Arbitrary Frege Arithmetic

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Abstract

This paper focuses on a less-known version of Abstractionism, that we'll call Arbitrary Logicism. It is obtained by means of a double revision of Frege's Logicist program: on the one side, weakening the Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary, I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such a non-canonical interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms.

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Abstractionist theories in philosophy of mathematics are systems composed by a logical theory augmented with an abstraction principle (AP), of form: $\forall X \forall Y(@X = @Y) \leftrightarrow E(X,Y)$[1] – that introduces, namely rules and implicitly defines, a term-forming operator @ by means of an equivalence relation E. As is well-known, the seminal abstractionist program, Frege's Logicism, failed[2]: Russell's Paradox proved its inconsistency and, *a fortiori*, its non-logicality. In the last century, both the issue of consistency and the issue of logicality have been resumed in the abstractionist debate (cf. [13], [7], [1], [4], [3]). More precisely, on the one side, different revisions of Frege's original system have been proposed in order to avoid Russell's Paradox and to obtain a consistent system that is strong enough to derive (at least, a relevant portion of) Peano Arithmetic. On the other side, given a semantical definition of logicality as permutation invariance, some abstraction principles have been proved to be logical ([1], [4]).

Nevertheless, many concerns are still open. Particularly, regarding the preliminary condition of consistency, the ways out of Russell's Paradox proposed so far do not precisely mirror a corresponding explanation of the origin of the contradiction and often imply a weakening of the hoped strength of the theory (cf. [11], [14], [6])[3]; regarding the issue of logicality, an undesired dilemma overshadows the abovementioned results: precisely in case of logical (i.e. permutation invariant) abstraction principles, their implicit *definienda* turn out to be non logical ([1]) – so preventing a full achievement of the Logicist goal.

My preliminary aim consists in arguing that these – apparently unrelated –problems have a common source in some unquestioned assumptions of Frege's project (inherited also by the following abstractionist programs). I argue that such assumptions are part of what we can call the Traditional view of abstraction, that includes the choice of classical logic as the base theory, with the related semantical consequence of full referentiality of the vocabulary, and the

[1]In the rest of the paper, I'll adopt this axiomatic version of AP. Given full Comprehension Axiom Schema (that will be assumed in the systems that we'll investigate), it is provably equivalent to the schematic form: $@x.\alpha(x) = @x.\beta(x) \leftrightarrow E(\alpha(x), \beta(x))$. Cf. [12]

[2]It was proposed with the foundational purpose to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms.

[3]In [5] and [2], second-order Peano Axioms are recovered but by appealing to stronger logical resources – i.e. double-sorted variables

choice of a so-called Canonical interpretation function for all the (both primitive and defined) expressions of the language.

In the rest of the talk, I show that by renouncing one or both of these problematic assumptions we can recover consistency and/or logicality. More precisely, I propose a double revision of Frege's Logicist program: on the one side, weakening the Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary (cf. [3]), I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such an arbitrary interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms. This means that this system – that we'll call Arbitrary Logicism, precisely renouncing to the Traditional assumptions mentioned above, is able to recover both Frege's goals of consistency and logicality.

The logical part of the language of Arbitrary Logicism, L_F, includes denumerably many first-order variables $(x, y, z, ...)$, denumerably many second-order variables $(X, Y, Z, ...)$, logical connectives $(\neg, \rightarrow)$ and a first-order existential quantifier $(\exists)$[4]. We can also usefully define a predicative monadic constant (E!), whose extension is equal to the range of identity: E!a $=_{def}$ $\exists x(x = a)$. The only non-logical primitive symbol is the term-forming operator ϵ which applies to monadic second-order variables to produce complex singular terms $(\epsilon(X))$[5].

The theory involves, as its logical part, the axioms and inference rules of non-inclusive negative free logic with identity (NF$^=$):

NF1) $\forall v\alpha \rightarrow (E!t \rightarrow \alpha(t/v))$;

NF2) $\exists vE!v$;

NF3) $s = t \rightarrow (\alpha \rightarrow \alpha(t//s))$[6];

NF4) $\forall v(v = v)$;

NF5) $P\tau_1, ..., \tau_n \rightarrow E!\tau_i$ (with $1 \leq i \leq n$);

∀I): $E!a...\phi(a/x) \vdash \forall x\phi$;

∃E): $\phi(a/x), E!a...\psi, \exists x\phi \vdash \psi$, where a is a new individual constant which does not occur in ϕ and ψ.

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[4]We can also define the other connectives and the universal quantifier $\forall xAx =_{def} \neg\exists x\neg Ax$.

