of ω-categorical structures and semantics of generalized quantifiers. J. Symb. Logic 64, 775 – 789, 1999.

[3] A. Ivanov and K. Majcher. Generic groups and wap. arXiv:2402.02143, 2024.

[4] A. S. Kechris and Ch. Rosendal. Turbulence, amalgamation and generic automorphisms of homogenoeus structures. Proc. Lond. Math. Soc. (3) 94, 302 – 350, 2007.

[5] K. Kanalas T. Kátay M. Elekes, B. Gehér and T. Keleti. Generic countably infinite groups. arXiv:2110.15902, 2021.