[5]Let D be the full first-order domain (then, the second-order domain is constituted by its power-set $\wp(D)$). The satisfaction clauses for the formulas of L_F are defined in terms of an evaluation function V and an assignment function I that ascribes elements of D to the first-order terms and elements of $\wp(D)$ to the second-order terms:

- $V(Pt_1, ..., t_n) = 1 \leftrightarrow I(t_1), ..., I(t_n) \in D \wedge < I(t_1), ..., I(t_n) > \in I(P)$; 0 otherwise;
- $V((s) = (t)) = 1 \leftrightarrow I(s), I(t) \in D \wedge I(s) = I(t)$; 0 otherwise;
- $V(E!t) = 1 \leftrightarrow I(t) \in D$; 0 otherwise;
- $V(\neg\alpha) = 1 \leftrightarrow V(\alpha) = 0$; 0 otherwise;
- $V(\alpha \wedge \beta) = 1 \leftrightarrow \alpha = 1 \wedge \beta = 1$; 0 otherwise;
- $V(\alpha \vee \beta) = 1 \leftrightarrow \alpha = 1 \vee \beta = 1$; 0 otherwise;
- $V(\forall v\alpha) = 1 \leftrightarrow \forall s \in D, V_{(t,s)}(\alpha(t/v)) = 1$ – where t is not in α and $V_{(t,s)}$ is the valuation function on the model $< D, I^* >$ such that $I^* = I$, except that $I^*(t) = s$.
- $V(\forall V\alpha) = 1 \leftrightarrow \forall S \subseteq D, V_{(T,S)}(\alpha(T/V)) = 1$ – where T is not in α and $V_{(T,S)}$ is the valuation function on the model $< D, I^* >$ such that $I^* = I$, except that $I^*(T) = S$.

[6]Where $\alpha(t//s))$ is the result of replacing one or more occurrences of s in A by t.

Additionally, the theory involves an axiom-schema of universal instantiation for second-order variables ($\forall X \phi(X) \rightarrow \phi(Y)$), a rule of universal generalisation (GEN), a second-order comprehension axiom schema (CA: $\exists X \forall x(Xx \leftrightarrow \alpha)$) and *modus ponens* (MP)[7].

The abstraction principle that characterizes this theory is obtained by weakening the right-to-left conditional of Basic Law V (BLV: $\forall F \forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx))$, i.e. BLVa (arbitrarily interpreted), by means of the condition of Permutation Invariance (cf. [1], [3]).

W-BLV: $\forall F \forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx) \wedge \epsilon(\pi(F)) = \pi(\epsilon F))$[8]

As well known, the ϵ operator (as defined by standard BLV), also arbitrarily interpreted, is not Permutation Invariant – because, roughly speaking, by being inconsistent it is unable to define or rule any function. We can emphasize that, given an arbitrary interpretation, Permutation Invariance fails precisely for the argument that determines its inconsistency. In other words, as can be pointed out for other consistent revisions of BLV, in any case in which it is safely restricted, ϵ turns out to satisfy Permutation Invariance, namely it is such that $\pi(\epsilon) = \epsilon$, i.e. $\forall X \forall y(\epsilon X = y \leftrightarrow \epsilon(\pi(X)) = \pi(y))$. Then, the second conjunct of the right-hand side of W-BLV requires that – no matter which object y is identical to ϵF – ϵ satisfies Permutation Invariance for the considered arguments[9].

Accordingly, W-BLV, as a bi-conditional, turns out to be satisfied by any concept instantiating the universal quantifier. On the one side, given an arbitrary interpretation of the abstraction operator, for any concept different from Russellian concept (R), $\pi(\epsilon) = \epsilon$. On the other side, we can consider Russell's Paradox as a *reductio ad absurdum* of the alleged truth of both the sides of the bi-conditional for the concept R: the contradiction proves that ϵR – as legitimately admitted on a free logical background – does not exist, namely it is a term devoid of denotation; accordingly, it is not identical to itself (so, falsifying the left-hand side of W-BLV) and, even if R, as any other concept, is co-extensional with itself, it falsifies Permutation Invariance of the operator[10]. Accordingly, also the right-hand side of W-BLV is false and also the instance of the bi-conditional for the concept R is verified.

Such a restricted version of W-BLV allows us to derive a corresponding restricted version of Hume's Principle. Nevertheless, the same restriction, on HP, is trivially satisfied by any instantiation, so it actually does not represent a weakening of the principle itself and allow us to derive the main arithmetical results, including Frege's Theorem.

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[7] From these axioms we can also derive the following theorems: T1) $\forall x E!x$; T2) $t = t \leftrightarrow E!t$; T3) $(\neg E!s \wedge \neg E!t) \rightarrow (\alpha \rightarrow \alpha(t//s))$.

[8] This abstraction principle is clearly circular because the extensional operator occurs on both the sides of the biconditional. The idea that circularity defeats the definitional role of such principle (or, in general, of implicit definitions) is controversial. Anyway, in this framework, what we need is a principle that rules the behavior of a new symbol of the language and W-BLV carries out this task.

[9] This revision of BLV (particularly of BLVa) is featured by a restriction that, with respect to many other (syntactical ones), is expressible into the language. Indeed, the permutation π of the operator or of the concepts mentioned in the right-hand side of the bi-conditional can be defined as abbreviation of the effects of any first-order bi-jective function f: $D_1 \rightarrow D_1$ on the entities (sets, relations or functions) further up in the type hierarchy.

[10] This last claim follows from the definition of π and the result of non-existence of ϵR: on the one side, $\epsilon(\pi(R)) = \epsilon(\{\pi(x) | x \in R\}) = \epsilon(X)$ – where X is any other concept (based on π); on the other side, $\pi(\epsilon R)$, given that ϵR is not denoting, is another well-formed term without denotation; then, the identity between ϵX (for any X that is obtained by means of a permutation of R) and the empty term $\pi(\epsilon R)$ is false.

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